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Christiaan Stolk

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U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Microlocal analysis and seismic imaging

Christiaan C. Stolk

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Abstract

We study certain Fourier integral operators arising in the inversion of data from reflection seismology.

1 Introduction

In reflection seismology one studies the subsurface of the earth using acoustic waves. Measurements are taken using sources and receivers of acoustic waves located at the surface. An important step in the processing of data is the inversion of the so called forward operator (or linearized forward operator) F, described precisely below. The operator F is a Fourier integral operator between distribution spaces

$$F : \mathcal{E}'(X) \to \mathcal{D}'(Y)$$

where X and Y are open subsets of $\mathbb{R}^n$ resp. $\mathbb{R}^m$, $n < m$. As $n < m$ this is a formally overdetermined problem, and one is not only interested in the inversion of F, but also for instance in a characterization of its range.

This paper is about the structure of F, and about factorizing this operator like

$$F = \bar{F} \circ \chi. \tag{1.1}$$

Here we search for $\bar{F}$ that is an invertible Fourier integral operator, and χ that is a natural injection of $\mathcal{E}'(X)$ into some larger space of distributions $\mathcal{D}'(X_0)$, $\dim X_0 = m$. In that case we will call $\bar{F}$ an extension of F, terminology introduced by Symes [13].

The question of constructing such an extension arises quite naturally. For example they can be used to construct explicitly the pseudodifferential "annihilators", introduced by Guillemin [4] to characterize the range of operators like F. In seismic data processing it is related to *prestack migration*.

The inverse problem of reflection seismology

Let us first describe the seismic experiment, and the model of the data that we will use. In a seismic experiment one puts sources at the surface of the earth, that emit a short pulse of acoustic waves. The waves propagate inside the medium, and part of the energy reflects where the mechanical properties of the medium vary strongly (e.g. at discontinuities). The waves that arrive back at the surface are observed.

We consider the linearized, high-frequency model usually used in processing the data. Throughout the paper $c = c(x)$ will denote a smooth function that has as its values a wavespeed as a function of the subsurface position. We will call this the background medium, or velocity model. The associated wave operator will be denoted by

$$P = c(x)^{-2}\partial_t^2 - \Delta.$$

The modeling starts with an *incoming wave field* $u_{\mathrm{i}} = u_{\mathrm{i}}(x, t, x_{\mathrm{s}})$. It depends on the source position denoted by x_{s}, in addition to the position and time variables (x, t). It is assumed to satisfy

$$Pu_{\mathrm{i}}(x, t) = w(t)\delta(x - x_{\mathrm{s}}) \text{ on } \mathbb{R}^n \times \mathbb{R}_+, \quad u_{\mathrm{i}} = 0 \text{ for } t < 0.$$

Here $w(t)$ is the source wavelet, that will be chosen equal to $w(t) = \delta(t)$. Next a reflected wave field u_{r} is defined. It is defined as the perturbation of u_{i} under a formal linearization, where $c(x)$ is replaced by $c(x)(1 + \frac{1}{2}r(x))$. This results in the following equation for u_{r}

$$Pu_{\mathrm{r}} = r(x)c(x)^{-2}\partial_t^2 u_{\mathrm{i}} \tag{1.2}$$

The perturbation $\frac{1}{2}c(x)r(x)$ is assumed to contain the discontinuities in the medium that cause the reflections. The modeled data is given by the reflected wave field

$$d(x_{\mathrm{s}}, x_{\mathrm{r}}, t) = u_{\mathrm{r}}(x_{\mathrm{r}}, t, x_{\mathrm{s}}), (x_{\mathrm{s}}, x_{\mathrm{r}}, t) \in Y,$$

where Y is the set of points $(x_{\mathrm{s}}, x_{\mathrm{r}}, t)$ for which measurements are done. Source and receiver points x_{s}, resp. x_{r} are contained in the surface $\mathbb{R}^{n-1} \times \{0\}$, therefore Y will be a submanifold of $\mathbb{R}^{2n-1}$.

One may consider different sets Y. In this paper we assume Y is given by $Y = X_{\mathrm{s}} \times X_{\mathrm{r}} \times]0, T[$, where X_{s} is an open subset of $\mathbb{R}^{n-1}$ containing the source locations, X_{r} is an open subset of $\mathbb{R}^{n-1}$ of receiver locations, and $]0, T[$ is the time interval that measurements are taken. We will identify $X_{\mathrm{s}} \subset \mathbb{R}^{n-1}$ with the subset $X_{\mathrm{s}} \times \{0\}$ of $\mathbb{R}^n$, and the same for $\mathbb{R}_{\mathrm{r}}$. Other possibilities include the case of a single source, say at $x_{\mathrm{s}} = 0$, for which $Y = \{(0, x_{\mathrm{r}}, t); (x_{\mathrm{r}}, t) \in Y' \subset \mathbb{R}^{n-1} \times \mathbb{R}_+\}$, the case of constant offset $Y = \{(x_{\mathrm{s}}, x_{\mathrm{s}} + h, t); (x_{\mathrm{s}}, t) \in Y' \subset \mathbb{R}^{n-1} \times \mathbb{R}_+\}$, where h is some fixed vector in $\mathbb{R}^{n-1}$. In $n = 3$ dimensions Y may be of codimension 1, as follows $Y = \{(x_{\mathrm{s}}, x_{\mathrm{r}}, t) \in \mathbb{R}^{2n-2} \times \mathbb{R}_+; v \cdot (x_{\mathrm{s}} - x_{\mathrm{r}}) = 0\}$, for some non-zero vector $v \in \mathbb{R}^{n-1}$ (single streamer geometry).

The modeled data depends on both c and r. Both are in general unknown, and are to be determined from the data. Here we will assume that c is given, and study the linear inverse problem for r, except in the last section, where we make a few remarks on the reconstruction of c. We therefore define the *forward map* to be the linear map

$$F : r \mapsto d.$$

We assume r is zero outside a bounded open subset X such that the closure $\overline{X}$ is contained in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_n > 0\}$.

Fourier integral operators and inversion

Let us recall a few facts about Fourier integral operators (see e.g. [3, 5, 16]). A Fourier integral operator mapping functions of $x \in X$ to functions of $y \in Y$ is an operator with Schwartz kernel given by a locally finite sum of terms

$$\sum_j \int_{\mathbb{R}^{N^{(j)}}} e^{i\phi^{(j)}(y,x,\theta)} a^{(j)}(y, x, \theta)\, d\theta, \tag{1.3}$$

for some $N^{(j)} \geq 1$. For simplicity, omit the index j. Here ϕ is called the phase function and is assumed smooth, homogeneous of order 1 in θ and non-degenerate, i.e.

$$d_\theta \phi(y, x, \theta) = 0 \Rightarrow d_{(y,x,\theta)} \frac{\partial \phi(y, x, \theta)}{\partial \theta_j} \text{ are linearly independent for } j = 1, \ldots, N.$$

The amplitude is a symbol in a class $S^m((Y \times X) \times \mathbb{R}^N)$, i.e. the class of functions in $C^\infty(Y \times X \times \mathbb{R}^N)$ such that for each compact set K and each pair of multi-indices α, β, there is a constant $C_{\alpha,\beta,K}$ with

$$\partial_\theta^\alpha \partial_{(x,y)}^\beta \leq C_{K,\alpha,\beta}(1 + \|\theta\|)^{m-\|\alpha\|}.$$

The essential support $\operatorname{esssupp} a$ is defined as the smallest conic subset of $Y \times X \times \mathbb{R}^N \backslash \{0\}$ outside which a is of class $S^{-\infty}$.

Associated with a non-degenerate phase function is a Lagrangian submanifold of the cotangent space $T^*(Y \times X)\backslash 0$, given by

$$\Lambda_\phi = \{(y, x, d_{(y,x)}\phi) \in T^*(Y \times X)\backslash 0$$

The canonical relation is defined by

$$\Lambda'_\phi = \{(y, x, \eta, \xi); (y, x, \eta, -\xi) \in \Lambda_\phi\}.$$

The global Fourier integral operator has canonical relation given by the union $\bigcup_j \Lambda'_{\phi(j)}$. We will sometimes denote the canonical relation also by Λ'_F. An important property is that different sets of phase functions may parameterize the same canonical relation. A Fourier integral operator can be represented by the different sets of phase functions as long as these phase functions parameterize the same canonical relation [3, theorem 2.3.4]. A second important property is that Fourier integral operators form a calculus, their composition is again a Fourier integral operator if the canonical relations can be composed in a clean or transversal way. For the precise results on the composition we refer to [3], theorem 2.4.1, and [5], section 25.2.

Under certain conditions the map F is a Fourier integral operator [8, 15]. The calculus of Fourier integral operators can then be used for the reconstruction of singularities of r. If certain conditions on c are satisfied, that depend on Y, a Fourier integral operator G can be constructed such that

$$GF = \text{ a pseudodifferential operator },$$

and the symbol of GF is 1 plus a term in $S^{-\infty}$ on a conic subset of $T^*(X)\backslash 0$, say Γ. Thus there is an approximate, high-frequency reconstruction of r in the sense that

$$\mathrm{WF}(Gd - r) \cap \Gamma = \emptyset,$$

where $\mathrm{WF}(\cdot)$ denotes Hörmanders wave front set. Such reconstructions for different choices of Y are given for example in [1, 15, 7].

The contents of the paper is as follows. In section 2 we study the connection with the paper [4]. We obtain there an extension for Fourier integral operators satisfying the Bolker condition. Our main result is in section 3, where we construct explicitly an extension for the seismic problem. In the fourth and last section we say a few words about the application in seismic data processing.

2 Microlocal extension when the Bolker condition is satisfied

In this section we describe the main steps in the construction of an extension when the Bolker condition of Guillemin [4] is satisfied. We then give a short discussion on the pseudodifferential "annihilators" introduced in that paper.

We assume $\Gamma \subset T^*(Y \times X)\backslash 0$ is a canonical relation, Y is a bounded open subset of $\mathbb{R}^m$, X a bounded open subset of $\mathbb{R}^n$, $n < m$, and π_X, π_Y are the projections from Γ to the factors $T^*(X)$ and $T^*(Y)$. This can be summarized in the diagram

$$\begin{array}{ccc} & \Gamma & \\ \pi_Y \swarrow & & \pi_X \searrow \\ T^*(Y)\backslash 0 & & T^*(X)\backslash 0 \end{array}$$

We have the following

Assumption 1. *(Bolker assumption [4]) The projection π_Y is an injective and proper immersion of Γ in $T^*(Y)\backslash 0$.*

The notation Σ will be used for the image of Γ under π_Y.

Standard coordinates

Let ρ be a point in Γ, and consider the tangent maps to the projections, $T_\rho\pi_X$ and $T_\rho\pi_Y$. By lemma 25.3.6 of [5], it follows that the rank λ_1 of the projection $T_\rho\pi_Y$, and λ_2 of $T_\rho\pi_X$ satisfy $\lambda_1 = \lambda_2 + \dim Y - \dim X$, so that Assumption 1 implies that the projection π_X

has everywhere maximal rank, and that π_X defines a fibration on Γ, and a fibration with isotropic fibers on Σ.

By (x, ξ) we denote coordinates on $T^*(X)$. Because π_X is a submersion, we can use (x, ξ) as local coordinates on Γ, and hence on Σ, together with a coordinate on the fiber, that we will call s. So, let $(x, s, \xi) \mapsto \rho(x, s, \xi)$ be such a parameterization of some subset U_0 on Σ. Then we have (using that $\frac{\partial \rho}{\partial s_j}$ is tangent to the fibers)

$$\sigma_Y\left(\frac{\partial \rho}{\partial \xi_j}, \frac{\partial \rho}{\partial x_k}\right) = \delta_{j,k}$$
$$\sigma_Y\left(\frac{\partial \rho}{\partial x_j}, \frac{\partial \rho}{\partial x_k}\right) = \sigma_Y\left(\frac{\partial \rho}{\partial \xi_j}, \frac{\partial \rho}{\partial \xi_k}\right) = \sigma_Y\left(\frac{\partial \rho}{\partial s_j}, \frac{\partial \rho}{\partial (s,x,\xi)_k}\right) = 0$$

Then there is an open subset U of $T^*(Y)\backslash 0$, such that $U_0 = U \cap \Sigma$, and a coordinate transformation

$$\Phi : U \ni (y, \eta) \mapsto (x, s, \xi, \sigma),$$

such that Σ is given locally by $\sigma = 0$. The proof is by an adaption of the proof of the Darboux theorem, and was done in [10], section 5. We let V be the image of U under Φ. (Standard coordinates for canonical relations were also discussed by Hörmander [5].)

Extension of F

We let U_1 be an open subset of U, such that $\overline{U_1}$ is contained in U, and $V_1 = \Phi(U_1)$.

Now let H be a Fourier integral operator of order 0 with canonical relation

$$\{(\Phi(y, \eta); y, \eta)\,;\, (y, \eta) \in U\},$$

whose principal symbol is nonzero on $\{(\Phi(y, \eta); y, \eta)\,;\, (y, \eta) \in U_1\}$. Then it follows from the composition theorem of Fourier integral operators [3, theorem 2.4.1], that HF is a Fourier integral operator with canonical relation given by

$$\{(x, s, \xi, 0; x, \xi)\,;\, (x, s, \xi, 0) \in V\}.$$

It follows that HF is an s-family of pseudodifferential operators, that we will denote by $A(x, s, D_x)$.

We define the map χ from (1.1) by

$$\chi : \mathcal{D}'(X) \to \mathcal{D}'(X \times S) : \chi f(x, s) = f(x).$$

Let $B = B(x, s, D_x)$ be a microlocal inverse of A for $(x, s, \xi, 0) \in V_1$. Then we have

$$BHFf \equiv \chi f \text{ on } V_1,$$

microlocally (the notation meaning that $\mathrm{WF}(BHFf - \chi f) \cap V_1 = \emptyset$). Denote by $\langle H \rangle^{-1}$ a Fourier integral operator with canonical relation

$$\{(y, \eta; \Phi(y, \eta))\,;\,(y, \eta) \in U\}$$

such that the symbol of the pseudodifferential operator $\langle H \rangle^{-1} H$ is 1 plus a term in $S^{-\infty}$ on U_1. We now let

$$\bar{F} = \langle H \rangle^{-1} A$$

The operator $\bar{F}$ is a microlocal extension, since

$$\bar{F}\chi f \equiv \langle H \rangle^{-1} ABHf \equiv f,$$

microlocally on V_1. It follows now that, if ψ_1 is a pseudodifferential cutoff with support contained in U_1, then we have

$$\psi_1 \bar{F} \chi = \psi_1 F.$$

Annihilators

Guillemin introduced the left ideal I_F of pseudodifferential operators on Y such that

$$W \in I_F \Leftrightarrow WF \text{ is a smoothing operator.}$$

Theorem 2 in [4] states that the characteristic variety of I_F is Σ, and that for each $(y, \eta) \in \Sigma$ there exist $l \stackrel{\text{def}}{=} \dim Y - \dim X$ commuting pseudodifferential operators $W_1, \ldots, W_l \in I_F$ whose symbols are defining functions of Σ near (y, η), such that I_F is generated, near (y, η), by the W_j's. If d is a compactly supported distribution on Y such that Wd is smooth for all $W \in I_F$, then

$$d = Ff + g,$$

where f is a compactly supported distribution on X and g is smooth. In this way the $W \in I_F$ characterize the functions in the range of F, modulo smooth functions.

With the operator $\bar{F}$, and H and B defined above, such pseudodifferential operators W_j, are directly constructed. Denote $\langle \bar{F} \rangle^{-1} = BH$, then we can set

$$W_j = \psi_1 \bar{F} D_{s_j} \langle \bar{F} \rangle^{-1}, j = 1, \ldots, \dim Y - \dim X.$$

The W_j are such that

$$W_j F \text{ is regularizing.}$$

3 Construction of an extension of F

In this section we construct an extension of the operator F. It will be slightly different from the description above. We will use that with a function $r = r(x)$ there is naturally associated a multiplication operator M_r mapping $f \mapsto rf$. With an operator on functions of X is associated a distribution kernel in $\mathcal{D}'(X \times X)$. In the extension here, χ will be a mapping of functions to the associated multiplication operators, or their distribution kernels, in a subspace of operators,

First some notation. Let the Green's function $G = G(x, y, t)$ be defined by

$$PG(x, y, t) = \delta(x - y)\delta(t), \text{ for } x \in \mathbb{R}^n, \quad G = 0, \text{ for } t < 0.$$

The solution to $Pu = g, u|_{t<0} = 0$ is then given by

$$\int_0^t \int_{\mathbb{R}^n} G(x, y, t - s) g(y, s) \, dy ds. \tag{3.1}$$

Instead of the incoming field due to a point source, we may consider the incoming wave field due to an arbitrary source $f(x_\mathrm{s}, t)$ at the boundary $x_n = 0$. We denote this by v_i, that satisfies

$$Pv_\mathrm{i}(x, t) = f(x_\mathrm{s}, t)\delta(x_n) \text{ on } \mathbb{R}^n \times \mathbb{R}_+, \quad v_\mathrm{i} = 0 \text{ for } t < 0.$$

Let v_r be a resulting reflected wavefield, again given by (1.2), with $u_\mathrm{r}, u_\mathrm{i}$ replaced by $v_\mathrm{r}, v_\mathrm{i}$. The data u_r of section 1 is the kernel of the linear map $f \mapsto v_\mathrm{r}$, in the sense that

$$v_\mathrm{r}(x_\mathrm{r}) = \int \int u(x_\mathrm{r}, t - s, x_\mathrm{s}) f(x_\mathrm{s}, s) \, dx_\mathrm{s} ds.$$

The operator $f \mapsto v_\mathrm{r}$ has a simple relation to the multiplication operator M_r. Denote by G the map $g \mapsto u$ given by (3.1). Let R_1 be the restriction $R_1 g = g|_{x_n=0}$, and I_1 the map multiplying a function on the boundary by the singular function of the boundary (i.e. if we write $x = (x', x_n), x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$, then $I_1 f(x,t) \stackrel{\text{def}}{=} \delta(x_n) f(x',t)$). Then we have

$$v_{\mathrm{r}} = \left(R_1 \circ G \circ M_r \circ M_{c^{-2}} \circ \partial_t^2 \circ G \circ I_1 \right) f. \tag{3.2}$$

(at this point we will not discuss whether this is well defined).

To proceed we discuss some properties of G. Denote by $p = p(x,t,\xi,\tau) = -c(x)^{-2}\tau^2 + \xi^2$ the principal symbol of P. Solutions to the wave equation $Pu = f$ have the property that $\mathrm{WF}(u) \setminus \mathrm{WF}(f) \subset \mathrm{Char}(p)$, and is invariant under flow defined there by the Hamilton vector field of H_p (Hörmander th 26.1.1). The solution curves of the Hamilton flow are called bicharacteristic, or null bicharacteristic if contained in $p^{-1}(0)$. The characteristic set $p^{-1}(0)$ is given by the set of points (x,t,ξ,τ) satisfying $\xi^2 = c^{-2}(x)\tau^2$. Outside the point $t = 0, x = y$, $G(x,y,t)$ is a Fourier integral with Lagrangian manifold (see [3], chapter 5)

$$\begin{aligned}\{(x,y,t,\xi,\eta,\tau)\,;\, t > 0, \eta^2 = c^{-2}(x)\tau^2,& \\ \text{and } (x,t,\xi,\tau) \text{ are on the bicharacteristic through } (y,0,-\eta,\tau)\}.&\end{aligned}$$

We also recall the reciprocity property

$$G(x,y,t) = G(y,x,t).$$

The Schwartz kernel of M_r is given by

$$K_{M_r}(x,y) = r(\frac{x+y}{2})\delta(x-y).$$

The kernel of the sequence of operators in (3.2) is given by a linear map acting on K_{M_r}. If Ψ is an operator $C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$, such that $G \circ \Psi \circ M_{c^{-2}} \circ \partial_t^2 \circ G$ is a well defined operator, then the Schwartz kernel of $G \circ \Psi \circ M_{c^{-2}} \circ \partial_t^2 \circ G$ is obtained by application of the following linear map, that we will call L to the kernel K_Ψ of Ψ

$$LK_\Psi(x,y,t) = \int G(x,u,t_0) c(y)^{-2} \partial_t^2 G(y,v,t-t_0) K_\Psi(u,v)\, du dv dt_0. \tag{3.3}$$

Here we used the reciprocity property of G.

Our first result is about L (by u^*, v^* we denote covectors associated with u, v)

Theorem 3.1. *Let U, V be disjoint bounded open subsets of $\mathbb{R}^n$, so that also their closures are disjoint, and similarly for U' and V'. The expression (3.3) defines a Fourier integral operator $C_0^\infty(U \times U') \to C^\infty(V \times V' \times]0, T[)$ with canonical relation*

$$\begin{aligned}\{(x, y, t+s, \xi, \eta, \tau; u, v, u^*, v^*) \in T^*(V \times V' \times]0, T[) \backslash 0 \times T^*(U \times U');\\ u^{*2} = c^{-2}(u)\tau^2, (x, t, \xi, \tau) \text{ on a bicharacteristic through } (u, 0, u^*, \tau),\\ v^{*2} = c^{-2}(v)\tau^2, (y, s, \eta, \tau) \text{ on a bicharacteristic through } (v, 0, v^*, \tau)\}. \end{aligned} \tag{3.4}$$

It extends to a continuous operator $\mathcal{E}'(U \times U') \to \mathcal{D}'(V \times V' \times]0, T[)$.

Proof. We first show the statement for L_0 defined by

$$L_0 K_\Psi(x, y, t) = \int G(x, u, t_0) G(y, v, t - t_0) K_\Psi(u, v)\, du dv dt_0.$$

We first give some properties of Fourier integral operators, and of G. A standard choice of phase function, that is convenient because it does not involve coordinate transformations, has been described by Maslov and Fedoriuk [6]. Suppose Λ is a Lagrangian submanifold of $T^*(\mathbb{R}^n)\backslash 0$, and let x_j, ξ_k, $1 \le j, k \le n$ be coordinates on $\mathbb{R}^n$. For each point in Λ, there is always a partition $I \cup J$ of $\{1, \dots, n\}$ such that the projection on the (x_I, ξ_J) coordinates (the vector valued function (x_I, ξ_J)) defines a local diffeomorphism. If I', J' are disjoint such that $I' \cup J' \subset \{1, \dots n\}$, and $(x_{I'}, \xi_{J'})$ define locally a submersive map, then I' and J' can be extended to a partition I, J with the previously described property. There is now a generating function $S = S(x_I, \xi_J)$, such that the Lagrangian manifold is locally given by

$$\xi_I = \frac{\partial S}{\partial x_I}(x_I, \xi_J), \qquad x_J = -\frac{\partial S}{\partial \xi_J}(x_I, \xi_J).$$

A local phase function is then given by $\phi(x, \xi_J) = S(x_I, \xi_J) + x_J \cdot \xi_J$.

In the case of G the projection is submersive on τ, x, and can be extended to a diffeomorphic projection on τ, x and y_I, η_J, for some partition $I \cup J$ of $\{1, \dots, n\}$. Using a generating function $S(x, y_I, \eta_J, \tau)$ we find a phase function of the form

$$\phi(x, y, t, \eta_J, \tau) = S(x, y_I, \eta_J, \tau) + \eta_J \cdot y_J + t\tau,$$

that locally describes Λ.

Consider now (3.3). The Schwartz kernel can be written as

$$K_{L_0}(x, y, t, u, v) = \int \hat{G}(x, u, \tau) \hat{G}(y, v, \tau') e^{i\frac{1}{2}t(\tau+\tau') + it_0(\tau-\tau')}\, d\tau d\tau' dt_0,$$

where $\hat{G}(x, y, \tau)$ is the time-Fourier transform of G. We observe that $\hat{G}(x, y, \tau)$ is bounded by $C(1+\tau^2)^{l/2}$ for some l, (x, y in some compact set). It follows by partial integration that this is a well defined distribution. The integration over t_0 can be done, yielding

$$\int \hat{G}(x, u, \tau)\hat{G}(y, v, \tau)e^{it\tau}\, d\tau.$$

Since U, U', V, V' are bounded sets, the G's are finite sums of local Fourier integral operators $\sum_j G^{(j)}$, with Maslov phase functions $\phi^{(j)}$ and amplitudes $a^{(j)}$, as introduced above. Let us consider the (j, k) contribution to the sum that is K_{L_0}. For brevity we write ϕ, ϕ' instead of $\phi^{(j)}, \phi^{(k)}$ etc. With this notation the (j, k) contribution to the Schwartz kernel K_{L_0} is given by

$$\int\int\int A(x, u, u_J^*, \tau)A'(y, v, v_{J'}^*, \tau)e^{i(S+S'-u_J\cdot u_J^*-v_{J'}\cdot v_{J'}^*+t\tau)}\, du_J^* dv_{J'}^* d\tau,$$

(where we have $u_J^* = -\eta_J$ compared to the formula above). Denote by Φ the phase function

$$\Phi = S + S' - u_J \cdot u_J^* - v_{J'} \cdot v_{J'}^* + t\tau.$$

It is easily verified that Φ is a non-degenerate phase function, since the matrix

$$\frac{\partial^2\Phi}{\partial(u_J, v_{J'}, t)\partial(u_J^*, v_{J'}^*, \tau)} \text{ has maximal rank.}$$

The corresponding Lagrangian manifold is given by

$$\begin{aligned}\Lambda'_{(j,k)} = \{&(x, y, t+s, \xi, \eta, \tau; u, v, u^*, v^*)\,;\\ &(x, u, t, \xi, -u^*, \tau) \in \Lambda_{G^{(j)}}, (y, v, s, \eta, -v^*, \tau) \in \Lambda_{G^{(k)}}\},\end{aligned}$$

Hence it is of the form given in the theorem.

We verify that the amplitude is a symbol. We have that $\|(\eta, \xi)\| \le C|\tau|$ on the Lagrangian manifold. Therefore for the amplitude

$$\partial^\alpha_{(x,y)}\partial^\beta_{(\eta_J,\tau)}A \le C_{\alpha,\beta}(1+\tau^2)^{1/2(k-|\beta|)},$$

for some constant k, and similar for A' with a constant k'. It follows that the product of amplitudes is also a symbol.

Since the canonical relation contains no elements of the form $(x, y, t, 0, 0, 0; u, v, u^*, v^*)$ and no elements of the form $(x, y, t, \xi, \eta, \tau; u, v, 0, 0)$ this Fourier integral operator is continuous $C_0^\infty(U \times U') \to C^\infty(V \times V' \times]0, T[)$ and $\mathcal{E}'(U \times U') \to \mathcal{D}'(V \times V' \times]0, T[)$.

Clearly, if $G(y,v,t)$ is a Fourier integral except at $x = y, t = 0$, then this is also true for $c(y)^{-2}\partial_t^2 G(y,v,t)$. This shows that the statement remains true for L. $\square$

Definition of $\bar{F}$ and χ

Next we will give χ and $\bar{F}$, the latter by composition of L with operators before and after, and give conditions such that $\bar{F}$ is a Fourier integral operator with canonical relation that is the graph of an invertible map.

We let $x \mapsto Z(x) \in \mathbb{R}$ define a family of hypersurfaces on a large open set Ω, containing X, X_{s} and X_{r}. We will also assume that $Z = 0$ on $x_n = 0$, $Z > 0$ on $x_n > 0$, $Z < 0$ when $x_n < 0$, and $\mathrm{grad}(Z) \neq 0$ on Ω.

We will assume that after a coordinate transformation, we can set $Z = x_n$. In the new coordinates the operator L remains a Fourier integral operator. However, the bicharacteristics of the canonical relation of G, are now given by the Hamilton flow of

$$p(x,t,\xi,\tau) = \sum_{j=1}^{n}\sum_{k=1}^{n} g^{j,k}(x)\xi_j\xi_k - \tau^2, \tag{3.5}$$

where the metric $g^{j,k}(x)$ is obtained from applying a coordinate transformation on the original metric $c(x)^2\delta^{j,k}$.

We will first formulate the results in the coordinates such that $x_n = Z$, using bicharacteristics with p given by (3.5). We define the open subset X_0 of $\mathbb{R}^{2n-1}$ by

$$X_0 = \{(x,y) \in X \times X\,;\, Z(x) = Z(y)\}.$$

A point in this set for $Z(x) = x_n$ is given by $(x,y,z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$ with $(x,z) \in X$ and $(y,z) \in X$. With this notation we define the operator χ in the factorization (1.1), by

$$\chi r(x,y,z) = r(x,z)\delta(x-y), \qquad (x,y,z) \in X.$$

It maps r to the Schwartz kernel of a family of operators on the surfaces $Z =$ constant. We define a map $I : \mathcal{D}'(\mathbb{R}^{2n-1}) \to \mathcal{D}'(\mathbb{R}^{2n})$, using the notation $x = (x', x_n), x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$, and similar for y,

$$Ig(x,y) = g(x',y',x_n)\delta(x_n - y_n).$$

We define also a restriction operator $R : C_0^\infty(\mathbb{R}^{2n+1}) \to C_0^\infty(\mathbb{R}^{2n-1})$ by

$$Ru(x,y,t) = u((x,0),(y,0),t), \qquad x,y \in \mathbb{R}^{n-1}.$$

We let

$$\bar{F} = R \circ L \circ I. \tag{3.6}$$

Fourier integral operator property of $\bar{F}$

It will be convenient to parameterize the null bicharacteristics. As pointed out above, the points (x,t,ξ,τ) in the characteristic set $p^{-1}(0)$ satisfy $\sum_{j=1}^n \sum_{k=1}^n g^{j,k}\xi_j\xi_k = \tau^2$. Hence those (ξ,τ) can be specified by $\alpha \in S^{n-1}$ and $\tau \in \mathbb{R}$ according to the formula $\xi = \Xi(x,\alpha,\tau) \stackrel{\text{def}}{=} -\tau \left(\sum_{j=1}^n \sum_{k=1}^n g^{j,k}\alpha^j\alpha^k\right)^{-1/2} \alpha$. The solution curves to the Hamilton field are such that τ is invariant. They can be parameterized by the time. The (x,ξ) components will be denoted by

$$\gamma(t,x,\alpha,\tau) = (\gamma_x(t,x,\alpha), \gamma_\xi(t,x,\alpha,\tau)),$$

(γ_x is independent of τ). A curve $t \mapsto \gamma_x(t,x,\alpha)$ is called a ray.

The elements of the canonical relation (3.4) are given by the points

$$\begin{aligned}(\gamma_x(t,x,\alpha), \gamma_x(s,y,\beta), t+s, \gamma_\xi(t,x,\alpha,\tau), \gamma_\xi(s,y,\beta,\tau), \tau;\\ x, y, \Xi(x,\alpha,\tau), \Xi(y,\beta,\tau))\end{aligned} \tag{3.7}$$

where $(x,y,\alpha,\beta,t,s,\tau)$ vary in an open subset of $U \times U' \times S^{n-1} \times S^{n-1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\backslash 0$, such that $\gamma_x(t,x,\alpha) \in V$, $\gamma_x(s,y,\beta) \in V'$. Consider the operator L with $U = U' = X$, V a neighborhood of X_s, and V' a neighborhood of X_r.

Lemma 3.2. *Under the assumption*

Each ray segment $t \mapsto \gamma_x(t,x,\alpha)$, $x \in X, t \in]0,T[$ intersects X_s at most once and transversally, and similarly for X_r, (3.8)

the operator $R \circ L$ is a Fourier integral operator with canonical relation

$$\begin{aligned}\{(\gamma_{x'}(t,x,\alpha), \gamma_{x'}(s,y,\beta), t+s, \gamma_{\xi'}(t,x,\alpha,\tau), \gamma_{\xi'}(s,y,\beta,\tau), \tau; x, y, \Xi(x,\alpha,\tau), \Xi(y,\beta,\tau))\,;\\ (t,x,\alpha) \text{ such that } \gamma_x(t,x,\alpha) \in X_\mathrm{s}, (s,y,\beta) \text{ such that } \gamma_x(s,y,\beta) \in X_\mathrm{r}\},\end{aligned} \tag{3.9}$$

where $\gamma_{x'} = (\gamma_{x_1}, \ldots, \gamma_{x_{n-1}})$, and similarly for $\gamma_{\xi'}$.

Proof. The restriction R is a Fourier integral operator with canonical relation given by, where we denote by $x' = (x_1, \dots, x_{n-1})$

$$\{(x', y', t, \xi', \eta', \tau; x, y, t, \xi, \eta, \tau)\,;\,(x, y, t, \xi, \eta, \tau) \in T^*(\mathbb{R}^{2n+1}), x_n = 0, y_n = 0\}$$

The composition of Fourier integral operators is described in [3, Theorem 2.4.1]. This includes a condition involving the support of the amplitudes, equation (2.4.8), which is satisfied in our case, since for (x, t) fixed, the set of y, such that $(x, y, t) \in \operatorname{supp}(G)$ is bounded. In addition it includes three conditions (2.4.9) to (2.4.11) on the canonical relations. Let us repeat them. Here X, Y, Z are open subsets of $\mathbb{R}^{n_X}, \mathbb{R}^{n_Y}, \mathbb{R}^{n_Z}$, such that $\Lambda'_{\phi_1} \subset T^*(X \times Y)\backslash 0$, $\Lambda'_{\phi_2} \subset T^*(Y \times Z)\backslash 0$. The conditions are

$$\eta \neq 0 \text{ if } (x, \xi, y, \eta) \in \Lambda'_{\phi_1} \text{ or } (y, \eta, z, \zeta) \in \Lambda'_{\phi_2}, \tag{3.10}$$

$$\xi \neq 0 \text{ or } \zeta \neq 0 \text{ if } (x, \xi, y, \eta) \in \Lambda'_{\phi_1} \text{ and } (y, \eta, z, \zeta) \in \Lambda'_{\phi_2}, \tag{3.11}$$

$$\Lambda'_{\phi_1} \times \Lambda'_{\phi_2} \text{ intersects } T^*(X) \times \operatorname{diag}(T^*(Y)) \times T^*(Z) \text{ transversally.} \tag{3.12}$$

The first of these is satisfied by the canonical relation of the restriction. The second conditions follows from the fact that $\xi \neq 0$ and $\eta \neq 0$ for all $(x, \xi, y, \eta) \in \Lambda'_L$. To verify the third condition we go to the parameterization above. The third condition is then equivalent to

$$\frac{\partial(\gamma_{x_n}(t, x, \alpha), \gamma_{x_n}(s, y, \beta))}{\partial(x, y, \alpha, \beta, t, s)} \text{ has rank 2 when } \gamma_{x_n}(t, x, \alpha) = \gamma_{x_n}(s, y, \beta) = 0,$$

which is clearly implied by (3.8). The composition of canonical relations yields (3.9), using (3.7). ☐

Next we consider the composition $(R \circ L) \circ I$. We denote $x' = (x_1, \dots, x_{n-1})$.

Lemma 3.3. *With the assumption (3.8), the composition $(R \circ L) \circ I$ is a Fourier integral operator with canonical relation consisting of the points*

$$\begin{aligned}(\gamma_{x'}(t, x, \alpha), \gamma_{x'}(s, y, \beta), t + s, \gamma_{\xi'}(t, x, \alpha, \tau), \gamma_{\xi'}(s, y, \beta, \tau), \tau;\\ x', y', x_n, \Xi'(x, \alpha, \tau), \Xi'(y, \beta, \tau), \Xi_n(x, \alpha, \tau) + \Xi_n(y, \beta, \tau)),\end{aligned} \tag{3.13}$$

such that $x_n = y_n$, $(x, \alpha, t) \in X \times S^{n-1} \times]0, T[$ such that $\gamma_x(t, x, \alpha) \in X_{\mathrm{s}}$ and $(y, \beta, s) \in X \times S^{n-1} \times]0, T[$ such that $\gamma_x(s, y, \beta) \in X_{\mathrm{r}}$, and $t + s < T$.

Proof. The map I is a Fourier integral operator with canonical relation

$$\{((u,z),(v,z),(u^*,\frac{\zeta}{2}+\theta),(v^*,\frac{\zeta}{2}-\theta);u,v,z,u^*,v^*,\zeta)\,;$$
$$(u,v,z,u^*,v^*,\zeta)\in T^*(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}),\theta\in\mathbb{R},(u^*,v^*,\zeta,\theta)\neq 0\} \quad (3.14)$$

As in the previous proof the condition on the support of the amplitudes is satisfied. The first condition on the canonical relations is satisfied, because of (3.14). The second condition on the canonical relation is satisfied since $\Lambda'_{R\circ L}$ has no elements of the form (x,ξ,y,η) with $\xi=0,\eta\neq 0$ or $\eta=0,\xi\neq 0$. From the parameterization of the canonical relation (3.9) it is clear that the composition of (3.9) and (3.14) is transversal. The composition of these canonical relations is given by the parameterization by (3.13) in the lemma. □

Local invertibility of the canonical relation

The source and receivers rays have take off velocity at x and y given by $-\tau^{-1}g^{j,k}\xi_k$, with $\xi=\Xi(x,\alpha,\tau)$ and $\xi=\Xi(y,\beta,\tau)$ resp. For an element of the canonical relation, we will use the condition that the sum of the n-th component of velocities is unequal zero.

$$\begin{gathered}\frac{\partial\gamma_{x_n}}{\partial t}(0,x,\alpha)+\frac{\partial\gamma_{x_n}}{\partial t}(0,y,\beta)=\\ -\tau^{-1}g^{n,k}\Xi_k(x,\alpha,\tau)-\tau^{-1}g^{n,k}\Xi_k(y,\beta,\tau)\neq 0.\end{gathered} \quad (3.15)$$

Here (x,y,α,β) are the parameters appearing in the parameterization (3.13).

Lemma 3.4. *If for some point of $\Lambda'_{\bar F}$ condition (3.15) is satisfied, then the linearizations of the natural projections of $\Lambda'_{\bar F}$ on $T^*(Y)\backslash 0$ and on $T^*(X_0)\backslash 0$ are invertible.*

Proof. By [5], Lemma 25.3.6, the corank of the projection of the canonical relation of Lemma 3.3 on the first factor $T^*(Y)\backslash 0$ is equal to that on $T^*(X_0)\backslash 0$. Therefore, it is sufficient to verify that the linearization of the projection on the second factor $T^*(X_0)\backslash 0$ is invertible. The parameters α,β,τ (with either $\tau<0$ or $\tau>0$) simply parameterize the $(\xi,\eta)\in\mathbb{R}^{2n}$, such that

$$\sum_{j,k}g^{j,k}(x)\xi_j\xi_k-\sum_{j,k}g^{j,k}(y)\eta_j\eta_k=0. \quad (3.16)$$

The cotangent vector $(\xi',\eta',\xi_n+\eta_n)$ occurring in the projection on the second factor $T^*(\mathbb{R}^{2n-1})$ is the projection of an element of this hypersurface along the lines given by

$\zeta \mapsto (\xi', \xi_n + \zeta, \eta', \eta_n - \zeta)$ on the hyperplane given by $\xi_n - \eta_n = 0$. It follows from some simple linear algebra, that the lines above must intersect the hypersurface given by (3.16) transversally, i.e.

$$\frac{\partial}{\partial \xi_n} \sum_{j,k} g^{j,k}(x)\xi_j\xi_k + \frac{\partial}{\partial \eta_n} \sum_{j,k} g^{j,k}(y)\eta_j\eta_k \neq 0.$$

Evaluation of the derivatives yields

$$\sum_j g^{j,n}(x)\xi_j + \sum_j g^{j,n}(y)\eta_j \neq 0.$$

Dividing by $-\tau$ we see that this holds if and only if (3.15) holds. $\square$

Global invertibility and summary of the result

We will now consider the case where not necessarily $Z(x) = x_n$, and define χ and I for this case. The operator $\bar{F}$ will again be given by (3.6). The definitions are non-unique in the sense that different multiplicative factors can be changed or added, so that the product $I\chi$ remains the same. Let $\kappa(x)$ be a coordinate transformation, such that $\kappa_n = Z$. Denote $\kappa'(x) = (\kappa_1, \ldots, \kappa_{n-1}(x))$, and $\tilde{X} = \kappa(X)$. We assume κ is defined on an open subset Ω of $\mathbb{R}^n$ that contains X, X_{s} and X_{r}, and is such that if there are two ray segments resp. from X to X_{s}, and from X to X_{r}, with total travel time $< T$, then the two rays are contained in Ω.

We denote by $(\tilde{x}, \tilde{z}) \in \mathbb{R}^{n-1} \times \mathbb{R}$ (or $(\tilde{y}, \tilde{z})$) the new coordinates. We define χ by

$$\chi r(\tilde{x}, \tilde{y}, \tilde{z}) = a(\tilde{x}, \tilde{y}, \tilde{z}) r(\kappa^{-1}(\tilde{x}, \tilde{z}))\delta(\tilde{x} - \tilde{y}),$$

with $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$, $(\tilde{x}, \tilde{z}) \in \tilde{X}$ and $(\tilde{y}, \tilde{z}) \in \tilde{X}$. We define I by

$$Ig(x, y) = b(x, y) g(\kappa'(x), \kappa'(y), \kappa_n(x))\delta(\kappa_n(x) - \kappa_n(y)),$$

where a, b are smooth multiplicative factors. We must have $I\chi r(x, y) = r(x)\delta(x - y)$. This yields

$$I\chi r(x, y) = r(x) a(\kappa'(x), \kappa'(y)\kappa_n(x)) b(x, y)\delta(\kappa(x) - \kappa(y)).$$

To fix a, b we now use that

$$\delta(\kappa(x) - \kappa(y)) = \left| \det \frac{\partial \kappa}{\partial x}(x) \right|^{-1} \delta(x - y).$$

This means we can set

$$a(\tilde{x}, \tilde{y}, \tilde{z}) = \left|\det \frac{\partial \kappa}{\partial x}(\kappa^{-1}(\tilde{x}, \tilde{z}))\right|,$$
$$b(x, y) = 1,$$

We will assume that

Assumption 2. *There is some $\epsilon > 0$, such that if $[0, t_1] \ni t \mapsto \gamma_1(t)$ is a ray from $x_{\rm s} \in X_{\rm s}$ to $x \in X$, and $[0, t_2] \ni t \mapsto \gamma_2(t)$ is a ray from $x_{\rm r} \in X_{\rm r}$ to $y \in X$, $t_1 + t_2 < T$, and $Z(x) = Z(y)$, then $\frac{\rm d}{{\rm d}t} Z(\gamma_j(t)) > \epsilon$ for all $t \in [0, t_j]$, $j = 1, 2$.*

Theorem 3.5. *With this assumption $\bar{F}$ is a Fourier integral operator, with canonical relation that is the graph of invertible canonical transformation from a subset of $T^*(X_0)\backslash 0$ to a subset of $T^*(Y)\backslash 0$.*

Proof. By a coordinate transformation this can be reduced to the case $Z(x) = x_n$, already discussed above.

Suppose $(x, \xi', \tau) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}$ is given, then there are 0, 1, or 2 different real solutions ξ_n to the equation $p(x, (\xi', \xi_n), \tau) = 0$. In case of 1 real solution, this correspond to a ray tangent to the hypersurface $Z = \text{constant}$. For the case of 2 real solutions, let these correspond to α_1, α_2. The two rays have non-zero n-th component of velocity $\frac{\partial \gamma_{x_n}}{\partial t}(0, x, \alpha_j)$, with different signs. So if this sign is known, then (x, ξ', τ) determines uniquely a null bicharacteristic. With the above assumption, in the parameterization of the canonical relation of Lemma 3.3, it follows that the rays in the canonical relation have negative vertical velocity component.

Assumption 2 implies the assumption (3.8). This shows that $\bar{F}$ is a Fourier integral operator by Lemmas 3.2 and 3.3. It also implies (3.15), which shows that the canonical relation is locally the graph of an invertible canonical transformation by Lemma 3.4. It remains to show that the projection π_1 from $\Lambda'_{\bar{F}}$ to $T^*(Y)\backslash 0$ is injective, and that the projection π_2 from $\Lambda'_{\bar{F}}$ to $T^*(X_0)\backslash 0$ is injective. We start with π_1. Let $(x_{\rm s}, x_{\rm r}, t, \xi_{\rm s}, \xi_{\rm r}, \tau)$ be in Y. Then $(x_{\rm s}, \xi_{\rm s}, \tau)$ determines uniquely the source bicharacteristic by the assumption and the previous alinea, and $(x, \Xi(x, \alpha, 0))$ must be on this bicharacteristic. Similarly $(x_{\rm r}, \xi_{\rm r}, \tau)$ determines a bicharacteristic, such that $(y, \Xi(y, \beta, 0))$ is on this bicharacteristic. The time t must be equal to the travel time along the bicharacteristic from $x_{\rm s}$ to x, and s equal to the time from x_{r} to y. Since these bicharacteristics intersect the hypersurfaces $Z =$ constant transversally, $t + s$ uniquely determines the value of Z, so it completely determines the point in $\Lambda'_{\bar{F}}$.

We next show that π_2 is injective. We will show that the map $(\alpha, \beta, \tau) \mapsto (\Xi'(x, \alpha, \tau), \Xi'(y, \beta, \tau), \Xi_n(x, \alpha, \tau) + \Xi_n(y, \beta, \tau))$ is injective, where (x, y) are considered as parameters. This map can be viewed as the composition of the two maps

$$(\alpha, \beta, \tau) \mapsto (\Xi'(x, \alpha, \tau), \Xi'(y, \beta, \tau), \tau)$$

and

$$(\xi', \eta', \tau) \mapsto (\xi', \eta', \xi_n + \eta_n),$$

where ξ_n is uniquely determined by the conditions $p(x, (\xi', \xi_n), \tau) = 0$, and that the corresponding bicharacteristic has negative n-th velocity component, and similar for η_n with $p(y, (\eta', \eta_n), \tau) = 0$. That those ξ_n, η_n are uniquely determined, shows that the first map is injective. Finally we must argue that the second map is injective. We will first give explicitly the formula for ξ_n, such that $p(x, (\xi', \xi_n), \tau) = 0$. Denote by g' the matrix $(g'^{j,k})_{j=1,\ldots,n-1,k=1,\ldots n-1}$, and denote $v = (g^{j,n})_{j=1,\ldots,n-1}$. The solutions ξ_n, such that $p(x, (\xi', \xi_n), \tau) = 0$, are given by

$$\xi_n = \frac{1}{g^{n,n}} \left(-v \cdot \xi' \pm \sqrt{(v \cdot \xi')^2 + g^{n,n}(\tau^2 - \xi' \cdot g' \cdot \xi')} \right). \tag{3.17}$$

Since the source and receiver rays have the same sign of the n-th component of velocity, it follows that the signs for ξ_n and η_n are the same, so that the map $\tau \mapsto \xi_n + \eta_n$ is strictly monotonous. This shows the injectivity of the second map. □

4 Application in reflection seismology

We will say a few words on the connections with the applications. The adjoint of $\bar{F}$ is a Fourier integral operator with the transposed canonical relation. Such operators are called *prestack migration operators*. Section 3 is related to downward-continuation seismic data processing, an important processing technique in practice. The standard geophysical reference for these methods is the book [2], for a mathematical discussion and more references see [11, 12]).

Applying an operator $\langle F\rangle^{-1}$ defined at the end of section 2, yields a set (s-family) of reconstructions of the singular part of r. Let us finally say a few words on why this is done. Recall that, in addition to r, also the background medium c is to be reconstructed. Both the forward operator, and the operator H depend on c. In the reconstruction of c one can therefore use the property that $\langle F\rangle^{-1}d$ must be independent of the parameter s (microlocally). *Migration velocity analysis* is the practice of finding a c so that this property is satisfied as far as practically possible.

Quantitave measures of the optimality, and algorithms to find an optimal c still form a highly challenging subject of research. Differential semblance optimization [14] is a method that involves the pseudodifferential annihilators introduced in section 2. It has the property that the resulting error function depends smoothly on the medium, which allows the use of efficient global optimization methods. The paper [9] does this using the extension of section 3, with some success in examples.

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