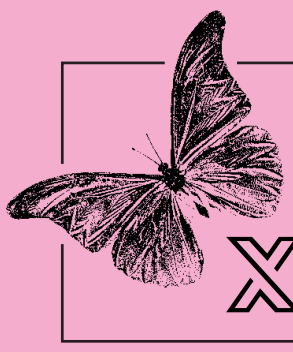

Centre de
Mathématiques
Laurent Schwartz
ECOLE
POLYTECHNIQUE

SEMINAIRE

Equations aux Dérivées Partielles

2000-2001

Christopher D. Sogge

Riemannian manifolds with maximal eigenfunction growth

Séminaire É. D. P. (2000-2001), Exposé n° XXIV, 16 p.

<http://sedp.cedram.org/item?id=SEDP_2000-2001____A24_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

RIEMANNIAN MANIFOLDS WITH MAXIMAL EIGENFUNCTION GROWTH

CHRISTOPHER D. SOGGE

1. Introduction.

The recent results in this talk are joint work with Steve Zelditch. The problem which motivates our work is to characterize compact Riemannian manifolds (M, g) with maximal eigenfunction growth as measured by $L^p(M)$ norms, especially sup-norms, $L^\infty(M)$. Before stating our results, let me go over some background material. I shall state some earlier results and also explain the role that eigenfunction bounds play in certain problems in harmonic analysis and partial differential equations.

Background: L^p norms of eigenfunctions

We first recall that the associated Laplace-Beltrami operator $\Delta = \Delta_g$, has eigenvalues $\{-\lambda_\nu^2\}$, where $0 \le \lambda_0^2 \le \lambda_1^2 \le \lambda_2^2 \le \dots$ are counted with multiplicity. Let $\{\phi_\nu(x)\}$ be an associated orthonormal basis of L^2-normalized eigenfunctions. If λ^2 is in the spectrum of $-\Delta$, let

$$V_\lambda = \{\phi_\lambda : \Delta\phi_\lambda = -\lambda^2\phi_\lambda\}$$

denote the corresponding eigenspace. We then measure the eigenfunction growth rate in terms of

$$L^p(\lambda, g) = \sup_{||\phi_\lambda||_2=1} ||\phi_\lambda||_{L^p}, \quad 2 < p \le \infty. \tag{1.1}$$

In the 1980's sharp estimates for this quantity were proved in [So1], [So2]. Specifically, the following result was proved in [So2].

Theorem 1.1. *Let (M, g) be a fixed Riemannian manifold of dimension $n \ge 2$. Then there is a constant C so that for $p > 2$*

$$\sup_{\phi\in V_\lambda} \frac{\|\phi\|_{L^p(M)}}{\|\phi\|_{L^2(M)}} \le C(1+\lambda)^{\delta(p)}, \tag{1.2}$$

where

$$\delta(p) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \le p \le \infty \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \le p \le \frac{2(n+1)}{n-1}. \end{cases} \tag{1.3}$$

Earlier, in [So1] these bounds were obtained in the special case where M is the round sphere S^n, and it was also shown that in this case they cannot be improved. Later, when we formulate open problems we shall explain the role of the numerology of the two cases $p \le 2(n+1)/(n-1)$ and $p > 2(n+1)/(n-1)$. Earlier work for spherical harmonics had been done by Bonami and Clerc [BC].

Let us sketch a proof of (1.2), which is somewhat different than the one in [So2], which was based on the resolvent. The one we shall give here is more in line with recent arguments in the subject.

Since the kernel of the operators that project onto V_λ cannot be explicitly computed except in certain special cases, the main idea is to reduce matters to proving an equivalent estimate involving an operator whose kernel can be computed. This step can be thought of as an operator-valued version of the Tauberian arguments that occur in the proof of the sharp Weyl theorem of Avacumovic [Av], Levitan [Le], and Hörmander [Ho 1].

To accomplish this, one fixes $\rho \in C_0^\infty$ satisfying $\hat\rho(0) = 1$, and $\rho(t) = 0$ if $|t| > R_M/2$, where R_M is the injectivity radius of M. If one then sets

$$\tilde\chi_\lambda f(x) = \frac{1}{2\pi}\int \rho(t) e^{it\lambda - it\sqrt{-\Delta}} f\, dt = \hat\rho(\lambda - \sqrt{-\Delta}) f, \tag{1.4}$$

then of course

$$\tilde\chi_\lambda f = f, \quad f \in V_\lambda,$$

and hence (1.2) would follow from showing that

$$\|\tilde\chi_\lambda f\|_p \le C(1+\lambda)^{\delta(p)} \|f\|_2. \tag{1.5}$$

Thus, the $\tilde\chi_\lambda$ can be thought of as *approximate spectral projection operators*. Since one can compute the kernel of the half-wave operators

$$U(t) = e^{-it\sqrt{-\Delta}} \tag{1.6}$$

with great precision when $|t|$ is smaller than $R_M/2$ one can compute the kernel of $\tilde\chi_\lambda$ explicitly and find that it is a sum of two terms

$$\tilde\chi_\lambda(x,y) = \lambda^{(n-1)/2} a_\lambda^\pm(x,y) \frac{e^{\pm i\lambda \mathrm{dist}(x,y)}}{(\lambda^{-1} + \mathrm{dist}(x,y))^{(n-1)/2}} + O(\lambda^{-\infty}), \quad \lambda > 1, \tag{1.7}$$

where the $a_\lambda^\pm$ belong to a bounded subset of C^∞ and $\mathrm{dist}(x,y)$ denotes the Riemannian distance between points $x, y \in M$.

Clearly,

$$\Big(\int |\tilde\chi_\lambda(x,y)|^2 dy\Big)^{1/2} \le C\lambda^{(n-1)/2}, \quad \lambda > 1,$$

and so (1.5) holds when $p = \infty$. It also holds, just by orthogonality when $p = 2$. Therefore, by applying the M. Riesz interpolation theorem, everything else would follow from establishing the estimate at the other endpoint, $p = 2(n+1)/(n-1)$:

$$\|\tilde\chi_\lambda f\|_p \le C(1+\lambda)^{1/p} \|f\|_2, \quad p = 2(n+1)/(n-1). \tag{1.8}$$

One can prove this by appealing to oscillatory integral theorems of Carleson-Sjölin [CS], Hörmander [Ho 2] and Stein [St]. The key fact is that if $\eta(x,y) \in C^\infty(M \times M)$ then an oscillatory integral operator like

$$T_\lambda h(x) = \int_M e^{i\lambda \mathrm{dist}(x,y)} \eta(x,y) h(y) dy,$$

enjoys the bounds

$$\|T_\lambda h\|_p \le C(1+\lambda)^{-n/p} \|h\|_p, \quad p = 2(n+1)/(n-1),$$

which leads to (1.8) since

$$\lambda^{\frac{n-1}{2}}\lambda^{-n\frac{n-1}{2(n+1)}} = \lambda^{\frac{n-1}{2(n+1)}}.$$

Note also that if $e_{\lambda_\nu}(f)$ denotes the projection of a function f onto the eigenspace with eigenvalue λ_ν, then this argument also leads to a stronger version of (1.2), namely, if

$$\chi_\lambda f = \sum_{\lambda_\nu \in (\lambda-1,\lambda]} e_{\lambda_\nu}(f), \tag{1.9}$$

then

$$\|\chi_\lambda f\|_p \le C(1+\lambda)^{\delta(p)}\|f\|_2, \quad 2 \le p \le \infty. \tag{1.10}$$

This bound can be shown to be sharp in all cases (see [So4]) in the sense that if $\|\chi\|_{2\to p}$ denotes the $L^2 \to L^p$ operator norm of χ_λ then

$$\limsup_{\lambda\to+\infty}(1+\lambda)^{-\delta(p)}\|\chi_\lambda\|_{p\to 2} > 0, \quad \forall\, 2 \le p \le \infty. \tag{1.11}$$

Applications

The main motivation for formulating and proving the above eigenfunction estimates was to obtain sharp bounds for Riesz means of eigenfunction expansions. Non-optimal bounds had been obtained by several authors, including Hörmander [Ho 1], who studied the $L^\infty \to L^\infty$ mapping properties.

Recall that the Riesz means of index $\delta > 0$ are given by

$$S^\delta_\lambda f(x) = \sum_{\lambda_\nu \le \lambda} (1-\lambda_\nu/\lambda)^\delta e_{\lambda_\nu}(f).$$

Note that S^0_λ is the familiar partial summation operator

$$S_\lambda f = \sum_{\lambda_\nu \le \lambda} e_{\lambda_\nu}(f). \tag{1.12}$$

Note also that as $\delta > 0$ the associated multipliers $\tau \to (1-\tau/\lambda)^\delta_+$ become increasingly more regular. Hence, one sees that the $L^p \to L^p$ mapping properties get better with increasing δ.

In Hörmander [Ho 1] it was shown that one has uniform bounds

$$\|S^\delta_\lambda f\|_\infty \le C\|f\|_\infty, \quad \text{if} \;\; \delta > n/2.$$

On the other hand, in the Euclidean version it was known that $\delta = (n-1)/2$ should be the critical index and not $\delta = n/2$. More generally, for other exponents $p > 2$ one can show that a uniform estimate of the form

$$\|S_\lambda f\|_p \le C\|f\|_p, \quad \delta > \max\{n(1/2-1/p)-1/2, 0\}, \tag{1.13}$$

would be sharp. Note that $\max\{n(1/2-1/p)-1/2, 0\}$ agrees with the power $\delta(p)$ in (1.2), precisely when $p \ge 2(n+1)/(n-1)$.

Fortunately as was shown in [So3], one can prove this estimate for $p \ge 2(n+1)/(n-1)$ using (1.10). Here too, the idea is to use what amount to operator-valued versions of the Tauberian argument behind the proof of the sharp Weyl formula (see [Ho 1]).

One firsts notices that S^δ_λ is given by the explicit formula

$$S^\delta_\lambda = c_\delta \lambda^{-\delta} \int (t+i0)^{-1-\delta} e^{it\lambda - it\sqrt{-\Delta}} dt.$$

To employ the above ideas, let $\rho \in C^\infty_0$ be as in (1.4), and define *approximate Riesz mean operators*

$$\tilde{S}^\delta_\lambda = c_\delta \lambda^{-\delta} \int \rho(t)(t+i0)^{-1-\delta} e^{it\lambda - it\sqrt{-\Delta}} dt.$$

Then just like in the case of $\tilde{\chi}_\lambda$, one can compute its kernel explicitly and see that it is a sum of two terms of the form

$$\tilde{S}^\delta_\lambda(x,y) = \lambda^{\frac{n-1}{2}-\delta} a^{\delta,\pm}_\lambda(x,y) \frac{e^{\pm i\lambda \mathrm{dist}(x,y)}}{(\lambda^{-1}+\mathrm{dist}(x,y))^{\frac{n-1}{2}+\delta}} + O(\lambda^{-\infty}), \quad \lambda > 1.$$

Based on this one immediately sees that the kernel is uniformly integrable in y when $\delta > (n-1)/2$, which means that the analog of (1.13) must hold for $p = \infty$ when S^δ_λ is replaced by $\tilde{S}^\delta_\lambda$. By using the same oscillatory integral operator bounds as before, one also sees that this variant of (1.13) must also hold for $p = 2(n+1)/(n-1)$. By interpolation, it also must be true for the range $2(n+1)/(n-1) \le p \le \infty$. Based on this, one would have (1.13) for this range of exponents if

$$\|R^\delta_\lambda f\|_p \le C\|f\|_p, \quad \delta \ge \max\{n(1/2-1/p)-1/2, 0\}, \quad p \ge 2(n+1)/(n-1),$$

if

$$R^\delta_\lambda = S^\delta_\lambda - \tilde{S}^\delta_\lambda.$$

This estimate is an easy consequence of (1.10). Morally,

$$R^\delta_\lambda \approx \lambda^{-\delta} \chi_\lambda, \tag{1.14}$$

and hence

$$\|R^\delta_\lambda f\|_p \le C\lambda^{-\delta+\delta(p)}\|f\|_2,$$

which leads to the desired bounds for this remainder term if $\delta \ge \delta(p) = n(1/2-1/p)-1/2$, $p \ge 2(n+1)/(n-1)$, after applying Hölder's inequality since M is compact. Equation (1.14) is a slight oversimplification. What one really does is notice that

$$R^\delta_\lambda f = \lambda^{-\delta} r^\delta(\lambda - \sqrt{-\Delta}) f,$$

where $|r^\delta(\tau)| \le C_N(1+|\tau|)^{-N}$, for every N, and so by orthogonality and (1.10)

$$\begin{aligned}\|R_\lambda f\|_p &\le C \sum_k \lambda^{-\delta} k^{\delta(p)} \|r^\delta_\lambda(\lambda - \sqrt{-\Delta})\chi_k f\|_2 \\ &\le C'_N \sum_k k^{\delta(p)} \lambda^{-\delta}(1+|k-\lambda|)^{-N}\|f\|_2 \le C' \lambda^{\delta(p)-\delta}\|f\|_2,\end{aligned}$$

assuming, as we may, that N is large.

It is not known if the estimates (1.13) hold for $2 < p < 2(n+1)/(n-1)$. Unfortunately, simple operator-valued Tauberian arguments like the one given above break down for this range of exponents, due to the fact that the $L^2 \to L^p$ bounds for χ_λ are not favorable for this range because $\delta(p) > \max\{n(1/2-1/p)-1/2, 0\}$ for $2 < p < 2(n+1)/(n-1)$. In two-dimensions, the version of (1.13) for the Euclidean case, $\mathbb{R}^2$, were proved 1970s by Carleson and Sjölin [CS]. It is frustrating that in the Riemannian case, one only knows (1.13) for the full range of exponents in certain special cases, such as S^2. On the

other hand, following an earlier idea of Bourgain, Minicozzi and the author [MS] showed that when $n \geq 3$ there can be geometric obstacles which preclude (1.13) holding for all $2 < p < 2(n+1)/(n-1)$. Still it would be very interesting to come up with geometric conditions that would allow improvements here over the $p \geq 2(n+1)/(n-1)$ bounds for Riesz means.

Other applications were made in proving Carleman and unique continuation theorems. See Jerison [J] and Jerison and Kenig [JK].

2. **Main Problem: Characterize Maximal Eigenfunction Growth.**

As we indicated before, Theorem 1.1 cannot be improved since the bounds (1.2) are sharp for the standard sphere. Thus, $L^p(\lambda, g) = \Omega(\lambda^{\delta(p)})$ for (S^n, g_{can}), where $\Omega(\lambda^{\delta(p)})$ means $O(\lambda^{\delta(p)})$ but not $o(\lambda^{\delta(p)})$. On the other hand, much better bounds hold in certain situations. For instance on the standard torus $\mathbb{R}^n/\mathbb{Z}^n$ one gets much better bounds where the power $\delta(p)$ can, for instance, be replaced by a lower power. In particular, $L^p(\lambda, g) = o(\lambda^{\delta(p)})$ in this case. Based on this, we say that S^n, but not $\mathbb{R}^n/\mathbb{Z}^n$, is a *Riemannian manifold with maximal eigenfunction growth* as measured by the sup-norm.

With this in mind we pose:

- **Problem**: *Determine the* (M, g) *for which* $L^\infty(\lambda, g) = \Omega(\lambda^{\frac{n-1}{2}})$.

Our main result, Theorem 2.1, implies a necessary condition on a compact Riemannian manifolds (M, g) with maximal eigenfunction growth: there must exist a point $x \in M$ for which the set

$$\mathcal{L}_x = \{\xi \in S^*_x M : \exists\, T : \exp_x T\xi = x\} \tag{2.1}$$

of directions of geodesic loops at x has positive surface measure. Here, exp is the exponential map (see below), and the measure $|\Omega|$ of a set Ω is the one induced by the metric g_x on $T^*_x M$. For instance, the poles x_N, x_S of a surface of revolution (S^2, g) satisfy $|\mathcal{L}_x| = 2\pi$. Note also that the geodesic loops do not have to close smoothly.

Theorem 2.1. *Suppose that* $|\mathcal{L}_x| = 0$. *Then given* $\varepsilon > 0$ *there exists a neighborhood* $\mathcal{N} = \mathcal{N}(\varepsilon)$ *of* x, *and a positive number* $\Lambda = \Lambda(\varepsilon)$, *so that*

$$\sup_{\phi \in V_\lambda} \frac{\|\phi\|_{L^\infty(\mathcal{N})}}{\|\phi\|_{L^2(M)}} \leq \varepsilon \lambda^{(n-1)/2}, \quad \lambda \in \textit{spec}\, \sqrt{-\Delta} \geq \Lambda. \tag{2.2}$$

If one has $|\mathcal{L}_x| = 0$ *for every* $x \in M$ *then*

$$\sup_{\phi \in V_\lambda} \frac{\|\phi\|_{L^p(M)}}{\|\phi\|_{L^2(M)}} = o(\lambda^{\delta(p)}), \quad p > \tfrac{2(n+1)}{n-1} \tag{2.3}$$

where $\delta(p)$ *is given by* (1.3).

Note that we are only able to obtain sufficient conditions for $L^p(\lambda, g) = o(\lambda^{\delta(p)})$ when $p > 2(n+1)/(n-1)$. We shall discuss open problems concerning the very interesting case where $2 < p < 2(n+1)/(n-1)$ at the end. We shall also give an example showing that the condition that $|\mathcal{L}_x| = 0$ for some $x \in M$ is not a necessary condition for (2.2).

We should also point out that the main part of Theorem 2.1 is the part corresponding to $p = \infty$. The remaining estimates for $2(n+1)/(n-1) < p < \infty$ just follow from interpolating between (2.3) with $p = \infty$ and (1.2) with $p = 2(n+1)/(n-1)$.

Before sketching the proof of Theorem 2.1, let us discuss its main hypothesis.

3. **Geometry of Loops.**

Let T^*M denotes the cotangent bundle and S^*M denotes the unit sphere bundle with respect to g of our compact Riemannian manifold (M, g). We denote by $\exp tH_p$ the geodesic flow of g, defined as the flow of the Hamiltonian vector field H_p of $p(x,\xi) = \sqrt{\sum g^{jk}(x)\xi_j\xi_k}$, the principal symbol for $\sqrt{-\Delta}$. By definition, $\exp tH_p$ is homogeneous, i.e. commutes with the natural $\mathbb{R}^+$-action, $r \times (x,\xi) = (x, r\xi)$ on $T^*M\backslash 0$. We also define the exponential map at x by $\exp_x \xi = \pi \circ \exp tH_p(x,\xi)$. These definitions are standard in microlocal analysis but differ from the usual geometer's definitions, which takes $p^2 = \sum_{i,j} g^{ij}(x)\xi_i\xi_j$ as the Hamiltonian generating the geodesic flow. The geometer's geodesic flow is not homogeneous.

Following [Ho IV], we begin by introducing the loop-length function on $T^*M\backslash 0$ given by

$$L^*(x,\xi) = \inf\{t > 0 : \exp_x t\xi = x\}, \tag{3.1}$$

where L^* is defined to be $+\infty$ if no such t exists. It is homogeneous of degree zero, so it is natural to consider the restriction of L^* to $S^*M = \{(x,\xi) : \sum g^{jk}(x)\xi_j\xi_k = 1\}$. A key fact for us is that L^* is a *lower semicontinuous function*, or equivalently that the function $1/L^*(x,\xi)$, which is defined to be zero when $L^*(x,\xi) = +\infty$, is an upper semicontinuous function on S^*M. For fixed $x \in M$ we define the set of loop directions at x by:

$$\mathcal{L}_x = \{\xi \in S^*_x M : 1/L^*(x,\xi) \neq 0\}. \tag{3.2}$$

The complimentary set

$$S^*_x M\backslash\mathcal{L}_x = \{\xi \in S^*_x M : 1/L^*(x,\xi) = 0\}$$

is the set of all unit vectors for which there is no geodesic loop with initial tangent vector ξ.

The lower semicontinuity of $L^*(x,\xi)$ plays a crucial role in everything. As a simple example, let us see how it can be used to prove the following result.

Theorem 3.1. *There exists a residual set $\mathcal{R}$ in the space $\mathcal{G}$ of C^∞ metrics with the Whitney C^∞ topology such that $|\mathcal{L}^g_x| = 0$ for every $x \in M$ when $g \in \mathcal{R}$.*

Proof. Recall that $|\mathcal{L}^g_x| = 0$ if and only if $\int_{S^x M} 1/L^*_g(x,\xi)d\xi = 0$, if $L^*_g(x,\xi)$ is the loop-length function defined before.

To make use of this, choose a coordinate patch $\Omega \subset M$ with coordinates $y = \kappa(x)$ ranging over an open subset of $\mathbb{R}^n$. We then fix $K \subset \kappa(\Omega)$ be compact and let

$$F(g) = \sup_{y\in K} \int_{S^{n-1}} \frac{d\sigma}{L^*_g(y,\xi)},$$

using the induced coordinates $\{y,\xi\}$ for $T^*\Omega \subset T^*M$. Here also, $d\sigma$ is the standard surface measure on S^{n-1}, and we are abusing notation a bit by letting $L^*_g(y,\xi)$ denote

the pushforward of L_g^* using κ. It then suffices to show that the set of metrics for which

$$\mathcal{G}_N = \{g : F(g) < 1/N\}$$

are open and dense.

Density just follows from the fact that $F(g) = 0$ for any non-Zoll real analytic metric. Such metrics are dense in $\mathcal{G}$.

The main step in proving that these sets are also open is to show that the function $f(g, y) = \int_{S^{n-1}} d\sigma / L_g^*(y, \xi)$ is upper semicontinuous on $\mathcal{G} \times K$. This holds since $1/L_g^*(y, \xi)$ is a positive, (locally) bounded upper semicontinuous function on $\mathcal{G} \times \Omega \times S^{n-1}$, if we equip $\mathcal{G}$ with the C^3-topology. Therefore, if $(g_j, y_j) \to (g, y)$, $y_j \in K$, we have

$$\sup_j \int_{S^{n-1}} \frac{d\sigma}{L_{g_j}^*(y_j, \xi)} \leq \int_{S^{n-1}} \sup_j \frac{d\sigma}{L_{g_j}^*(y_j, \xi)}$$
$$\implies \limsup_j \int_{S^{n-1}} \frac{d\sigma}{L_{g_j}^*(y_j, \xi)} \leq \int_{S^{n-1}} \limsup_j \frac{d\sigma}{L_{g_j}^*(y_j, \xi)} \leq \int_{S^{n-1}} \frac{d\sigma}{L_g^*(y, \xi)},$$

using the dominated convergence theorem and upper semicontinuity.

Now let us prove that the sets $\mathcal{G}_N$ are open. Let $g \in \mathcal{G}_N$. By definition of F, $f(g, y) < \frac{1}{N}$ for each $y \in K$. Since f is upper-semicontinuous, the set $\{f < \frac{1}{N}\}$ is open, so there exist $\delta(y)$ such that $B_{\delta(y)}(y) \times B_{\delta(y)}(g) \subset \{f < \frac{1}{N}\}$, if $B_\delta(y)$ and $B_\delta(g)$ denote the δ Euclidean and C^3 balls of y and g, respectively. Here, g is fixed so we do not indicate the dependence of the δ's on it. As y varies over K, the balls $B_{\delta(y)}(y)$ give an open cover of K, and by compactness there exists a finite subcover $\{B_{\delta(y_j)}(y_j),\ \ j = 1, \ldots, N\}$. Let $\delta = \min_j \delta(y_j)$, so that $B_\delta(y_j) \times B_\delta(g) \subset \{f < \frac{1}{N}\}$ for $j = 1, \ldots, N$. If $g' \in B_\delta(g)$, then $f(y, g') < \frac{1}{N}$ for all $y \in K$. Hence $F(g') < \frac{1}{N}$ and so $\mathcal{G}_N$ is open. □

The following appears to be a new geometric result:

Corollary 3.2. *$L^\infty(\lambda, g) = o(\lambda^{(n-1)/2})$ for a generic Riemannian metric on any manifold.*

One deduces this result simply by combining Theorem 2.1 and Theorem 3.1.

To further clarify the geometry of loops, we introduce the following notions:

Definition 3.3. *We put:*

- $\Gamma_x^T := \exp T H_p\ T_x^* M \cap T_x^* M$.
- $S\Gamma_x^T = \Gamma_x^T \cap S_x^* M$.
- $\Gamma_x = \exp_x^{-1}(x) = \{T\xi : \xi \in S\Gamma_x^T\} \subset T_x^* M$;
- $Lsp_x = \{|T| : \Gamma_x^T \neq \emptyset\} \subset \mathbb{R}^+$
- $\mathcal{L}_x := \{\xi \in S_x^* M : \exists t \in Lsp_x, \exp_x t\xi = x\} \subset S_x^* M$

In this list, we have mixed together notions involving the geometer's geodesic flow (the Hamiltonian flow of p^2) and the microlocal analyst's geodesic flow (that of p). Thus, Γ_x^T is a homogeneous closed subset of $T_x^* M$ (invariant under the $\mathbb{R}^+$-action), and $S\Gamma_x^T$ is the 'base' of the cone. On the other hand, we define $\exp_x$ in the usual geometer's, rather than as $\pi_x \circ \exp t H_p|_{T_x^* M}$. For instance, we have: $TS\Gamma_x^T = \{T\xi : |\xi| = 1, \exp xT\xi = x\}$.

It is natural to say that Lsp_x is the 'loop-length spectrum' at the point x, i.e. the set of lengths of geodesic loops. The set $\mathcal{L}_x$, already introduced, is the set of loop directions at x. The (non-homogeneous) set Γ_x is the set of loop (co)-vectors. In general, the sets Γ_x^t, Γ_x need not be smooth manifolds. Later in this section, we shall consider restrictions on (M, g) which imply smoothness properties of loopsets, and study a variety of examples.

Let us now present a few simple examples to illustrate the loopset notions.

3.1. *Flat tori.* Let $\mathbb{R}^n/\Gamma$ be a flat torus. A geodesic loop at x is a helix which returns to x. Loops on flat tori are always closed geodesics and they correspond to lattice points in Γ. Thus,

$$\Gamma_x = \Gamma, \qquad \mathcal{L}_x = \{\frac{\gamma}{|\gamma|} : \gamma \in \Gamma\}, \quad Lsp_x = \{|\gamma|, \ \gamma \in \Gamma\}, \quad \}.$$

We observe that Γ_x is a countable discrete subset of $T_x^*\mathbb{R}^n/\Gamma$, that $\mathcal{L}_x$ is a countable dense subset of S_x^*M, and that Γ_g is a countable set of embedded Lagrangean components each diffeomorphic to $\mathbb{R}^n/\Gamma$. Also, $\mathcal{C}_\Delta = \cup_{\gamma\in\Gamma}\{(|\gamma|, \tau, x, 0) : x \in M\}$. This example is non-generic because (among other things) every loop is a closed geodesic. Flat tori are examples of metrics without conjugate points. We now generalize the discussion.

3.2. *Manifolds without conjugate points.* Let (M, g) denote a manifold without conjugate points, i.e. a Riemannian manifold such that each exponential map $\exp_x$ is is a covering map. Manifolds of non-positive curvature are examples, so this class of metrics is open on any M (thought it may be empty for some M). By definition, there are no Jacobi fields along any loop satisfying $Y(0) = Y(1) = 0$, so the Jacobi operator is non-degenerate. Equivalently, each loop functional L_x is a Morse function on Ω_x. Thus, for all x, Γ_x is a countable discrete set of points, $\mathcal{L}_x$ is countable and Γ_x^T is a finite set. Unlike the case of flat tori, loops are not generally closed geodesics.

3.3. *Surfaces of revolution.* Surfaces of revolution provide a simple (but non-trivial) class of examples for which the loopsets may be explictly determined. Topologically, the surfaces must be either spheres or tori. It would take us too far afield to discuss the geometry of loops and eigenfunctions on general surfaces of revolution, so we only give a few indications of how to determine the loops explicitly and refer the reader to the articles [CV] [KMS][TZ] [Z] for further background.

A 2-sphere of revolution is a Riemannian sphere (S^2, g) with an action of S^1 by isometries. We may write $g = dr^2 + a(r)^2 d\theta^2$ in geodesic polar coordinates at one of the poles (fixed points) of the S^1 action. The poles are always self-conjugate points, and are points at which eigenfunctions attain the maximal bounds.

Tori of revolution generalize flat tori but non-flat cases must have conjugate points (Hopf's theorem). The metric on a torus of revolution may be written in standard angle coordiantes as $g = dx^2 + a(x)^2 d\theta^2$. Tori of revolution have no poles, and as we will see in Theorem 5.1 eigenfunctions need not attain the maximal bounds. In fact, although we do not prove it here, tori of revolution *never* have maximal eigenfunction growth.

3.4. *Tri-axial ellipsoids.* We recall that $E_{a_1,a_2,a_3} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1\}$ with $0 < a_1 < a_2 < a_3$. Jacobi proved that the geodesic flow of E_{a_1,a_2,a_3} (for any (a_1, a_2, a_3)) is completely integrable in 1838. The two integrals of the motion are the

length $H(x,\xi) = |\xi|_x$ and the so-called Joachimsthal integral J. More recent discussions of the geodesics of the ellipsoid can be found in [A], [K], and in [CVV] (§3).

There are four distinguished umbilic points $\pm P, \pm Q$ which occur on the middle closed geodesic $\{x_2 = 0\}$. All geodesics leaving P arrive at $-P$ at the same time, then leave $-P$ and return to P at the same time (see [K], Theorem 3.5.16 or [A]). Thus, the tri-axial ellipsoid is an example of a Y_ℓ^m-metric which is not a Zoll metric. At all other points $x \in E_{a_1,a_2,a_3}$, the set of initial directions of geodesics which return to x is countable [K].

3.5. *Zoll surfaces.* Now let us consider the extreme case of Zoll metrics on S^2, i.e metrics all of whose geodesic are closed [Besse]. Among such metrics, there is an infinite dimensional family of surfaces of revolution. There is an even larger class with no isometries.

We may suppose with no loss of generality that the least common period of the geodesics equals 2π. Then $\Gamma_x^{2\pi} = S_x^* S^2$ for every x, and $\Gamma_x = \cup_{n=1}^\infty 2\pi n S_x^* S^2$. As x varies we obviously get $\Gamma = \cup_{n=1}^\infty 2\pi n S^* S^2$.

As will be discussed in §7, although geodesics are recurrent at every x, we do not expect eigenfunction blow-up to occur everywhere, or even anywhere in general.

4. **Real Analytic Metrics.**

Using Theorem 2.1 and a standard result from geometry we can characterize real analytic manifolds with maximal eigenfunction growth.

Theorem 4.1. *Suppose that (M,g) is real analytic and that $L^\infty(\lambda, g) = \Omega(\lambda^{(n-1)/2})$. Then (M,g) is a Y_ℓ^m -manifold, i.e. a pointed Riemannian manifold (M,m,g) such that all geodesics issuing from the point m return to m at time ℓ. In particular, if dim M = 2, then M is topologically a 2-sphere S^2.*

For the definition and properties of Y_ℓ^m -manifolds, we refer to [Besse] (Chapter 7). By a theorem due to Bérard-Bergery (see [BB, Besse], Theorem 7. 37), Y_ℓ^m manifolds M satisfy $\pi_1(M)$ is finite and $H^*(M, \mathbb{Q})$ is a truncated polynomial ring in one generator. This of course implies $M = S^2$ (topologically) when $n = 2$. We remark that the loops are not assumed to close up smoothly. An interesting example to keep in mind here is the tri-axial ellipsoid discussed before, $E_{a_1,a_2,a_3} = \{(x_1,x_2,x_3) \in \mathbb{R}^2 : \frac{x_1^2}{a_1^2} + \frac{x_1^2}{a_1^2} + \frac{x_1^2}{a_1^2} = 1\}$, with $a_1 < a_2 < a_3$.

To prove Theorem 4.1, we require the following result whose proof we postpone for the moment.

Lemma 4.2. *L^* is constant on smooth submanifolds of Γ_x.*

Proof of Theorem 4.1: By assumption, there exists a constant $C > 0$ and a subsequence of L^2-normalized eigenfunctions $\{\phi_{\lambda_j}\}$ such that $||\phi_{\lambda_j}||_\infty \geq C\lambda_j^{\frac{(n-1)}{2}}$. This contradicts the last statement of Theorem 2.1, hence there exists a point m such that $|\mathcal{L}_m| > 0$.

Since g is real analytic, $\exp : T_m^* M \to M$ is a real analytic map, hence Γ_m is an analytic set. In any local coordinate patch $U \subset M$ containing m, Γ_m is the zero set of a pair of real-valued real-analytic functions, i.e.has the form $(f_1(\xi), \ldots, f_n(\xi)) = (m_1, \ldots, m_n)$.

The solutions are the same as for $(f_1(\xi) - m_1)^2 + \cdots + (f_n(\xi) - m_n)^2 = 0$, so Γ_m is the zero set of a real analytic function.

It is well-known (see [BM, H, L, S]) that the zero set of a real analytic function is locally a finite union of embedded real analytic submanifolds $Y_i^{k_i}$ of dimensions $1 \leq k_i \leq n-1$. Thus, for each ξ there exists a ball $B_\delta(\xi)$ such that

$$\Gamma_m \cap B_\delta(\xi) = \cup_{i=1}^d Y_i^{k_i}. \tag{4.1}$$

We claim that for some (ξ, δ) there exists a component Y_i^{n-1} of dimension $n-1$. If not, Γ_m is of Haussdorf dimension $\leq n-2$. But then its radial projection $\rho : \Gamma_m \to S_m^* M$ would also have Haussdorf dimension $\leq n-2$. In fact, each ray through the origin in $T_m^* M$ intersects Γ_m in at most countably many points. So the radial projection preserves the dimension. But this contradicts the fact that $\Lambda_m = \rho(\Gamma_m)$ has positive measure.

Now let $\Xi \subset \Gamma_m$ be an open embedded real analytic hypersurface of T^*M. Consider the rays $\{t\xi : 0 \leq t \leq 1\} \subset T_m^* M$ and the union

$$C = \cup_{\xi \in \Xi} \{t\xi : 0 \leq t \leq 1\}.$$

Thus, each ray in C exponentiates to a geodesic loop which returns at $t = 1$. By Lemma 4.2 the length $|\xi|$ of each loop must be a constant independent of $\xi \in \Xi$.

We conclude that $|\xi| = \ell$ for some $\ell \in \mathbb{R}^+$ and $\xi \in Y$. But this equation is real analytic, hence must hold on all of Ξ; hence $\Xi \subset \ell S_m^* M$. Again, by real analyticity, $\ell S_m^* M \subset \Gamma_m$. This is the same as saying that (M, g) is a Y_ℓ^m-manifold.

□

Proof of Lemma 4.2: It is sufficient to prove the result for smooth curves in Γ_x. This can be seen by using either the first variation formula for the lengths of a one-parameter family of geodesics, or by using a symplectic argument. For the sake of brevity, we shall only present the former.

First Variation proof: Let $\alpha(s)$ denote a smooth curve in Γ_x, and let $\gamma_s(t) = \exp_x tL^*(x, \alpha(s)) \frac{\alpha(s)}{|\alpha(s)|}$. By definition, $\gamma_s(0) = x, \gamma(1) = x$ and $L^*(x, \alpha(s)) = \int_0^1 |\gamma_s'(t)| dt$. Since γ_s is a geodesic for each s, we have

$$\begin{aligned} &\tfrac{d}{ds} \int_0^1 |\gamma_s'(t)| dt = \tfrac{1}{L^*(x,\alpha(s))} \int_0^1 \langle \tfrac{D}{ds} \gamma_s'(t),\ \gamma_s'(t) \rangle dt \\ &= \tfrac{1}{L^*(x,\alpha(s))} \int_0^1 \langle \tfrac{D}{dt} Y_s(t),\ \gamma_s'(t) \rangle dt \\ &= -\tfrac{1}{L^*(x,\alpha(s))} \int_0^1 \langle Y_s(t),\ \tfrac{D}{dt} \gamma_s'(t) \rangle dt = 0. \end{aligned}$$

Here, $Y_s(t) = \frac{d}{ds}\gamma_s(t)$ is the Jacobi field associated to the variation, and $\frac{D}{Dt}$ is covariant differentiation along γ_s. In integrating $\frac{D}{dt}$ by parts, the boundary terms vanished because $Y_s(0) = Y_s(1) = 0$ as the endpoints of the variation were fixed. □

5. **Converse Results.**

Theorem 2.1 says that $L^\infty(\lambda, g) = \Omega(\lambda^{(n-1)/2})$, then there must be a point $x \in M$ for which $|\mathcal{L}_x| > 0$. However, the naive converse to this sup-norm result is simply false.

Theorem 5.1. *There exists C^∞ Riemannian tori of revolution (T^2, g) such that*

$$\exists\, x:\ |\mathcal{L}_x| > 0, \quad \textit{but} \ \frac{\|\phi_\lambda\|_\infty}{\|\phi_\lambda\|_2} = o(\lambda^{(n-1)/2}).$$

The example involves $M = S^1 \times S^1$ with certain metrics

$$g_a = dx^2 + (a(x))^2 d\theta^2,$$

where (x, θ), $-1 \le x \le 1$, $-\pi \le \theta \le \pi$ are the angle variables. We shall assume that $a(x)$ is an even positive function on $[-1, 1]$. Moreover, we shall assume that the metric contains an "equatorial band", by which we mean that there is an $\varepsilon > 0$ so that

$$a = \sqrt{1 - x^2}, \quad x \in (-\varepsilon, \varepsilon).$$

We shall also assume that for $2\varepsilon < |x| \le 1$, $a(x)$ is constant. Using Sturm-Liouville analysis one see that one can construct such metrics with the property that every eigenspace, V_λ, of $\sqrt{-\Delta}$ has consisting of L^2-normalized eigenfunctions of the form

$$\Phi_{n,\lambda} = e^{in\theta}\phi_{n,\lambda}(x).$$

Based on this one can prove Theorem 5.1. One notices that clearly if (x, θ) is a point in the equatorial band where $|x| < \varepsilon$, then $|\mathcal{L}_{x,\theta}| > 0$. On the other hand because of the special form of the $\Phi_{n,\lambda}$ it is not hard to prove that

$$\|\Phi_{n,\lambda}\| \le C\lambda^{1/2-\sigma},$$

for some $\sigma > 0$, which of course means that $L^\infty(\lambda, g) = O(\lambda^{1/2-\sigma})$ and hence is $o(\lambda^{1/2})$.

6. **Sketch of proof of Theorem 2.1.**

The proof of Theorem 1.1 is a much more refined version of the proof of Theorem 2.1. We follow an idea of Ivrii [Iv1] to the extent possible. Our analysis is also related to that of Duistermaat and Guillemin [DG].

Let us set things up. We first recall that all of the L^p bounds in Theorem 2.1 would follow from showing that

$$|\mathcal{L}_x| > 0,\ \forall x \in M \implies \|\phi_\lambda\|_\infty = o(\lambda^{(n-1)/2}), \quad \phi_\lambda \in V_\lambda.$$

By compactness, this in turn follows from showing that if we fix $z \in M$ satisfying $|\mathcal{L}_z| = 0$ then for every $\varepsilon > 0$ there is a neighborhood $\mathcal{N}_\varepsilon(z)$ of z and a $\Lambda(\varepsilon)$ such that

$$\sup \frac{|e_\lambda(f)(x)|}{\|f\|_2} = \sup_{\|f\|_2=1} |e_\lambda(f)(x)| \le \varepsilon\lambda^{(n-1)/2}, \quad x \in \mathcal{N}_\varepsilon(z), \quad \lambda \ge \Lambda(\varepsilon). \tag{6.1}$$

On the other hand, by the converse to Schwarz's inequality

$$\begin{aligned}
\sup_{\|f\|_2=1} |e_\lambda(f)(x)| &= \sup_{\|f\|_2=1} \Big| \int_M \sum_{\lambda_\nu=\lambda} \phi_{\lambda_\nu}(x)\overline{\phi_{\lambda_\nu}(y)} f(y)\, dy \Big| \\
&= \Big(\int_M \Big| \sum_{\lambda_\nu=\lambda} \phi_{\lambda_\nu}(x)\overline{\phi_{\lambda_\nu}(y)} \Big|^2 dy \Big)^{1/2} \\
&= \Big(\sum_{\lambda_\nu=\lambda} |\phi_{\lambda_\nu}(x)|^2 \Big)^{1/2}.
\end{aligned}$$

Thus, we need

$$\sum_{\lambda_\nu=\lambda} |\phi_{\lambda_\nu}(x)|^2 \le \varepsilon \lambda^{(n-1)/2}, \quad x \in \mathcal{N}_\varepsilon(z).$$

Note that

$$\sum_{\lambda_\nu=\lambda} |\phi_{\lambda_\nu}(x)|^2 = N(\lambda; x) - N(\lambda - 0; x),$$

where

$$N(\lambda; x) = \sum_{\lambda_\nu \le \lambda} |e_{\lambda_\nu}(x)|^2,$$

is the *local Weyl term*, whose trace gives $N(\lambda)$, the number of eigenvalues counted with multiplicity that are $\le \lambda$. Thus, since pointwise eigenfunction estimates are measured by the *jumps* of $N(\lambda; x)$ we conclude that we would be done if we could show that

$$N(\lambda; x) = c(x)\lambda^n + R(\lambda; x),$$

where the local Weyl remainder satisfies

$$|R(\lambda; x)| \le \varepsilon \lambda^{(n-1)/2}, \quad x \in \mathcal{N}_\varepsilon(z), \quad \lambda \ge \Lambda(\varepsilon). \tag{6.2}$$

To continue, notice that $N(\lambda; x)$ is the restriction to the diagonal of the kernel of the partial summation operator $S_\lambda f = \sum_{\lambda_\nu \le \lambda} e_{\lambda_\nu}(f)$ introduced in §1. Recalling the formulas that related S_λ to the wave group shows that the Stieljies-derivative, $dN(\lambda; x)$ has $U(t; x, x)$ as its Fourier transform, where $U(t, x, y)$ denotes the kernel of $e^{-it\sqrt{-\Delta}}$, that is,

$$\widehat{dN}(t; x) = U(t; x, x).$$

Recall that for fixed $t \neq 0$, the function $x \to U(t; x, x)$ has as its wave front set

$$\{(x, \xi) \in T^*M \backslash 0, \ \exists \text{ loop of length } t \text{ through } x \text{ with initial tangent vector } \xi\}.$$

Model Case

To show how one can use this, let us first handle a model case of (6.2). Let us assume that there are *no loops of length* $\le L$ *through points* $x \in \mathcal{N}_\varepsilon(z)$. It then follows that

$$t \to U(t; x, x) = \widehat{dN}(t; x)$$

is smooth for $0 < |t| < L$.

Because of this fact we can invoke a Tauberian lemma used by Ivrii [Iv1] which says that if $N(\lambda)$ is an *increasing* function and if $g(\lambda)$ is a function satisfying $|g'(\lambda)| \le A\lambda^{n-1}$ as well as $\hat{g}(t) = \hat{N}(t)$, $|t| \le L$, then $|N(\lambda) - g(\lambda)| \le C(A)\lambda^{n-1}/L$.

Since

$$N(\lambda; x) = \frac{1}{2\pi} \int e^{it\lambda} U(t; x, x)(t + i0)^{-1} dt,$$

we apply the lemma with

$$g(\lambda; x) = \frac{1}{2\pi} \int e^{it\lambda} U(t; x, x)\rho(t/\lambda)(t + i0)^{-1} dt,$$

where $\rho(t) \in C_0^\infty$ equals one for $|t| < 1/2$ but vanishes when $|t| > 1$. If L is large, the Tauberian lemma leads to the favorable bounds

$$|N(\lambda; x) - g(\lambda; x)| \le C(A)\lambda^{(n-1)/2}/L.$$

One cannot compute $g(\lambda; x)$, but can compute

$$h(\lambda; x) = \frac{1}{2\pi} \int e^{it\lambda} U(t; x, x) \rho(t/\delta)(t + i0)^{-1} dt,$$

if δ is smaller than half the injectivity radius. On the other hand the hypotheses imply that $|g(\lambda; x) - h(\lambda; x)| = O(\lambda^{-\infty})$. Finally, it is routine to use the calculations for h to see that

$$h(\lambda; x) = c(x)\lambda^n + \tilde{R}(\lambda; x),$$

where

$$|\tilde{R}(\lambda; x)| \leq C\lambda^{n-1}/L,$$

which leads to (6.2) with $\varepsilon \approx 1/L$ in this model case where we have made the very restrictive hypotheses that there are no closed loops of length $< L$.

Microlocal Variation: Proof of (6.2)

Recall that we wish if we assume that $\mathcal{L}_z$ has zero measure and if $\varepsilon > 0$ is small then the jumps of $N(x, \lambda)$ will be $\leq \varepsilon\lambda^{n-1}$ for all large λ and x in a neighborhood $\mathcal{N}_\varepsilon(z)$ of z. To do this we use the lower semicontinuity to see that we decompose

$$I = B(x, D) + b(x, D),$$

where B and b are zero-order pseudodifferential operators having the property that when $x \in \mathcal{N}_\varepsilon(z)$ and $B(x, \xi) \neq 0$ there are *no closed loops of length* $\leq \varepsilon^{-1}$ *through* x *with initial tangent vector* ξ, while on the other hand $\int_{|\xi|=1} |b(y, \xi)|\, d\xi < \varepsilon$.

Note then that

$$N(\lambda; x) = \sum_{\lambda_\nu \leq \lambda} |\phi_{\lambda_\nu}(x)|^2 = \sum_{\lambda_\nu \leq \lambda} |B\phi_{\lambda_\nu}(x)|^2 + \sum_{\lambda_\nu \leq \lambda} |b\phi_{\lambda_\nu}(x)|^2 + \sum_{\lambda_\nu \leq \lambda} B\phi_{\lambda_\nu}(x)\overline{b\phi_{\lambda_\nu}(x)} + \sum_{\lambda_\nu \leq \lambda} b\phi_{\lambda_\nu}(x)\overline{B\phi_{\lambda_\nu}(x)}.$$

After using the same Tauberian lemma and making a tedious calculation, one sees that the jumps of the first two terms are each $O(\varepsilon\lambda^{(n-1)/2})$. This argument cannot work for the last two terms, though. However, since both involve b which satisfies $\int_{|\xi|=1} |b(y, \xi)|\, d\xi < \varepsilon$, one can use the Cauchy-Schwartz inequality along with the Tauberian arguments from the older proof of the sharp Weyl formula of Hörmander [Ho 1] to see that the jumps of the last two terms also must be $O(\varepsilon\lambda^{(n-1)/2})$, which finishes the proof. □

Remark. We should mention that Safarov [Saf] studied remainder estimates in the local Weyl laws earlier. He also saw pointed out the role that the existence or non-existence of a positive measure of loops plays in bounds like (6.1).

7. Further problems and conjectures.

Let us mention some related natural problems which remain open.

Problem 1: First, what is a sufficient condition for maximal eigenfunction growth at a given point x or at some point of (M, g)?

As mentioned above, $|\mathcal{L}_x| > 0$ is not sufficient to imply the existence of a sequence of eigenfunctions blowing up at x at the maximal rate. In fact, to our knowledge, the only Riemannian manifolds known to exhibit maximal eigenfunction growth are surfaces of revolution and compact rank one symmetric spaces. In the case of surfaces of revolution, invariant eigenfunctions must blow up at the poles because all other eigenfunctions vanish. In the case of compact rank one symmetric spaces, the exceptionally high multiplicity of eigenspaces allows for the construction of eigenfunctions of maximal sup norm growth. In each case, $\mathcal{L}_x$ has full measure, but the mechanisms producing maximal eigenfunction growth involve something more. All of these examples are completely integrable and eigenfunctions with maximal sup-norms may be explicitly constructed by the WKB method. Let us describe the symplectic geometry, because it turns out that we are tantalizingly close to proving it must occur in the real analytic case.

The eigenfunctions with maximal sup norms in these cases are actually oscillatory integrals (quasimodes) associated to (geodesic flow)-invariant Lagrangean submanifolds diffeomorphic to $S^*_x M \times S^1 \sim S^{n-1} \times S^1$. They are the images of $S^1 \times S^*_m M$ under the Lagrange immersion

$$\iota : S^1 \times S^*_m M \to T^*M\backslash 0, \quad \iota(t, x, \xi) = G^t(x, \xi),$$

where $G^t : T^*M\backslash 0 \to T^*M\backslash 0$ is the geodesic flow. Under the natural projection $\pi : S^*M \to M$, $\pi : \iota(S^1 \times S^*_m M) \to M$ the sphere $S^*_x M$ is 'blown down' S^*M to m. As discussed in [TZ] (and elsewhere), singularities of projections of Lagrangean submanifolds correspond closely to sup norms of the associated quasimodes, and such blow-downs give the maximal growth rate of the associated quasimodes. Existence of a quasimodes of high order attached to an invariant Lagrangean $S^*_x M \times S^1$ may therefore be the missing condition.

In the real analytic case, we showed in Theorem 4.1 that all geodesics issuing from some point m return to m at a fixed time ℓ. If, as is widely conjectured, they are smoothly closed curves at m, then these geodesics fill out a Lagrangean submanifold of the form $\iota(S^1 \times S^*_m M)$ above. To complete the conjectured picture, we would to show that there exist eigenfunctions which are oscillatory integrals associated to this Lagrangean. It is automatic that quasimodes (approximate eigenfunctions) can be constructed, but it is not necessarily the case that they approximate actual eigenfunctions.

We do not expect such quasimodes always to approximate eigenfunctions, nor do we expect maximal eigenfunction growth in all situations where $\mathcal{L}_x$ has full measure, even at all points x. In other words, we do not expect maximal eigenfunction growth on all Zoll manifolds (manifolds all of whose geodesics are closed). It is quite conceivable that (non-rotational) Zoll spheres do not have maximal eigenfunction growth, even though the converse estimate $R(\lambda; x) = \Omega(\lambda^{\frac{n-1}{2}})$ holds. Indeed, it is known ([V]) that on the standard S^2, almost every orthonormal basis of eigenfunctions satisfies $||\phi_\lambda|| = O(\sqrt{\log \lambda})$, even though special eigenfunctions (zonal spherical harmonics) have maximal eigenfunction growth. It is possible that eigenfunctions of typical Zoll surfaces resemble such typical bases of spherical harmonics rather than the special ones with extremal eigenfunction growth. In short, the converse direction appears to be a difficult open problem.

In a future article with J. Toth, we plan to study L^p-norms of eigenfunctions and quasimodes directly using oscillatory integral and WKB formulae. As a very special case, we will improve the sup norm bound on eigenfunctions of tori of revolution to $\lambda^{1/4}$.

At the opposite extreme is the problem of characterizing compact Riemannian manifolds with minimal eigenfunction growth, such as occurs on a flat torus. In [TZ] it is proved that in the integrable case, the only examples are flat tori and their quotients.

Problem 2: Characterize (M, g) with maximal L^p-norms of eigenfunctions.

We have left this problem open for $p \leq \frac{2(n+1)}{n-1}$, and we expect the condition on (M, g) to change at the the critical Lebesgue exponent $p = \frac{2(n+1)}{n-1}$. Indeed, as was shown in [So2], the 'geometry' of extremal eigenfunctions changes at this exponent. Specifically (cf. [So3], p. 142-144, [So1]), for $p > \frac{2(n+1)}{n-1}$ eigenfunctions concentrated near a point tend to have extreme L^p norms, while for $2 < p < \frac{2(n+1)}{n-1}$ ones concentrated along stable closed geodesics tend to have this property; if $p = \frac{2(n+1)}{n-1}$, at least in the case of the round sphere, both types give rise to $\Omega(\lambda^{\delta(p)})$ bounds.

Bourgain [B] has constructed a metric on a 2-torus of revolution for which the maximal L^6 bound is attained although $|\mathcal{L}_x| = 0$ for all x. The eigenfunctions are similar to the highest weight spherical harmonics on S^2 which concentrate on the equator. Note that this is the Lebesgue exponent where one expects the behavior of eigenfunctions which maximize this quotient to change.

Thus, we do not expect '$|\mathcal{L}_x| > 0$' to be a relevant mechanism in producing large L^p norms below the critical exponent. In some sense, existence of stable elliptic closed geodesics is more likely to be involved. We plan to study the problem elsewhere.

References

[A] V.I. Arnold, *Mathematical methods of classical mechanics*. Graduate Texts in 1998.

[Av] V. I. Avakumovič, Über die Eigenfunktionen auf geshchlossenen Riemannschen Mannigfaltigkeiten. Math. Z. **65** (1956), 327–344.

[BB] L. Bérard-Bergery, Quelques exemples de variétés riemanniennes où toutes les géodésiques issues d'un point sont fermés et de même longueur, suivis de quelques résultats sur leur topologie. Ann. Inst. Fourier (Grenoble) 27 (1977), 231–249.

[Besse] A. L. Besse, *Manifolds all of whose geodesics are closed. With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan.* Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 93. Springer-Verlag, Berlin-New York, 1978.

[BM] E. Bierstone and P.D. Milman, Semianalytic and subanalytic sets. Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 5–42.

[BC] A. Bonami and J.-C. Clerc, Sommes de Cesìo et multiplicateurs des développements en harmoniques sphériques, Trans. Amer. Math. Soc. 183 (1973), 223–263.

[B] J. Bourgain, Eigenfunction bounds for compact manifolds with integrable geodesic flow, IHES preprint.

[CS] L. Carleson and P. Sjölin, Oscillatory integrals and a multiplier problem for the disc, Studia Math. 44 (1972), 287–299.

[CV] Y. Colin de Verdière, Spectre conjoint d'opérateurs pseudo-différentiels qui commutent. II. Le cas intégrable, Math. Z. 171 (1980), no. 1, 51–73.

[CVV] Y. Colin de Verdiere and S. Vu Ngoc, Singular Bohr-Sommerfeld rules for 2D integrable systems, http://www-fourier.ujf-grenoble.fr/prepublications.html

[DG] J. J. Duistermaat and V. W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. **29** (1975), 39–79.

[H] R. M. Hardt, Stratification via corank one projections. Singularities, Part 1 (Arcata, Calif., 1981), 559–566, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, R.I., 1983.

[Ho 1] L. Hörmander, The spectral function of an elliptic operator. Acta Math. **121** (1968), 193–218.

[Ho 2] L. Hörmander, Oscillatory integrals and multipliers on FL^p. Ark. Mat. 11, 1–11. (1973).

[Ho IV] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Volume IV*, Springer-Verlag Berlin Heidelberg, 1983.

[Iv1] V. Ivriĭ, The second term of the spectral asymptotics for a Laplace Beltrami operator on manifolds with boundary. (Russian) Funktsional. Anal. i Prilozhen. 14 (1980), no. 2, 25–34.

[J] D. Jerison, Carleman inequalities for the Dirac and Laplace operators and unique continuation, Adv. in Math. 62 (1986), no. 2, 118–134.

[JK] D. Jerison and C. Kenig, Unique continuation and absence of positive eigenvalues for Schrödinger operators, Ann. of Math. (2) 121 (1985), no. 3, 463–494.

[K] W. Klingenberg, *Riemannian Geometry*, de Gruyter Studies in Mathematics 1, de Gruyter, New York (1982).

[KMS] D. Kosygin, A. Minasov, Ya. G. Sinaĭ, Statistical properties of the spectra of Laplace-Beltrami operators on Liouville surfaces. (Russian) Uspekhi Mat. Nauk 48 (1993), no. 4(292), 3–130; translation in Russian Math. Surveys 48 (1993), no. 4, 1–142

[Le] B. M. Levitan, On the asymptoptic behavior of the spectral function of a self-adjoint differential equaiton of second order. Isv. Akad. Nauk SSSR Ser. Mat. **16** (1952), 325–352.

[L] S. Lojasiewicz, "Ensembles semi-analytiques", Preprint, I.H.E.S., Bures-sur-Yvette, 1964.

[MS] W. Minicozzi and C. D. Sogge, Negative results for Nikodym maximal functions and related oscillatory integrals in curved space. Math. Res. Lett. 4 (1997), no. 2-3, 221–237.

[Saf] Y. Safarov, Asymptotics of a spectral function of a positive elliptic operator without a nontrapping condition. (Russian) Funktsional. Anal. i Prilozhen. 22 (1988), no. 3,53–65, 96 translation in Funct. Anal. Appl. 22 (1988), no. 3, 213–223 (1989).

[S] L. Simon, Lecture notes (unpublished, 1992).

[So1] C. D. Sogge, Oscillatory integrals and spherical harmonics. Duke Math. J. **53** (1986), 43–65.

[So2] C. D. Sogge, Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds. J. Funct. Anal. **77** (1988), 123–134.

[So3] C. D. Sogge, On the convergence of Riesz means on compact manifolds. Ann. of Math. (2) 126 (1987), no. 2, 439–447.

[So4] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Univ. Press, Cambridge, 1993.

[SZ] C. D. Sogge, and S. Zelditch, *Riemannian manifolds with maximal eigenfunction growth*, http://front.math.ucdavis.edu/math.AP/0103172 .

[St] E. M. Stein, Oscillatory integrals in Fourier analysis, *Beijing lectures in harmonic analysis* (Beijing, 1984), 307–355, Ann. of Math. Stud., 112, Princeton Univ. Press, Princeton, NJ, 1986.

[TZ] J.A.Toth and S.Zelditch, Sup norms of eigenfunctions in the completely integrable case (preprint, 2000).

[V] J. VanderKam, L^∞ norms and quantum ergodicity on the sphere. Internat. Math. Res. Notices (1997), 329–347; Correction to: "L^∞ norms and quantum ergodicity on the sphere", Internat. Math. Res. Notices (1998), 65.

[Z] S. Zelditch, The inverse spectral problem for surfaces of revolution, JDG 49 (1998), 207-264.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA