SEMINAIRE

Equations aux Dérivées Partielles

2004-2005

Wei-Min Wang

Quasi Periodic Solutions of Nonlinear Random Schrödinger Equations

Séminaire É. D. P. (2004-2005), Exposé n° XI, 11 p.

<http://sedp.cedram.org/item?id=SEDP_2004-2005____A11_0>

QUASI PERIODIC SOLUTIONS OF NONLINEAR RANDOM SCHRÖDINGER EQUATIONS

W-M. Wang
CNRS, Universite Paris-Sud

The nonlinear random Schrödinger equation.

We seek time quasi-periodic solutions to the nonlinear random Schrödinger equation

$$i\frac{\partial}{\partial t}u = (\epsilon\Delta + V)u + \delta|u|^{2p}u \qquad (p > 0), \tag{1.1}$$

on $\mathbb{Z}^d \times [0, \infty)$, where $0 < \epsilon, \delta \ll 1$, Δ is the discrete Laplacian:

$$\begin{aligned} \Delta_{ij} &= 1, \quad |i-j|_{\ell^1} = 1, \\ &= 0, \quad \text{otherwise,} \end{aligned} \tag{1.2}$$

$V = \{v_j\}_{j\in\mathbb{Z}^d}$, the potential, is a family of *time independent* independently identically distributed (i.i.d.) random variables with common distribution $g = \tilde{g}(v_j)dv_j$, $\tilde{g} \in L^\infty$. The probability space is taken to be

$$\mathbb{R}^{\mathbb{Z}^d} \text{ with measure } \prod_{j\in\mathbb{Z}^d} g(v_j) = \prod_{j\in\mathbb{Z}^d} \tilde{g}(v_j)dv_j, \ \tilde{g} \in L^\infty. \tag{1.3}$$

$V = \{v_j\}_{j\in\mathbb{Z}^d}$ serve as parameters for the nonlinear problem in (1.1).

Given an initial condition $u(0)$ in $\ell^2(\mathbb{Z}^d)$, one of the central questions is whether $u(t)$ remains localized for all t, i.e., if $u(0) \in \ell^2(\mathbb{Z}^d)$, $\forall\epsilon$, can one find R, such that

$$\|u(t)\|_{\ell^2(\{\mathbb{Z}\backslash[-R,R]\}^d)} < \epsilon, \ \forall t? \tag{1.4}$$

(From now on, we write $|\,|$ for $|\,|_{\ell^1}$, $\|\,\|$ for $\|\,\|_{\ell^2}$.) When $\epsilon = \delta = 0$, the answer to (1.4) is affirmative. Since $u(0) = \sum_{j\in\mathbb{Z}^d} a_j\delta_j$, $a_j \to 0$, as $|j| \to \infty$, $u(t) =$

$\sum_{j\in\mathbb{Z}^d} a_j\delta_j e^{-iv_j t}$ is almost-periodic (infinite number of frequencies) and the upper bound in (1.4) is trivialy verified.

In this talk, we describe a recent result in [BW2], where for appropriate initial conditions $u(0)$, time quasi-periodic solutions to (1.1) were constructed. The answer to (1.4) is therefore affirmative for such $u(0)$'s. Before we enter into the heart of the matter, we first address question (1.4) to

The linear random Schrödinger equation.

When $\delta = 0$, (1.1) reduces to the linear random Schrödinger equation:

$$\begin{aligned} i\frac{\partial}{\partial t}u &= (\epsilon\Delta + V)u, \\ &\overset{\text{def}}{=} Hu \end{aligned} \tag{1.5}$$

on $\mathbb{Z}^d \times [0, \infty)$. When $0 < \epsilon \ll 1$, it is well known from the works in [FS, GMP, FMSS, vDK, AM, AFHS, AENSS] etc. that the upper bound in (1.4) is satisfied. This is customarily called Anderson localization (A.L.) after the physicist P. Anderson [An]. Since the potential is time independent: $V(j,t) = V(j)$, properties of time evolution can be deduced from the spectral properties of H, which we summarize below.

Let $\sigma(H)$ be the spectrum of H. For H defined in (1.5),

$$\sigma(H) = [-2\epsilon d, 2\epsilon d] + \operatorname{supp} g,\ a.s. \tag{1.6}$$

(Recall the probability space defined in (1.3).) [CFKS, PF]. If $0 < \epsilon \ll 1$ and the probability measure satisfies (1.3), then almost surely the spectrum of H is pure point, $\sigma(H) = \sigma_{pp}$, with exponentially localized eigenfunctions: ϕ_j, $j \in \mathbb{Z}^d$.

Given $u(0) \in \ell^2(\mathbb{Z}^d)$, we decompose $u(0)$ as $u(0) = \sum_{j\in\mathbb{Z}^d} a_j\phi_j$. So

$$u(t) = \sum_{j\in\mathbb{Z}^d} a_j\phi_j e^{-i\omega_j t}, \tag{1.7}$$

where ω_j are the eigenvalues for the eigenfunctions ϕ_j. $u(t)$ is almost-periodic and verifies the upper bound in (1.4). So equation (1.5) has A.L.

Some motivations for studying equation (1.1).

Schrödinger equations are equations that describe physical systems, which typically correspond to a n-body problem. The linear equation in (1.5) is a 0^{th} order appproximation, where the n-body interaction is lumped into the effective potential V. Quantum mechanically, $|u|^2$ is interpreted as particle density, so the nonlinear term in (1.1) can be interpreted as modelling particle-particle interaction. (The nonlinear term in (1.1) can be more general and of convolution type. It will not affect our construction below.) This is sometimes called the Hartree-Fock approximation (cf. [O, LL, S]) and is a first order approximation to the original n-body problem. This is our first motivation to study (1.1). Other physical motivations along this line appear in [FSW].

In particular, our method permits us to construct quasi-periodic solutions for the Landau-Lifschitz equations on nonlinear classical spin waves with a large random external magnetic field. Thus

$$\dot{S}_j = S_j \times [(\Delta S)_j + h_j] \qquad (j \in \mathbb{Z}^d)$$

where S_j are unit vectors in $\mathbb{R}^3$ and $h_j = V_j \vec{e}_3$ say; with $V = (V_j)_{j\in\mathbb{Z}^d}$ a large random potential.

As explained in [FSW], we may then seek for a solution $S_j \approx e_3$ and the perturbation is subject to an equation of the form (1.1), but with a nearest neighbor convolution nonlinearity instead of the local one $|u|^{2p}u$ (see [FSW] for details). As mentioned before, (1.1) was chosen as a model but the method described in the paper is sufficiently robust to cover in particular any nonlinearity with finite range interactions.

Our second motivation originates from KAM type of stability questions for infinite dimensional dynamical systems. (For results in the standard KAM context, see e. g. [E].) (1.1) is a Hamiltonian PDE. It can be recast as the equation of motion corresponding to a Hamiltonian of a perturbed $\mathbb{Z}^d$-system of coupled harmonic oscillators with i.i.d. random frequencies (see (2.2, 2.3)). When $\delta = 0$, the linear system has pure point spectrum: $\sigma(H) = \sigma_{pp}$. This corresponds to the KAM tori scenerio. A natural question is the stability of such invariant tori under small

$(0 < \delta \ll 1)$ perturbations, which leads to construction of quasi-periodic or almost periodic solutions to (1.1).

Remark. Previously in [AF, AFS], solutions to the nonlinear eigenvalue problem

$$(\epsilon\Delta + V)\phi + \delta|\phi|^{2p}\phi = E\phi \qquad \text{on} \quad \ell^2(\mathbb{Z}^d)$$

were found, which give the time periodic solutions to (1.1) of the particular form

$$u(j,t) = \phi(j)e^{-iEt}.$$

A sketch of the construction.

We expand in the Fourier basis: $e^{in\cdot\omega t}\delta_k(j)$ and as an ansatz, seek solutions of the form

$$u(\ell,t) = \sum_{(j,n)\in\mathbb{Z}^{d+\nu}} \hat{u}(j,n)e^{in\cdot\omega t}\delta_j(\ell), \tag{1.8}$$

with the initial condition

$$u(\ell,0) = \sum_{k=1}^{\nu} a_k\delta_k(\ell), \quad \text{satisfying} \sum_{k=1}^{\nu} |a_k| \ll 1, \tag{1.9}$$

where in (1.9), we identify $\{j_k\}_{k=1}^{\nu}$ with $\{1,...,\nu\}$, δ_k with δ_{j_k} $(k = 1,...,\nu)$. The unperturbed frequencies are therefore $\omega = \omega(\mathcal{V}) = \mathcal{V} \in \mathbb{R}^\nu$, where $\mathcal{V} \stackrel{\text{def}}{=} \{v_{j_k}\}_{k=1}^{\nu}$ are the random potentials at sites $j_k \in \mathbb{Z}^d$.

Substituting (1.8) into (1.1), we obtain the following equation for the Fourier coefficients:

$$(n\cdot\omega + \epsilon\Delta_j + V_j)\hat{u}(j,n) + \delta[(\hat{u} * \hat{v})^{*p} * \hat{u}](j,n) = 0, \tag{1.10}$$

where $\hat{v}(j,n) = \bar{\hat{u}}(j,-n)$, the convolution $*$ is in the n variable only, $*p$ denotes the p-fold convolution and we added the subscript j to operators that originated from $\ell^2(\mathbb{Z}^d)$. We also write the equation for $\hat{v}$:

$$(-n\cdot\omega + \epsilon\Delta_j + V_j)\hat{v}(j,n) + \delta[(\hat{u} * \hat{v})^{*p} * \hat{v}](j,n) = 0. \tag{1.11}$$

Combining (1.10, 1.11), we then have a closed system of equations for $y = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$, which we write as

$$F(y) = 0. \tag{1.12}$$

Equation (1.12) is a $\mathbb{Z}^{d+\nu}$ system of equations. Let $y_0 = y(t = 0)$.

$$\text{supp } y_0 = \{j_k,\ -e_k\}_{k=1}^{\nu} \cup \{j_k,\ e_k\}_{k=1}^{\nu},$$

where e_k are the unit vectors of $\mathbb{Z}^\nu$. We seek solutions to (1.12) with y fixed at the initial condition on supp y_0. We make a Lyapunov-Schmidt decomposition as in [CW1,2, B1,3]. Let $y_0 = y(t = 0)$. The equations

$$F(y) = 0|_{\mathbb{Z}^{d+\nu}\backslash \text{supp}\, y_0} \qquad \text{on} \quad \ell^2(\mathbb{Z}^{d+\nu}\backslash \text{supp}\, y_0)$$

are the so called P-equations, the rest are the Q-equations. The P-equations are used to determine $y(j, n)$ on $\{\text{supp}\, y_0\}^c$. On supp y_0, $y(j, n)$ are held fixed at the initial condition from (1.9). Instead the ν Q-equations determine $\omega = \omega(\mathcal{V})$.

We use a Newton scheme to solve the P-equations (for more details, see section 3). This leads to investigate the invertibility of the linearized operators $F_i'(y_i)$, where y_i is the i^{th} approximate solution, F_i' is F' restricted to $[-M^{i+1}, M^{i+1}]^{d+\nu}$ $(i \geq 0)$ for appropriate M.

The random potentials $\mathcal{V} = \{v_{j_k}\}_{k=1}^{\nu} \in \mathbb{R}^\nu$ are the parameters in the problem. Invertibility of $F_i'(y_i)$ are assured by appropriate incisions in $\mathcal{V}$. Similar to the linear case in [BW1], this is done by using semi-algebraic set techniques to control the complexity of the sigular sets and Cartan type of theorem for analytic matrix valued functions to control the measure.

The main difference with the linear case in [BW1] is that F_i' are evaluated at different y_i. But due to rapid convergence of the Newton scheme, made possible by estimates on $F_{i'}'(y_{i'})$ for $i' < i$, this is within the margin of estimates.

Solving the P-equations iteratively is the main part of the work. The solutions to the P-equations are then substituted into the Q-equations to determine $\omega = \omega(\mathcal{V})$ iteratively by using the implicit function theorem. We obtain time quasi-periodic solutions of the form (1.8) to (1.1), which are exponentially localized (both in the spacial and Fourier space) to the initial condition (1.9), with modified frequencies $\omega = \omega(\mathcal{V})$, which are $(\epsilon + \delta)$-close to the unperturbed frequencies $\mathcal{V} = \{v_{j_k}\}_{k=1}^{\nu}$.

We therefore have

Statement of the Theorem.

Theorem 1. *Consider the nonlinear random Schrödinger equation*

$$i\frac{\partial}{\partial t}u = (\epsilon\Delta + V)u + \delta|u|^{2p}u, \qquad (p \in \mathbb{N}^+), \tag{1.14}$$

where Δ is the discrete Laplacian defined in (1.2), $V = \{v_j\}_{j\in\mathbb{Z}^d}$ is a family of i.i.d. random variables with common distribution g satisfying (1.3). Fix $j_k \in \mathbb{Z}^d$, $k = 1, \cdots, \nu$. Let $\mathcal{R} = \{j_k\}_{k=1}^{\nu} \subset \mathbb{Z}^d$, $\mathcal{V} = \{v_\alpha\}_{\alpha\in\mathcal{R}} \in \mathbb{R}^\nu$. Consider an unperturbed solution of (1.14) with ϵ, $\delta = 0$,

$$u_0(y,t) = \sum_{k=1}^{\nu} a_k e^{-iv_{j_k}t}\delta_{j_k}(y),$$

with $\sum_{k=1}^{\nu}|a_k|$ sufficiently small. Let $a = \{a_k\}_{k=1}^{\nu}$.

For $0 < \epsilon \ll 1$, $\exists X_\epsilon \subset \mathbb{R}^{\mathbb{Z}^d}\backslash\mathbb{R}^\nu$ of positive probability, such that for $0 < \delta \ll 1$,if we fix $x \in X_\epsilon$, there exists $\mathcal{G}_{\epsilon,\delta}(x;a) \subset \mathbb{R}^\nu$, Cantor set of positive measure. There is $\omega = \omega_{\epsilon,\delta}(\mathcal{V};a)$ smooth function defined on $\mathcal{G}_{\epsilon,\delta}(x;a)$, such that if $\mathcal{V} \in \mathcal{G}_{\epsilon,\delta}(x;a)$, then

$$u_{\epsilon,\delta,x}(y,t) = \sum_{(j,n)\in\mathbb{Z}^{d+\nu}} \hat{u}(j,n)e^{in\cdot\omega t}\delta_j(y) \tag{1.15}$$

is a solution to (1.14), satisfying

$$\begin{aligned} &\hat{u}(j_k, -e_k) = a_k, \\ &\sum_{(j,n)\notin\mathcal{S}} e^{c(|n|+|j|)}|\hat{u}(j,n)| < \sqrt{\epsilon+\delta} \qquad (c > 0), \\ &|\omega - \mathcal{V}| < c(\epsilon+\delta), \end{aligned} \tag{1.16}$$

for some $c > 0$, and where $\{e_k\}_{k=1}^{\nu}$ are the basis vectors for $\mathbb{Z}^\nu$ and $\mathcal{S} = \{j_k, -e_k\}_{k=1}^{\nu} \subset \mathbb{Z}^{d+\nu}$. The sets X_ϵ and $\mathcal{G}_{\epsilon,\delta}(x;a)$ satisfy

$$\textit{Prob } X_\epsilon \to 1, \quad \textit{mes } \mathbb{R}\backslash\mathcal{G}_{\epsilon,\delta}(x;a) \to 0 \quad \textit{as} \quad \epsilon+\delta \to 0.$$

Remark. The set $X_\epsilon \in \mathbb{R}^{\mathbb{Z}^d}\backslash\mathbb{R}^\nu$ only depends on ϵ; while the set $\mathcal{G}_{\epsilon,\delta}(x;a) \in \mathbb{R}^\nu$ depends on ϵ, δ, $x \in X_\epsilon$ (the random potentials in X_ϵ) and a (the initial amplitude).

Corollary. *There exists* $X_{\epsilon,\delta} \subset \mathbb{R}^{\mathbb{Z}^d}$ *of positive probability,* $0 < \epsilon \ll 1$, $0 < \delta \ll 1$, *satisfying*

$$\textit{Prob } X_{\epsilon,\delta} \to 1 \quad \textit{as} \quad \epsilon + \delta \to 0,$$

such that for initial amplitudes a sufficiently small, there are quasi-periodic solutions to (1.14).

An insertion into a larger picture.

The theorem presented above is proven for i.i.d. random potentials $V = \{v_j\}_{j\in\mathbb{Z}^d}$. The construction used to prove the theorem is, however general. It only requires an *eigenvalue separation property* on the linear operator H, (aside from ϵ and δ being small). For example, in the i.i.d. random case, this is manefested as:

if Λ_1, Λ_2 are two subsets at scale L and $\Lambda_1 \cap \Lambda_2 = \emptyset$, then

$$\text{dist}\,(\sigma(H_1), \sigma(H_2)) \geq e^{-L^\sigma}\ (0 < \sigma < 1),$$

on a probability subset of measure at least $1 - e^{-L^{\sigma'}}$ $(0 < \sigma' < \sigma)$. (1.17)

When V is a given function, similar separation property could be obtained from number theoretical considerations (see e.g., [B3-5]).

The construction of time quasi-periodic (or almost-periodic) solutions needs a parameter. This parameter can sometimes be extracted from amplitude-frequency modulation, see e.g., [B1, 3, KP]. Nonlinear random Schrödinger equation is an equation endowed with a family of parameters. The separation property (1.17) can be easily obtained by a first order variation. So it is a natural medium to enact such constructions.

The continuum Schrödinger equations (linear or nonlinear) are a more frequently studied subject. The discrete nonlinear Schrödinger equation presented here should be seen as the analogue of the continuum nonlinear Schrödinger equation in a compact domain, e.g., on a torus. The $\mathbb{Z}^d$ lattice therefore can be seen as the indices of the eigenvalues or eigenfunctions for the underlying linear Schrödinger operator.

Time quasi-periodic solutions have been constructed for the continuum nonlinear Schrödinger or wave equation in 1-D, on a finite interval with either Dirichlet or periodic boundary conditions. See for example, the works of Bourgain, Kuksin, Pöschel and Wayne in [B1, KP, W]. In [B3], time quasi-periodic solutions are constructed for the 2-D nonlinear Schrödinger equation on $\mathbb{T}^2$.

The construction presented here is related to those in [B1-5], which use a Newton scheme directly on the equations. This direct approach is originated by Craig and Wayne in [CW1,2]. It has the advantage of not relying on the underlying Hamiltonian structure. The Hamiltonian structure does assure, however, the reality of the frequency ω during the iteration.

We remark also that the present method, as it stands, does not yet extend to the construction of almost-periodic solutions. This is because our point of perturbation is the equation

$$i\frac{\partial}{\partial t}u = Vu.$$

In order to construct almost-periodic solutions, we will need more informations on the spectrum of the linear operator $H = \epsilon\Delta + V$.

In [B2], the construction of almost-periodic solutions was made possible by the precise knowledge of the spectrum of the linear operator and the fact that the perturbation is quartic (in the Hamiltonian). In the present case it is quadratic. Almost-periodic solutions have also been constructed by Pöschel [Pö2] in the case of a nonlinear Schrödinger equation, where the nonlinearity is "nonlocal".

PDE's (such as (1.1)) typically correspond to the so called "short range" (but not finite range) case. In the "finite range" case, which typically corresponds to perturbation of integrable Hamiltonian systems, almost-periodic solutions have been constructed in e.g., [FSW, Pö1, CP] among others.

Results like those in the Theorem 1 are nice, in the sense that they give very detailed information on the solutions. But the initial conditions are very special. It is natural to inquire what happens to more generic initial conditions. We end the talk by presenting a result in that direction. This is the theorem in [BW3].

We consider a slightly tempered equation in $1-d$:

$$i\dot{q}_j = v_j q_j + \epsilon(q_{j-1} + q_{j+1}) + \lambda_j q_j |q_j|^2 = 0, \quad j \in \mathbb{Z},$$

where we take for instance $V = \{v_j\}$ to be independent randomly chosen variables in $[0,1]$ (uniform distribution). The multiplier $\{\lambda_j\}_{j\in\mathbb{Z}}$ satisfies the condition

$$|\lambda_j| < \epsilon(|j|+1)^{-\tau}$$

with $\tau > 0$ fixed and arbitrarily small. Note that $\tau = 0$, $\lambda_j = 1$ for all $j \in \mathbb{Z}$, is the standard lattice random Schrödinger equation.

We have the following bound on the discrete equivalent of H^1 norm:

Theorem 2. *Given $\tau > 0$, $\kappa > 0$ and taking $0 < \epsilon < \epsilon(\tau,\kappa)$, the following is true almost surely in V. If at $t = 0$, the initial datum $\{q_j(0)\}_{j\in\mathbb{Z}}$ satisfies*

$$\sum_{j\in\mathbb{Z}} j^2 |q_j(0)|^2 < \infty$$

then

$$\sum_{j\in\mathbb{Z}} j^2 |q_j(t)|^2 < t^\kappa \; as \, t \to \infty.$$

The bound in Theorem 2 shows that if there is propagation, it is very slow $\sim t^{\kappa/2}$. If the initial datum is in H^1, then the growth of H^1 norm in time cannot be faster than $t^{\kappa/2}$. (Recall also that $\sim t^{1/2}$ is diffusive, $\sim t$ is ballistic.)

References

[**AENSS**] M. Aizenman, A. Elgart, S. Naboko, S. Shankar, G. Stoltz,, (in preparation) (2002).

[**AFHS**] M. Aizenman, R. Friedrich, D. Hundertmark, S. Shankar, *Constructive fractional-moment criteria for localization in random operators*, Phys. A **279** (2000), 369-377.

[**AM**] M. Aizenman, S. Molchanov, *Localization at large disorder and at extreme energies: an elementary derivation*, Commun. Math. Phys. **157** (1993), 245.

[**AF**] C. Albanese, J. Fröhlich, *Periodic solutions of some infinite-dimensional Hamiltonian systems associated with non-linear partial difference equations I*, Commun. Math. Phys. **116** (1988), 475-502.

[**AFS**] C. Albanese, J. Fröhlich, T. Spencer, *Periodic solutions of some infinite-dimensional Hamiltonian systems associated with non-linear partial difference equations II*, Commun. Math. Phys. **119** (1988), 677-699.

[**An**] P. Anderson, *Absence of diffusion in certain random lattices*, Phys.Rev. **109** (1958), 1492.

[**Ba**] S. Basu, *On bounding the Betti numbers and computing the Euler characteristic of semi-algebraic sets*, Discrete Comput. Geom. **22** (1999), 1-18.

[**B1**] J. Bourgain, *Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE*, IMRN **11** (1994), 475-497.

[**B2**] J. Bourgain, *Construction of approximate and almost periodic solutions of perturbed linear Schrödinger and wave equations*, GAFA **6** (1996), 201-230.

[**B3**] J. Bourgain, *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*, Ann. Math **148** (1998), 363-439.

[**B4**] J. Bourgain, *Nonlinear Schrödinger equations*, Park City Lectures, 1999.

[**B5**] J. Bourgain, *Green's function estimates for latttice Schrödinger operators and applications*, Lectures at UC, Irvine and UCLA, 2000-2001.

[**BGS**] J. Bourgain, M. Goldstein, W. Schlag, *Anderson localization for Schrödinger operators on $\mathbb{Z}^2$ with quasi-periodic potential*, Acta Math. **188** (2002), 41-86.

[**BW1**] J. Bourgain, W.-M. Wang, *Anderson localization for time quasi-periodic random Schrödinger and wave equations*, Commun. Math. Phys. **248** (2004), 429-466.

[**BW2**] J. Bourgain, W.-M. Wang, *Quasi periodic solutions of nonlinear random Schrödinger equations*, (submitted) (2004).

[**BW3**] J. Bourgain, W.-M. Wang, *Diffusion bound for a nonlinear Schrödinger equation*, (preprint) (2004).

[**CP**] L. Chierchia, P. Perfetti, *Maximal almost-periodic solutions for Lagrangian equations on infinite dimensional tori*, Seminar on Dynamical Systems. Eds. S. Kuksin, V. Lazutkin, J. Pöschel. Birkhäuser, Basel (1994), 203-212.

[**CW1**] W. Craig, C. E. Wayne, *Newton's method and periodic solutions of nonlinear equations*, Commun. Pure Appl. Math. **46** (1993), 1409-1498.

[**CW2**] W. Craig, C. E. Wayne, *Periodic solutions of nonlinear Schrödinger equations and the Nash-Moser method*, Hamiltonian Mechanics, 103-122, Nato Adv. Sci. Inst. Ser. B Phys. 331, Plenum, NY, 1994.

[**CFKS**] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators*, Springer-Verlag, 1987.

[**vDK**] H. von Dreifus, A. Klein, *A new proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **124** (1989), 285-299.

[**E**] L. H. Eliasson, *Perturbations of stable invariant tori for Hamiltonian systems*, Ann. Scuola Norm. Sup. Pisa CI. Sci **15** (1988), 115-147.

[**FMSS**] J. Fröhlich, F. Martinelli, E. Scoppola, T. Spencer, *Constructive proof of localization in Anderson tight binding model*, Commun. Math. Phys. **101** (1985), 21-46.

[**FS**] J. Fröhlich, T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Commun. Math. Phys. **88** (1983), 151-184.

[**FSW**] J. Fröhlich, T. Spencer, C. E. Wayne, *Localization in disordered, nonlinear dynamical systems*, J. Stat. Phys. **42** (1986), 247-274.

[**GMP**] Ya. Gol'dsheid, S. Molchanov, L. Pastur, *Pure point spectrum of stochastic one dimensional Schrödinger operators*, Func. Anal. Appl **11, 1** (1977).

[**KP**] S. Kuksin, J. Pöschel, *Invariant Cantor manifolds of quasi-periodic osillations for a nonlinear Schrödinger equation*, Ann. Math. **143** (1996), 149-179.

[**Le**] Ya. B. Levin, *Lectures on Entire Functions*, Transl. of Math. Monographs, vol. 150, AMS, Providence, RI, 1996.

[**LZ**] B. M. Levitan, V. V. Zhikov, *Almost-Periodic Functions and Differential Equations*, von Nostrand Reinhold, NY, 1971.

[**LL**] E. H. Lieb, M. de Llano, *Solitons and the delta function fermion gas in the Hartree-Fock theory*, J. Math. Phys. **19** (1978), 860.

[**O**] A. W. Overhauser, *Structure of nuclear matter*, Phys. Rev. Lett. **4** (1960), 415.

[**Pö1**] J. Pöschel, *Small divisors with spatial structure in infinite dimensional Hamiltonian systems*, Commun. Math. Phys. **127** (1990), 351-393.

[**Pö2**] J. Pöschel, *On the construction of almost periodic solutions for a nonlinear Schrödinger equation*, Ergod. Th. and Dynam. Sys. **22** (2002), 1537-1549.

[**PF**] L. Pastur, A. Figotin, *Spectra of Random and Almost Periodic Operators*, Springer, 1992.

[**S**] B. Sriram Shastry, *Exact solutions of a nonlinear eigenvalue problem in one dimension*, Phys. Rev. Lett. **50** (1983), 633.

[**W**] C. E. Wayne, *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Commun. Pure Appl. Math. **127** (1990), 479-528.