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Bohr–Sommerfeld quantization condition for non-selfadjoint operators in dimension 2.

Joint work with A.Melin

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0. Introduction.

We will work on $\mathbf{R}^n$, mainly in the case when $n = 2$. The Bohr–Sommerfeld condition is a very efficient tool for finding the eigenvalues of a semi-classical differential or pseudodifferential operator in dimension 1, mainly in the self-adjoint case. Let us start by recalling the rough statement of this. See for instance [LaLi], [HeRo], and one of the exercises in [GrSj].

Let $p(x, \xi, z)$ be a smooth symbol on $\mathbf{R}^2_{x,\xi}$ which is bounded together with all its derivatives and which depends smoothly on $z \in \mathrm{neigh}\,(0, \mathbf{R})$ (i.e for z in some real neighborhood of 0 in $\mathbf{R}$.) Assume that $p \geq$ Const. > 0 near infinity, and that $p^{-1}(\cdot, 0)(0)$ is a simple closed curve $\gamma(0)$ on which $d_{x,\xi}p$ is non-vanishing. Let $\gamma(z) = p^{-1}(\cdot, z)(0)$ and let

$$I(z) = \int_{\gamma(z)} \xi dx \tag{0.1}$$

be the action. Let $P^w(z) = p^w(x, hD_x, z)$ be the Weyl quantization of $p(x, h\xi, z)$ Then the Bohr–Sommerfeld condition tells us that there exists a real-valued function

$$\theta(z; h) \sim \theta_0 + \theta_1(z)h + \theta_2(z)h^2 + ..., \ h \to 0$$

in $C_0^\infty(\mathrm{neigh}\,(0, \mathbf{R}_z))$, with ($\theta_j$ smooth in the same neighborhood and) θ_0 equal to a half-integer, such that $P^w(z)$ is non-invertible precisely when

$$I(z) = (k - \theta(z; h))2\pi h, \tag{BS}$$

for some integer k. If the z-dependence is non-degenerate in a suitable sense, this gives a sequence of real "eigenvalues" (solving (BS)), each separated from its right and left neighbors by a distance which is of the order of h. In this result, the assumptions near infinity can be generalized, what is important is only that we have ellipticity there.

If we drop the assumption that p be real, but assume instead that p is close to being real (in a suitable sense) and that p extends to a bounded holomorphic function in a tubular neighborhood of $\mathbf{R}^2$, and depends holomorphically on $z \in \mathrm{neigh}\,(0, \mathbf{C})$, then $I(z)$ can still be defined by integrating along a closed curve in $p^{-1}(\cdot, z)(0)$ close to the real domain, and becomes a holomorphic function of z. Then we can find a holomorphic function $\theta(z; h)$ with an asymptotic expansion as above, so that (BS) determines the "eigenvalues" z of $P^w(z)$, i.e. the values for which the operator is not bijective. See [FuRa] for this type of result, as well as more advanced ones related to critical points.

In higher dimensions ($n > 1$), it is still possible to describe the eigenvalues provided that we assume that $p(\cdot; z)$ is completely integrable and that we have a corresponding quantum complete integrability condition. Roughly, we then have that $p^{-1}(\cdot,z)(0)$ is foliated into an $(n-1)$-parameter family of Lagrangian (n-dimensional) torii $\Lambda_{z,z_2,..,z_n}$ for $z_2,..,z_n \in \mathrm{neigh}\,(0,\mathbf{R})$. Let $I_j(z,z_2,..,z_n)$, $j=1,2,..,n$, be the actions, i.e. the integrals of the canonical 1-form $\xi\cdot dx$ along the n fundamental cycles of the torus $\Lambda_{z,z_2,...,z_n}$. Then (under suitable assumptions that we do not recall in detail here), $P^w(z)$ is non-invertible precisely when there exist $z_2,...,z_n$, such that the following Bohr–Sommerfeld (Einstein, Keller, Maslov) quantization condition is fulfilled:

$$I_j(z)=(k_j-\theta^j(z,z_2,..,z_n;h))2\pi h, \tag{BS}$$

for some integers k_j. Here θ^j are smooth functions of $z,z_2,...,z_n$ with asymptotic expansions in powers of h as above. See [Vu] and further references given there.

In non-completely integrable situations, it is still possible to describe some part of the spectrum in some h-dependent set, by means of a BS-condition. This can be achieved by using Birkhoff normal forms also in the quantized version, sometimes in combination with the KAM theorem. See Lazutkin [Laz], Colin de Verdière [Co], Sjöstrand [Sj], Kaidi–Kerdelhué [KaKe], Popov [Po1,2].

In [MeSj], we obtained some rather philosophical results about estimating determinants of h-pseudodifferential operators and the present work started as an attempt to show that those results are optimal in some more special situations. The attempt was successful, but what we obtained is probably of independent interest: For a class of *non self-adjoint* pseudodifferential operators *in dimension 2* with analytic symbols and with no complete integrability assumption, we have:

1) A complex KAM theorem without exceptional sets caused by small divisors,

2) A BS-condition which gives all eigenvalues in some h-independent open set set in $\mathbf{C}$.

1. A complex KAM-theorem.

In the remainder of this talk we take $n=2$. Let $p(x,\xi)$ be holomorphic and bounded in a tubular neighborhood of $\mathbf{R}^4_{x,\xi}$. Assume that on $\mathbf{R}^4$ we have:

$$\Gamma_0:=\mathbf{R}^4\cap p^{-1}(0)\text{ is connected, }|p|\ge\frac{1}{C}\text{ for }|(x,\xi)|\ge C, \tag{1.1}$$

$$dp,\ d\overline{p}\text{ are linearly independent at the points of }\Gamma_0, \tag{1.2}$$

$$|\frac{1}{i}\{p,\overline{p}\}|\text{ is sufficiently small on }\Gamma_0 \tag{1.3}$$

Here in (1.3), we adopt the convention that we consider a family of functions p which fulfill all the other assumptions uniformly. As usual,

$$\{a,b\}=\sum_1^2\big(\frac{\partial a}{\partial\xi_j}\frac{\partial b}{\partial x_j}-\frac{\partial a}{\partial x_j}\frac{\partial b}{\partial\xi_j}\big)$$

denotes the Poisson bracket of the two functions a and b.

If we strengthen (1.3) to the assumption that the Poisson bracket $\frac{1}{i}\{p,\overline{p}\}$ vanishes on Γ_0, then this set becomes a Lagrangian manifold carrying a complex elliptic vectorfield $H_p = H_{\mathrm{Re}\, p} + iH_{\mathrm{Im}\, p}$. It is a well-known fact that a smooth compact surface carrying an elliptic (complex) vectorfield must be (diffeomorphic to) a torus, so we conclude in this case that Γ_0 is a Lagrangian torus.

Under the general assumptions above, we can project H_p to Γ_0 and still get an elliptic vectorfield on this surface, so we still have a torus which however will not be Lagrangian in general.

Theorem 1.1. *In a small complex neighborhood of $\Gamma_0 = p^{-1}(0) \cap \mathbf{R}^4$, there exists a smooth torus $\Gamma \subset p^{-1}(0)$ of real dimension 2, close to Γ_0 in the C^1 sense, with $\sigma_{|_\Gamma} = 0$, such that $I_j(\Gamma) \in \mathbf{R}$ for $j = 1, 2$.*

Here $\sigma = \sum_1^2 d\xi_j \wedge dx_j$ denotes the complex symplectic (2,0)-form, and the actions are defined as in the case of real Lagrangian torii, by integration of the canonical (1,0)-form $\xi \cdot dx$ along two fundamental cycles. From the proof, we actually get a family of torii on which σ vanishes, parametrized by one of the actions. The other action becomes a holomorphic function of the selected one, and there is a unique real value of the parameter for which both actions are real. The torus in the theorem is not unique, but the corresponding flow-out, $\{\exp \widehat{tH_p}(\rho);\, t \in \mathbf{C},\, |t| < \frac{1}{C}\}$, $C \gg 0$, becomes a complex Lagrangian manifold, and different choices of Γ give rise to the same Λ near the real domain. Here we treat $H_p = \sum \left(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\right)$ as a complex vectorfield of type (1,0) and if v is a general vectorfield of type (1,0), we let $\widehat{v}$ denote the corresponding real vectorfield (in the complex domain) which acts the same way as v as a differential operator on holomorphic functions; $\widehat{v} = v + \overline{v}$.

We notice that E. Chirka [Ch] has studied complex foliations under non-compactness assumptions (so that the leafs are open rather than compact as in our case). As was pointed out to me by D. Bambusi and S. Graffi, the fact that one can sometimes get less problems with small denominators in the complex domain, has been used by Bazzani–Turchetti [BaTu] and by Marmi–Yoccoz.

The remainder of this section is an outline of the proof of Theorem 1.1. We say that a multivalued function on a torus or on a small neighborhood of a torus is grad-periodic if its gradient is a single-valued function. Let x_1, x_2 be grad-periodic, real and analytic on Γ_0, with the property that (x_1, x_2) induces a diffeomorphism: $\Gamma_0 \to \mathbf{R}^2/L$ for some lattice L. Extend x_1, x_2 to be real and analytic grad-periodic functions in a neighborhood of Γ_0 with

$$\{x_1, x_2\} = 0.$$

Then

$$\sigma_{|_{\Gamma_0}} = f(x) dx_1 \wedge dx_2,$$

with f small: $f = \mathcal{O}(\epsilon)$ (in a suitable space of functions). Since σ is exact, its restriction to Γ_0 is also exact, and it follows that

$$\sigma_{|_{\Gamma_0}} = d(\gamma_1 dx_1 + \gamma_2 dx_2),\ \ \gamma_j = \mathcal{O}(\epsilon).$$

Let ξ_j be the single-valued functions defined near Γ_0, which solve:

$$\xi_{j|_{\Gamma_0}} = \gamma_j,\ \ H_{x_j}\xi_k = -\delta_{j,k}.$$

The following result is then quite straight forward to show:

Lemma 1.2. *x, ξ form a system of symplectic coordinates.*

In these coordinates Γ_0 takes the form $\xi = \gamma(x)$, where we may view γ as an L-periodic function on $\mathbf{R}^2$. We can then write

$$p(x,\xi) = \sum_1^2 a_j(x)(\xi_j - \gamma_j(x)) + \underbrace{\sum_{j,k} b_{j,k}(x,\xi)(\xi_j - \gamma_j(x))(\xi_k - \gamma_k(x))}_{F(x,\xi-\gamma(x))=\mathcal{O}((\xi-\gamma)^2)}.$$

Look for $\Gamma = \Gamma_\phi$ of the form $\xi = \phi'_x(x)$, where ϕ is grad-periodic and complex-valued. Then $\sigma_{|_\Gamma} = 0$ and we get the eikonal equation

$$p(x, \phi'(x)) = 0, \tag{1.4}$$

which can also be written as

$$Z\phi + F(x, \phi' - \gamma) + r(x) = 0, \tag{1.5}$$

where $Z = \sum a_j(x)\frac{\partial}{\partial x_j}$ is an elliptic vectorfield, and $r = -\sum a_j(x)\gamma_j(x) = \mathcal{O}(\epsilon)$. Try $\phi = \widetilde{\epsilon}\psi$, $\epsilon \ll \widetilde{\epsilon} \ll 1$. Then we get

$$Z\psi + \widetilde{\epsilon}G(x, \psi' - \frac{\gamma}{\widetilde{\epsilon}}, \widetilde{\epsilon}) + \frac{r(x)}{\widetilde{\epsilon}} = 0,$$

where

$$G(x,\xi,\widetilde{\epsilon}) = \frac{1}{\widetilde{\epsilon}^2}F(x, \widetilde{\epsilon}\xi)$$

is bounded in some fixed neighborhood of Γ_0, when $\widetilde{\epsilon}$ tends to 0.

Since Z is an elliptic vectorfield on a torus, it is well-known that there exist new (grad-periodic) coordinates x_1, x_2, so that

$$Z = A(x)\frac{\partial}{\partial \overline{x}},\ \ \frac{\partial}{\partial \overline{x}} = \frac{1}{2}(\frac{\partial}{\partial x_1} - \frac{1}{i}\frac{\partial}{\partial x_2}),$$

where $A(x)$ is a non-vanishing function. After division with A we get with new functions G and r:

$$\frac{\partial \psi}{\partial \overline{x}} + \widetilde{\epsilon}G(x, \psi' - \frac{\gamma}{\widetilde{\epsilon}}, \widetilde{\epsilon}) + \frac{r}{\widetilde{\epsilon}} = 0. \tag{1.6}$$

Look for $\psi = \psi_{\text{per}} + ax + b\overline{x} = ax + u(x)$ with ψ_{per} periodic, where we let the parameter a vary in some neighborhood of 0 in $\mathbf{C}$, and try to solve (1.6) by a simple iteration putting $u_0 = 0$ and solving the linear Cauchy-Riemann equations:

$$\frac{\partial u_{j+1}}{\partial \overline{x}} + \widetilde{\epsilon} G(x, a + u_j'(x) - \frac{\gamma}{\widetilde{\epsilon}}, \widetilde{\epsilon}) - \frac{r}{\widetilde{\epsilon}} = 0, \tag{1.7}$$

with $u_j = \psi_{j,\text{per}} + b_j \overline{x}$. Wee can apply standard estimates for Calderon–Zygmund integral operators and see that this procedure converges for the norm $\|\psi'_{j,\text{per}}\|_{C^m} + |b_j|$, with $m > 0$ non-integer. See [BeJoSc]. In this construction it is also easy to understand how the actions vary with the parameter a, and this allows us to see that there is a unique a for which both actions are real.

2. The BS-condition.

Now let $p(x, \xi, z)$ be a holomorphic family of p's as above for z in some neighborhood of zero in the complex plane. Let $P(z) = p^w(x, hD_x, z)$ be the h-Weyl quantization of $p(\cdot, z)$. Then we can choose $\Gamma = \Gamma(z)$ associated to $p(\cdot, z)$ as in Theorem 1.1, depending smoothly on z. The map

$$z \mapsto I(z) = (I_1(z), I_2(z)) \in \mathbf{R}^2,\ \ I_j(z) = I_j(\Gamma(z)), \tag{2.1}$$

is therefore smooth and it can be showed that it satisfies a kind of modified d-bar system. Our main result is

Theorem 2.1. *There exists $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$ such that the following holds for z in some neighborhood Ω of 0 in $\mathbf{C}$ and for $h > 0$ small enough:*

1) There exists a function $\theta(z; h) \sim \theta_0 + \theta_1(z)h + ...$ in $C_0^\infty(\Omega)$ such that $P^w(z) : L^2 \to L^2$ is non-bijective precisely when

$$\frac{I(z)}{2\pi h} = k - \theta(z; h) \text{ for some } k \in \mathbf{Z}^2. \tag{2.2}$$

2) When I is a local diffeomorphism, all eigenvalues (i.e. solutions to (2.2)) are simple in a suitable sense.

The proof uses the machinery of H_Φ-spaces and associated operators (see [Sj2,3]), and we can here only give some hints about the main ideas.

Using that the actions of $\Gamma(z)$ are real one first sees that there exists an IR-manifold $\Lambda_z \subset \mathbf{C}^4$ close to $\mathbf{R}^4$ which contains $\Gamma(z)$. (Recall that an IR-manifold is a smooth submanifold of $\mathbf{C}^4$ of real dimension 4, on which the restriction of σ is real and non-degenerate.) Using a semi-classical Bargman transform, we can define a corresponding Hilbert space $H(\Lambda_z)$ on which $P^w(z)$ acts with leading symbol $p_{|\Lambda_z}$. (On the transform side this is a weighted space of entire functions.) Now $\Gamma(z)$ can be viewed as a "real" Lagrangian submanifold of $\Lambda(z)$, and the spectral problem becomes microlocalized to a small neighborhood of $\Gamma(z)$. Here we cannot go into any details at all, and we just mention that at one point we get a simple spectral problem on the torus, namely to determine if the equation

$$(A(x)\frac{\partial}{\partial \overline{x}}+V(x))u=v(x) \tag{2.3}$$

is solvable say for L-periodic functions, where L is a lattice in $\mathbf{C}$, and A, V are L-periodic with A is non-vanishing. After division by A and conjugation by a factor of the form e^{ϕ}, with ϕ smooth and periodic, we get the equivalent problem with new functions u, v:

$$(\frac{\partial}{\partial \overline{x}}+(\frac{\partial \phi}{\partial \overline{x}}+\frac{V}{A}))u=v. \tag{2.4}$$

Here we choose ϕ such that

$$\frac{\partial \phi}{\partial \overline{x}}+\frac{V}{A}=\mathcal{F}(\frac{V}{A})(0),$$

the Fourier coefficient of V/A at 0. Then (2.4) becomes an equation with constant coefficients, which can be completely analyzed on the level of Fourier series. In this way one encounters the Bohr–Sommerfeld condition.

3. Saddle point resonances.

Consider the operator

$$P=-\frac{h^2}{2}\Delta+V(x),\ x\in\mathbf{R}^2, \tag{3.1}$$

where V is a real-valued analytic potential, which extends holomorphically to a set $\{x\in\mathbf{C}^2;\ |\mathrm{Im}\, x|<\frac{1}{C}\langle\mathrm{Re}\, x\rangle\}$, with $V(x)\to 0$, when $x\to\infty$ in that set. The resonances of P can be defined in an angle $\{z\in\mathbf{C};\ -2\theta_0<\arg z\le 0\}$ for some fixed $\theta_0>0$ as the eigenvalues of $P_{|_{e^{i\theta_0}\mathbf{R}^n}}$. In [HeSj], they were also defined, using an FBI transform. (See also Lahmar-Benbernou–Martinez [LahMa], for a simplified version of the theory.)

Let $E_0>0$. Let $p(x,\xi)=\xi^2+V(x)$. We assume that the union of trapped H_p-trajectories in $p^{-1}(E_0)\cap\mathbf{R}^4$ is reduced to a single point (x_0,ξ_0). Necessarily, $\xi_0=0$ and after a translation, we may also assume that $x_0=0$. (Recall for instance from [GeSj] that a trapped trajectory is a maximally extended trajectory which is contained in a bounded set.) It follows that 0 is a critical point for V and that $V(0)=E_0$. Assume,

$$0 \text{ is a non-degenerate critical point of } V, \text{ of signature } (1,-1). \tag{3.2}$$

After a linear change of coordinates in x and a corresponding dual one in ξ, we may assume that

$$p(x,\xi)-E_0=\frac{\lambda_1}{2}(\xi_1^2+x_1^2)+\frac{\lambda_2}{2}(\xi_2^2-x_2^2)+\mathcal{O}((x,\xi)^3),\ (x,\xi)\to 0. \tag{3.3}$$

Under the assumptions above, but without any restriction on the dimension and without the assumption on the signature in (3.3), we determined in [Sj4] all resonances in a disc $D(E_0, Ch)$ for any fixed $C > 0$, when $h > 0$ is small enough. Under the same assumptions plus a diophantine one on the eigenvalues of $V''(0)$, Kaidi and Kerdelhué [KaKe] determined all resonances in a disc $D(E_0, h^\delta)$ for any fixed $\delta > 0$ and for $h > 0$ small enough. In the two dimensional case, their diophantine condition follows from (3.2), and we recall their result in that case.

Theorem 3.1. *([KaKe]). Under the assumptions from (3.1) to (3.2), let $\lambda_j > 0$ be defined in (3.3). Fix $\delta > 0$. Then for $h > 0$ small enough, the resonances in $D(E_0, h^\delta)$ are all simple and coincide with the values in that disc, given by:*

$$z = f(2\pi h(k - \theta_0); h),\ \ k \in \mathbf{N}^2, \tag{3.4}$$

where $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$ is fixed, and $f(\theta; h)$ is a smooth function of $\theta \in \mathrm{neigh}\,(0, \mathbf{R}^2)$, with

$$f(\theta; h) \sim f_0(\theta) + hf_1(\theta) + h^2 f_2(\theta) + ..,\ \ h \to 0, \tag{3.5}$$

in the space of such functions. Further,

$$f_0(\theta) = \frac{1}{2\pi}(\lambda_1\theta_1 - i\lambda_2\theta_2) + \mathcal{O}(\theta^2).$$

For $\epsilon_0, C_1 > 0$, we consider now the set

$$h^\delta \le |z| < \frac{1}{C_1},\ -\frac{\pi}{2} + \epsilon_0 < \arg z < -\epsilon_0, \tag{3.6}$$

Theorem 3.2. *The description of the resonances in Theorem 3.1 extends to the set of z in (3.6), provided that C_1 there is sufficiently large as a function of $\epsilon_0 > 0$ and that $h > 0$ is small enough.*

The proof consists in combining Theorem 2.1, a scaling argument and the theory of [HeSj].

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