SEMINAIRE

Equations aux Dérivées Partielles

1999-2000

Sergiu Klainerman

A Commuting Vectorfields Approach to Strichartz type Inequalities and Applications to Quasilinear Wave Equations

A Commuting Vectorfields Approach to Strichartz type Inequalities and Applications to Quasilinear Wave Equations

S. Klainerman

June 16, 2000

A large body of knowledge about wave equations can be traced down to two fundamental facts concerning the standard linear wave equations in Minkowski space-time $\mathbf{R}^{n+1}$,

$$\Box\, \phi = m^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

with $m_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$ the standard Minkowski metric.

The first is the well known energy identity,

$$E[\phi](t) = E[\phi](0) \tag{0.1}$$

where,

$$E[\phi](t) = \int_{\mathbf{R}^n} \left(|\partial_t \phi(t, x)|^2 + |\partial_1 \phi(t, x)|^2 + \cdots + |\partial_n \phi(t, x)|^2 \right) dx.$$

Therefore, for $\partial\phi = (\partial_t\phi, \partial_1\phi, \ldots, \partial_n\phi)$,

$$\|\partial\phi(t)\|_{L^2} \le \|\partial\phi(0)\|_{L^2} \tag{0.2}$$

The second, which I will refer to as the *basic dispersive inequality*, has the form,

$$|\phi(t)|_{L^\infty} \le c t^{-\frac{n-1}{2}} \|\nabla^{\frac{n-1}{2}}\partial\phi(0)\|_{L^1} \tag{0.3}$$

In fact 0.3 is not quite right, the correct estimate holds if we replace the L^∞ norm on the left by the BMO-norm, or, the L^1 norm on the right by the Hardy norm $\mathcal{H}^1$. The inequality 0.3 is true however, as it stands, if the Fourier transform of the data $\phi(0) = f$, $\partial_t\phi(0) = g$ have their Fourier transform supported in a dyadic shell $\frac{\lambda}{2} \le |\xi| \le 2\lambda$ for some fixed $\lambda \in 2^{\mathbf{N}}$.

Interpolating between these two basic facts one derives the so called Strichartz-Brenner result,

$$\|\phi(t)\|_{L^r} \le c|t|^{-\gamma(r)} \|\nabla^\sigma \partial\phi(0)\|_{L^{r'}}$$

with $\gamma(r) = (n-1)(\frac{1}{2} - \frac{2}{r})$, $\frac{1}{r} + \frac{1}{r'} = 1$, $r \ge 2$ and scaling condition $\frac{n}{r} = -\gamma(r) - \sigma - 1 + \frac{n}{r'}$. This leads, by a standard TT^* argument, Hardy -Littlewood-Sobolev inequalities and an application of the Littlewood-Paley theory, to the generalized Strichartz inequality,

$$\|\phi\|_{L_t^q L_x^r} \leq c\|\partial\phi(0)\|_{H^\sigma} \tag{0.4}$$

$$\frac{2}{q} \leq \gamma(r), q \geq 2, \qquad (q, r, n) \neq (\infty, 1, 3)$$

$$\sigma = n(\frac{1}{2} - \frac{2}{r}) - 1 - \frac{1}{q}$$

The latter plays a crucial role in many recent advances of the theory of nonlinear wave equations. Observe that the steps involved in deriving 0.4, at fixed frequency, from the energy identity and dispersive inequality are quite soft, they can be traced back to the Duhamel's principle and uniqueness of the initial value problem[1]. Both apply to general linear wave equations with variable coefficients[2] and require very little regularity of the coefficients. Thus the main building blocks of the Strichartz type inequalities are 0.1 and 0.3.

The identity 0.1, and the corresponding L^2 estimate, can easily be derived from the Fourier representation of solutions. The beauty and power of the identity, however, is that it can be derived directly , in physical space, by a simple integration by parts argument. Thus energy type estimates are extremely versatile, they can be applied to large classes of linear and nonlinear equations. On the other hand the classic derivation of the dispersive inequality is based on the method of stationary phase applied to the specific representation of solutions as Fourier integral operators. In more complicated situations the Fourier representation of solutions, or rather approximate solutions, may be quite difficult to derive and not very natural. The main goal of this paper is to outline a method of proof which avoids an explicit representation of solutions, see [Kl4] for details.

The dispersive inequality provides two types of information:

1. The precise decay rate of $\|\phi(t)\|_{L^\infty}$ as $t \to \infty$.

2. Improved regularity properties of $\|\phi(t)\|_{L^\infty}$ for $t > 0$.

It is well known that as far as the asymptotic behavior is concerned 0.3 is not very useful in applications to nonlinear wave equations. A more effective procedure to derive the asymptotic properties of solutions of the wave equation is based on generalized energy estimates, obtained by the commuting vectorfields method, together with global Sobolev inequalities. We shall make here a quick review of this procedure. As far as improved regularity is concerned the estimate 0.3 gains, for $t > 0$, $\frac{n-1}{2}$ derivatives when compared to the Sobolev embedding $L^\infty(\mathbf{R}^n) \subset W^{1,n}(\mathbf{R}^n)$. It thus may seem that the methods discussed here, based on Sobolev estimates, are not relevant to questions concerning regularity. The main observation discussed below is that the decay estimates based on commuting vectorfields do actually imply, after a suitable localization in phase space[3], the dispersive inequality 0.3.We present a different, direct, approach to the derivation of 0.3 based only on energy estimates, commuting vectorfields, generalized energy estimates and an appropriate localization. In

---

[1]See Theorem 1.2. for a straightforward derivation of 0.4 from 0.1 and 0.3

[2]The uniqueness of the I.V.P. is also a consequence of the basic energy inequality

[3]The localization method, which is the key in the proof of Theorem 1.1, was used in a different context by O. Liess [L]. The essence of his idea was that, after localization to the unit dyadic region in Fourier space, the $L^1 - L^\infty$ dispersive inequality follows from a weighted $L^2 - L^\infty$ inequality. This is done easyly by a further localization in physical space. I am grateful to T. Tao for pointing this important fact to me.

what follows I will discuss this approach for the standard wave equation in Minkowski space. I will also outline how to adapt the new approach to the case of variable coefficients wave equations. I indicate how the method can be applied to derive Strichartz type results for linear wave equations with nonsmooth coefficients and improved regularity results for quasilinear wave equations in the spirit of the works of H.Smith [S1], [S2], Chemin-Bahouri [B-C1], [B-C2] and D.Tataru [T1], [T2]. These results are stated in the Theorems A,B,C below. This paper is a short summary of [Kl4].

0.1 Commuting vectorfields and global Sobolev inequalities

Let ϕ be the solution of the initial value problem for the standard wave equation,

$$\Box \, \phi = 0$$
$$\phi(0) = f, \; \partial_t \phi(0) = g \tag{0.5}$$

As discussed in the introduction it it is possible to show, using the explicit form of the fundamental solution as a Fourier integral operator, that for any $k \geq 0$,

$$||\nabla^k \phi(t)||_{L^\infty} \leq C|t|^{-\frac{n-1}{2}} \tag{0.6}$$

as $|t|$ goes to infinity. According to 0.3 the constant C depends on the L^1 of appropriate number of derivatives of the data f, g. In what follows we review the commuting vectorfields method for deriving the decay rate 0.6. The idea is to use the energy identity 0.1 together with commuting vectorfields and a global form of the classical Sobolev inequalities.

The Minkowski space-time $\mathbf{R}^{n+1}$ is equipped with a family of Killing and conformal Killing vector fields,

$$T_\mu = \partial_\mu$$
$$O_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$
$$S = t\partial_t + x^i \partial_i \tag{0.7}$$
$$K_\mu = -2x_\mu S + <x, x> \partial_\mu$$

Here x^μ, denote the standard variables $x^0 = t$, $x^1, \ldots, x^n$, $m^{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$ the Minkowski metric and $x_\mu = m_{\mu\nu} x^\nu$. The operator $\Box$ is defined by $\Box = m^{\alpha\beta} \partial_{\alpha\beta}$. The Killing vector fields T_μ and $O_{\mu\nu}$ commute with $\Box$ while S preserves the space of solutions in the sense that $\Box \, \phi = 0$ implies $\Box \, S\phi = 0$ as $[\Box, S] = 2\Box$. We split the operators $O_{\mu\nu}$ into the angular rotation operators $^{(ij)}O = x_i \partial_j - x_j \partial_i$ and the boosts $^{(i)}L = x_i \partial_t + t\partial_i$, for $i, j, k = 1, \ldots, n$. Recall the energy norm 0.1,

$$E[\phi](t) = \left(\int |\partial_t \phi(t, x)|^2 + |\partial_1 \phi(t, x)|^2 \cdots + |\partial_n \phi(t, x)|^2 dx \right)^{\frac{1}{2}}.$$ Based on the commutation properties described above we define the following "generalized energy norms"

$$E_{k+1}[\phi] = \left(\sum_{X_{i_1}, \ldots, X_{i_j}} E^2[X_{i_1} X_{i_2} ... X_{i_j} \phi] \right)^{\frac{1}{2}} \tag{0.8}$$

with the sum taken over $0 \leq j \leq k$ and over all Killing vector fields $T, \Omega_{\mu\nu}$ as well as the scaling vector field S.

The crucial point of this method is that the quantities E_k, $k \geq 1$ are conserved by solutions to 0.5. Therefore, if for all $0 \leq k \leq s$ the data f, g verify,

$$\int (1 + |x|)^{2k} \left(|\nabla^{k+1} f(x)|^2 + |\nabla^k g(x)|^2 \right) dx \leq C_s \qquad (0.9)$$

for a constant $C_s < \infty$, then for all t,

$$E_{s+1}[\phi](t) \leq C_s. \qquad (0.10)$$

The desired decay estimates of solutions to 0.5 can now be derived from the following global version of the Sobolev inequalities(see [Kl1],[Kl2],[Ho]):

Proposition 0.1 *Let ϕ be an arbitrary function in R^{n+1} such that $E_s[\phi]$ is finite for some integer $s > \frac{n}{2} + 1$. Then for $t > 0$*

$$|\partial \phi(t, x)| \leq (1 + t + |x|)^{-\frac{n-1}{2}} (1 + |t - |x||)^{-\frac{1}{2}} E_s[u]. \qquad (0.11)$$

Therefore if the data f, g in 0.5 satisfy 0.9 , for $0 \leq k \leq s$ with some $s > \frac{n}{2}$, then for all $t \geq 0$,

$$|\partial \phi(t, .)|_{L^\infty} \leq C_s \frac{1}{(1 + t + |x|)^{\frac{n-1}{2}} (1 + |t - |x||)^{\frac{1}{2}}} \qquad (0.12)$$

Clearly this estimate implies 0.6. In fact it provides more information outside the wave zone $|x| \sim t$ which fit very well with the expected propagation properties of the linear equation $\Box \, \phi = 0$.

In what follows we show that in fact the commuting vectorfields method implies the dispersive inequality 0.3. The key ingredient in the proof is a simple phase-space localization argument which I borrow from [L].

Theorem 0.1 *The commuting vectorfields method implies the dispersive inequality 0.3.*

Without loss of generality we may assume that $\partial_t \phi = g = 0$ and that the Fourier transform of $f = \phi(0)$ is supported in the shell $\frac{\lambda}{2} \leq |\xi| \leq 2\lambda$ for some $\lambda \in 2^{\mathbf{N}}$. By a simple scaling argument we may in fact assume $\lambda = 1$. Since $\hat{\phi}$, the Fourier transform of ϕ relative to the space variables x, is also supported in the same shell it suffices to prove the estimates for $\nabla \phi$ or $\nabla^k \phi$. Next we cover $\mathbf{R}^n$ by an union of discs D_I centered at points $I \in \mathbf{Z}^n$ with integer coordinates such that each D_I intersects at most a finite number c_n of discs D_J with c_n depending only on the dimension n. Consider a smooth partition of unity $(\chi_I)_{I \in \mathbf{Z}^n}$ with supp $\chi_I \subset D_I$ and each χ_I positive. Clearly we can arrange to have, for all k,

$$\sum_{I \in \mathbf{Z}^n} |\nabla^k \chi_I(x)| \leq C_{k,n} \qquad (0.13)$$

uniformly in $x \in \mathbf{R}^n$. For $k = 0$ we have in fact $C_{k,n} = 1$.

Now set, $f_I = \chi_I \cdot f$, and ϕ_I the corresponding solution to 0.5 with data $\phi_I(0) = f_I, \partial_t \phi_I(0) = 0$. Clearly $f = \sum_I f_I$, $\phi = \sum_I \phi_I$. It suffices to prove that for all I,

$$\|\nabla^k \phi_I(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+k+1} \|D^j f_I\|_{L^1} \qquad (0.14)$$

with a constant $C_{n,k}$ depending only on n and k. Indeed if 0.14 holds true we easily infer that,

$$\|\nabla^k \phi(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+k+1} \| \sum_I \nabla^j \chi_I\|_{L^\infty} \|f\|_{L^1}$$

and therefore, in view of 0.13

$$\|\nabla^k \phi(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}} \|f\|_{L^1}$$

It therefore remains to check 0.14. Without loss of generality, by performing a space translation, we may assume that $I = 0$. Applying the Prop. 0.11 to $\psi = \nabla \phi_0$ we derive, for s_* the first integer strictly larger than $\frac{n}{2} + 1$,

$$\begin{aligned} \|\psi(t)\|_{L^\infty} &\le c(1+t)^{-\frac{n-1}{2}} E_{s_*}[\phi_0](t) \\ &\le c(1+t)^{-\frac{n-1}{2}} E_{s_*}[\phi_0](0). \end{aligned}$$

Since the support of ϕ_0 is included in in the ball of radius 1 centered at the origin we have,

$$E_{s_*}[\phi_0](0) \le C_n \sum_{j=0}^{s_*+1} \|D^j f_0\|_{L^2}.$$

Finally, according to the standard Sobolev inequality in $\mathbf{R}^n$, $\|f\|_{L^2} \le c\|\nabla^{\frac{n}{2}} f\|_{L^1}$, we conclude with,

$$\|\psi(t)\|_{L^\infty} \le c(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+2+1} \|D^j f_0\|_{L^1}$$

as desired.

In what follows I will present a way of deriving the weaker decay estimate $t^{-1+\epsilon}$ in any dimension $n \ge 3$, using only the Morawetz vectorfield $K_0 = (t^2 + |x|^2)\partial_t + 2tx^i \partial_i$ and its associated first order operator $K_0 \phi + (n-1)t\phi$. Let

$$Q_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} m_{\alpha\beta}(m^{\mu\nu}\partial_\mu \phi \partial_\nu \phi)$$

the energy momentum tensor associated to the equation $\Box \phi = 0$ with $m_{\mu\nu}$ the Minkowski metric of $\mathbf{R}^{n+1}$. If ϕ is a solution of the equation we have, $\partial^\beta Q_{\alpha\beta} = 0$. We recall the following classical fact, see [C-K1],

Proposition 0.2 *Let ϕ be a solution of $\Box\phi = 0$ and $Q_{\alpha\beta}$ the corresponding energy momentum tensor. Let X be a conformal Killing vectorfield, i.e. $^{(X)}\pi = \mathcal{L}_X m = \Omega m$, and $tr\pi = m^{\alpha\beta}\pi_{\alpha\beta}$. It is easy to check that $\Box\Omega = 0$; in fact, in the particular case of $X = K_0$, $\Omega = 4(n+1)t$. Let $\bar{P}_\alpha = Q_{\alpha\beta}X^\beta + \frac{n-1}{4(n-1)}tr\,^{(X)}\pi\phi\partial_\alpha\phi - \frac{n-1}{8(n+1)}\partial_\alpha(tr\,^{(X)}\pi)\phi^2$. Then, if $\Box\phi = 0$,*

$$\partial^\alpha\bar{P}_\alpha = 0.$$

Applying the proposition to $\Box\phi = 0$ and $X = K$ and integrating the corresponding divergence free equation on a time slab $[t_0, t] \times \mathbf{R}^n$ we infer the following[4]:

Proposition 0.3 *Let $\bar{Q}(K_0, T_0) = Q(K_0, T_0) + (n-1)t\phi\partial_t\phi - \frac{n-1}{2}\phi^2$, with $T_0 = \partial_t$ the unit normal to Σ_t and ϕ a solution to $\Box\phi = 0$.*

i.) The following conformal conservation law holds true,

$$\int_{\Sigma_t} \bar{Q}(K_0, T_0) = \int_{\Sigma_{t_0}} \bar{Q}(K_0, T_0) \tag{0.15}$$

ii.) Moreover we have,

$$\int_{\Sigma_t} \bar{Q}(K_0, T_0) = \frac{1}{4}\left(\int_{\Sigma_t} \underline{u}^2(L'\phi)^2 + \int_{\Sigma_t} 2(t^2 + r^2)|\nabla\!\!\!/\,\phi|^2 + \int_{\Sigma_t} u^2(\underline{L}'\phi)^2 \right) \tag{0.16}$$

where $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$, $u = t - r$, $\underline{u} = t + r$ and $\underline{u}L'(\phi) = \underline{u}L(\phi) + (n-1)\phi$, $u\underline{L}'(\phi) = u\underline{L}(\phi) + (n-1)\phi$.

iii.) Also, if $n \geq 3$, there exists a constant $c > 0$ such that,

$$\int_{\Sigma_t} \bar{Q}(K_0, T_0) \geq c\left(\int_{\Sigma_t} \underline{u}^2(L\phi)^2 + \int_{\Sigma_t} 2(t^2 + r^2)|\nabla\!\!\!/\,\phi|^2 + \int_{\Sigma_t} u^2(\underline{L}\phi)^2 \right) \tag{0.17}$$

To prove the last two parts of the proposition one starts by expressing K, T as linear combinations of the null vectorfields $L, \underline{L}$,

$$K_0 = \frac{1}{2}(\underline{u}^2 L + u^2\underline{L}) \tag{0.18}$$

$$T_0 = \frac{1}{2}(L + \underline{L}) \tag{0.19}$$

with $u = t - r$, $\underline{u} = t + r$. Observe that u is a special solution of the Eikonal equation $m^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$. This will play an important role in the variable coefficient case.

We easly check the formulas:

$$
\begin{aligned}
Q_{LL} = Q(L, L) &= L(\phi)^2 \\
Q_{L\underline{L}} = Q(L, \underline{L}) &= |\nabla\!\!\!/\,\phi|^2 \\
Q_{\underline{L}\underline{L}} = Q(\underline{L}, \underline{L}) &= L(\phi)^2
\end{aligned}
$$

---

[4]Part i and ii of the proposition are due to C. Morawetz [M]. For part iii see [Kl3], pages 310–313.

where $\nabla\!\!\!\!/\,\phi$ denotes the induced covariant derivatives on the spheres of intersection between the level surfaces of t and those of r. Thus,

$$
\begin{aligned}
Q(K_0, T_0) &= \frac{1}{4}\Big(\underline{u}^2 Q_{LL} + (u^2 + \underline{u}^2)Q_{L\underline{L}} + u^2 Q_{\underline{LL}}\Big) \\
&= \frac{1}{4}\Big(\underline{u}^2 L(\phi)^2 + (u^2 + \underline{u}^2)|\nabla\!\!\!\!/\,\phi|^2 + u^2 \underline{L}(\phi)^2\Big)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{\Sigma_t} \bar{Q}(K_0, T_0) &= \int_{\Sigma_t} \frac{1}{4}\Big(\underline{u}^2 (L\phi)^2 + (u^2 + \underline{u}^2)|\nabla\!\!\!\!/\,\phi|^2 + u^2 (\underline{L}\phi)^2\Big) \\
&+ (n-1)\int_{\Sigma_t} t\partial_t\phi\phi - \frac{n-1}{2}\int_{\Sigma_t} \phi^2
\end{aligned}
$$

One then proceeds by a carefull integration by parts procedure. One shows, for example, that in dimension $n \geq 3$,

$$
\int_{\Sigma_t} \bar{Q}(K_0, T_0) \geq c \int_{\Sigma_t} \Big(|\phi|^2 + \underline{u}^2 |L\phi|^2 + (t^2 + r^2)|\nabla\!\!\!\!/\,\phi|^2 + u^2 |\underline{L}\phi|^2\Big).
$$

To derive the desired decay estimate we make use of the following Lemmas:

Lemma 0.1 *Let $u(x)$ be a smooth, compactly supported function on $\mathbf{R}^n$, $n \geq 3$. For any $p > n-1$, $\sigma \geq 1 + \frac{n}{2} - \frac{n}{p}$, we have*

$$
|u(x)| \leq C\frac{1}{|x|^{\frac{n-1}{p}}}\Big(\|r\nabla\!\!\!\!/\,u\|_{H^\sigma} + \|u\|_{H^\sigma}\Big) \tag{0.20}
$$

where $\nabla\!\!\!\!/\,$ denotes the induced covariant derivative along the spheres $r = const$.

To prove the Lemma we write, in polar coordinates $x = r\xi$ with $\xi \in \mathbf{R}^{n-1}$,

$$
u(r\xi)^p = -p\int_r^\infty \partial_r u(\lambda\xi)u^{p-1}(\lambda\xi)d\lambda
$$

Hence,

$$
\int_{|\xi|=1} u(r\xi)^p d\sigma(\xi) \leq c\frac{1}{r^{n-1}}\int_{\mathbf{R}^n} |\nabla u(y)||u(y)|^{p-1}dy.
$$

Hence, for $\sigma \geq \frac{n}{2} - \frac{n}{p} + 1$,

$$
\int_{|\xi|=1} u(r\xi)^p d\sigma(\xi) \leq c\frac{1}{r^{n-1}}(\|\nabla u\|^p_{L^p(\mathbf{R}^n)} + \|u\|^p_{L^p(\mathbf{R}^n)}) \leq c\frac{1}{r^{n-1}}\|u\|^p_{H^\sigma}.
$$

Finally, using the Sobolev inequality on the unit sphere S^{n-1} we infer that, for $x = r\xi$ and $r \neq 0$,

$$
|u(x)| \leq c_n\Big(\|u(r\cdot)\|_{L^p(S^{n-1})} + \|(r\nabla\!\!\!\!/\,u)(r\cdot)\|_{L^p(S^{n-1})}\Big)
$$

which combined with the inequality above proves the desired result.

Now let

$$\mathcal{E}^2[\phi](t) \;=\; \int_{\Sigma_t} \Big(|\phi|^2 + \underline{u}^2 |L\phi|^2 + (t^2 + r^2)|\nabla\!\!\!/\,\phi|^2 + u^2|\underline{L}\phi|^2 \Big) \tag{0.21}$$

$$\mathcal{E}^2_{k+1}[\phi](t) \;=\; \sum_{0 \le i \le k} \mathcal{E}^2[\nabla^i \phi] \tag{0.22}$$

In view of the Lemma 0.1 we immediately derive the result of the Proposition below in the exterior region $|x| \ge \frac{t}{2}$. For the interior region $|x| \le \frac{t}{2}$ the result follows from the fact that,

$$\mathcal{E}^2[\phi](t) \ge ct^2 \int_{|x| \le \frac{t}{2}} |\nabla\phi|^2$$

combined with the standard $H^s(\mathbf{R}^n) \subset L^\infty, s > \frac{n}{2}$ Sobolev embedding. Thus,

Proposition 0.4 *Let $\phi(t,x)$ be a smooth function in $\mathbf{R}^{n+1}$ compactly supported in x for each fixed $t \ge 0$. The following inequality holds true for any $n \ge 3$, $p > n-1$ and $k \ge 3 + \frac{n}{2} - \frac{n}{p}$*

$$\|\partial\phi(t)\|_{L^\infty} \le c(1+t)^{-\frac{n-1}{p}} \mathcal{E}_k(t).$$

In view of the conservation of the integrals $\int_{\Sigma_t} \bar{Q}(K_0, T_0)$ as well as[5] $\int_{\Sigma_t} \bar{Q}(T_0, T_0)$ applied to the standard derivatives $\partial_t, \partial_1, \ldots \partial_n$ of solutions to $\Box\,\phi = 0$, as well as part iii of 0.2 we obtain the following:

Proposition 0.5 *Let $\Box\,\phi = 0$ subject to the initial conditions $\phi(0) = f, \partial_t\phi(0) = g$ with f, g smooth and compactly supported in the ball $|x| \le 2$. Then, for all $t \ge 0$, $k > 3 + \frac{n}{2} - \frac{n}{n-1}$*

$$\|\partial\phi(t)\|_{L^\infty} \le C(1+t)^{-1+\epsilon}\Big(\|f\|_{H^k} + \|g\|_{H^{k-1}} \Big). \tag{0.23}$$

0.2 Main Theorems and their reduction to Dispersive Inequalities

In what follows we state the main results see[Kl4], Theorems A–C. These are not new[6] they are due in fact to the combined pioneering efforts of H. Smith [S1], [S2], Bahouri-Chemin [B-C1],[B-C2] and D.Tataru[7]. The method of proof, however, is very different. Instead of constructing parametrices we rely on a variation of the vectorfield approach presented in the previous section. The crucial new ingredients are the construction of a modified Morawetz vectorfield and generalized conformal energy estimates.

---

[5] This is needed to control small $t \ge 0$.

[6] Using a variation of the approach described here, Rodniansky and I(see [Kl-R]) were recently able to improve the result of Theorem C from $\sigma > \frac{1}{6}$, due to Tataru(see [T2]), to $\sigma > \frac{2-\sqrt{3}}{2}$ for $n = 3$.

[7] The precise statements of Theorem A and B and the optimal result of Theorem C is due to Tataru, see [T2]. His results connected to Theorems A, B are however more general.

Theorem A *Consider the wave operator $\Box'_h = -\partial_t^2 + h^{ij}\partial_i\partial_j$ defined in a space-time slab $\mathcal{D}_T = [0,T] \times \mathbf{R}^n$, $n \geq 3$. Assume that the coefficients $h = (h^{ij})_{i,j=1}^n$ verify the following assumptions:*

A1 *For all $(t,x) \in \mathcal{D}_T$, $\xi \in \mathbf{R}^n$,*

$$C^{-1}|\xi|^2 \leq h^{ij}(t,x)\xi_i\xi_j \leq C|\xi|^2$$

A2 *For all $0 \leq i \leq k$, and some fixed constant $\mu \geq 1$,*

$$T^i\|\partial^{1+i}h\|_{L_t^1 L_x^\infty(\mathcal{D}_T)} \leq c\mu^{2i}$$

Then,

$$\|\partial\phi\|_{L_t^2 L_x^\infty(\mathcal{D}_T)} \leq CT^\epsilon\left(\mu^{\frac{k}{k+1}}\|\partial\phi\|_{L_t^\infty \dot{H}^s(\mathcal{D}_T)} + \mu^{-\frac{k}{k+1}}\|\Box'_h\phi\|_{L_t^1 \dot{H}^s(\mathcal{D}_T)}\right) \tag{0.24}$$

for any $s = \frac{n-1}{2} + \epsilon$.

Theorem B *Assume that Theorem A holds for a fixed $k \geq 1$. Consider a metric h which verifies only the assumptions A1 and A2 for $k = 0$, $\mu = 1$;*

$$\|\partial h\|_{L_t^1 L_x^\infty(\mathcal{D}_T)} \leq c.$$

Then,

$$\|\partial\phi\|_{L_t^2 L_x^\infty(\mathcal{D}_T)} \leq C\left(T^\sigma\||D|^\sigma\partial\phi\|_{L^\infty \mathcal{H}^s(\mathcal{D}_T)} + T^{-\sigma}\||D|^{-\sigma}\Box'_h\phi\|_{L^1 \mathcal{H}^s(\mathcal{D}_T)}\right) \tag{0.25}$$

for any $s = \frac{n-1}{2}$ and $\sigma > \frac{k}{2(2k+1)}$.

Here and throughout the paper whenever we write $\|\psi\|_{L_t^q B}$, or simply $\|\psi\|_{L^q B}$, we mean $\left(\int \|\psi(t)\|_B^q dt\right)^{\frac{1}{q}}$ with B a Banach norm with respect to the space variables $x = (x^1, \ldots, x^n)$. Theorem B has an immediate application to quasilinear equations of the form,

$$\Box'_{h(\phi)}\phi = N(\phi, \partial\phi) \tag{0.26}$$

subject to the initial conditions at $t = 0$,

$$\phi(0) = \varphi_0 \quad \partial_t\phi(0) = \varphi_1 \tag{0.27}$$

Here $\Box'_{h(\phi)}\phi = -\partial_t^2\phi + h^{ij}(\phi)\partial_i\partial_j\phi$. Assume that $h(\phi) = (h^{ij}(\phi))_{i,j=1}^n$ is a smooth matrix valued function of ϕ. Assume also that N is a smooth function of $\phi, \partial\phi$ and depending quadratically on $\partial\phi$.

Theorem C *Assume that Theorem A is valid for some fixed $1 \leq k$ and $\mu = 1$. Consider the initial value problem 0.27 for the quasilinear wave equation 0.26 in $\mathbf{R}^{n+1}$, $n \geq 3$ Assume that the coefficients $h_{ij}(\phi)$ verify;*

$$c^{-1}|\xi|^2 \leq h^{ij}(\phi)\xi_i\xi_j \leq c|\xi|^2 \tag{0.28}$$

uniformly for $|\phi| \leq M$ and $\xi \in \mathbf{R}^n$.

Assume also that the initial data in 0.27 verify the assumptions $(\varphi_0, \varphi_1) \in H^s \times H^{s-1}$ *with* $s = \frac{n}{2} + \frac{1}{2} + \sigma$ *for* $\sigma > \frac{k}{2(2k+1)}$. *Moreover assume that* $\|\varphi_0\|_{L^\infty(\mathcal{D}_T)} \leq \frac{M}{2}$. *Then there exists a time* $T > 0$ *and a unique solution of 0.26, 0.27 verifying,*

$$
\begin{aligned}
\phi &\in L^\infty([0,T]; H^s) \cap Lip([0,T]; H^{s-1}) \\
\partial\phi &\in L^2([0,T]; L^\infty)
\end{aligned}
$$

and $\|\phi\|_{L^\infty(\mathcal{D}_T)} \leq M$.

Remark 1 *A more general sharper form of Theorem A, for all wave admissible Strichartz exponents, has been proved by Tataru in [T2] for $k = 1$. In particular, the optimal known result in conection to Theorem C is $\sigma = \frac{1}{6}$. The first improved regularity result is due to Chemin-Bahouri [B-C1]. They later improved the result in [B-C2].*

Proof of Theorem B from Theorem A:

Step 1. It clearly suffices to prove Theorem B for $T = 1$. Moreover we can reduce the proof to the following dyadic case[8],

Set $h_\lambda \approx S_{\frac{\lambda}{16}}h$ and $\phi_\lambda = \Delta_\lambda \phi$ with Δ_λ the standard frequency cut-off operator corresponding to the space-time Fourier region $\frac{1}{2}\lambda \leq |\tau| + |\xi| \leq 2\lambda$. Set $\Box'_\lambda = \Box'_{h_\lambda}$. Then, for all λ suficiently large, say $\lambda \geq 2^8$, it suffices to prove that

$$
\|\partial\phi_\lambda\|_{L_t^2 L_x^\infty(\mathcal{D})} \leq C\lambda^s \left(\lambda^\sigma \|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})} + \lambda^{-\sigma}\|\Box'_\lambda \phi_\lambda\|_{L_t^1 L_x^2(\mathcal{D})}\right) \tag{0.29}
$$

with $\mathcal{D} = \mathcal{D}_1$.

Step 2. Split $\Box'_\lambda \phi_\lambda = \Box'_{\lambda^a}\phi_\lambda + R_\lambda$, for some $0 \leq a \leq 1$ to be chosen later. Here $\Box'_{\lambda^a}\phi_\lambda = -\partial_t^2 + h_{\lambda^a}^{ij}\partial_i\partial_j$ with $h_\lambda^a = S_{\frac{1}{16}\lambda^a}h$. Observe that

$$
\|R_\lambda\|_{L_t^1 L_x^2(\mathcal{D})} \leq \lambda^{1-a}\|\partial h\|_{L_t^1 L_x^\infty(\mathcal{D})}\|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})}
$$

Therefore the estimate 0.29 follows from the following,

$$
\|\partial\phi_\lambda\|_{L_t^2 L_x^\infty(\mathcal{D})} \leq C\lambda^s \left(\lambda^\sigma \|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})} + \lambda^{-\sigma}\|\Box'_{\lambda^a}\phi\|_{L_t^1 L_x^2(\mathcal{D})}\right) \tag{0.30}
$$

provided that $\sigma = \frac{1-a}{2}$. On the other hand it is easy to see that the metric $h_{\lambda^a} = S_{\frac{1}{16}\lambda^a}h$ verifies the conditions, A1, A2 of Theorem A. More precisely,

$$
\|\partial^{1+i}h_{\lambda^a}\|_{L_t^1 L_x^\infty(\mathcal{D})} \leq \lambda^{ai}\|\partial h\|_{L_t^1 L_x^\infty(\mathcal{D})} \leq \mu^{2i}.
$$

with $\mu = \lambda^{\frac{a}{2}}$.

Finally 0.30 follows from Theorem A applied to the metric h_λ^a for $\mu = \lambda^{\frac{a}{2}}$ and a chosen such that $\frac{a}{2}\frac{k}{k+1} = \frac{1-a}{2}$, i.e. $a = \frac{k+1}{2k+1}$. Therefore $\sigma = \frac{1-a}{2} = \frac{k}{2(2k+1)}$ as desired.

Step 3. It suffices to prove Theorem A for the special case $\mu = 1$. This is essentially the proof of Tataru in [T2]

---

[8]The reduction is standard, see [B-C1], [B-C2] and [T1], [T2].

Sketch of the proof of Theorem A for $\mu = 1$.

Step 1. Proceeding precisely as the proof of Theorem B it suffices to prove the dyadic version of the Strichartz estimate of Theorem A($\mu = 1$) for $T = 1$ and λ sufficiently large.

$$\|\partial\phi_\lambda\|_{L_t^2 L_x^\infty(\mathcal{D})} \leq C\lambda^s \left(\|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})} + \| \Box'_\lambda \phi_\lambda\|_{L_t^1 L_x^2(\mathcal{D})} \right) \tag{0.31}$$

with $s = \frac{n-1}{2} + \epsilon$ and $\mathcal{D} = \mathcal{D}_1$. We have, roughly , $h_\lambda = S_{\frac{\lambda}{16}} h$, $\phi_\lambda = \Delta_\lambda \phi$ and $\Box'_\lambda = \Box'_{h_\lambda}$.

Step 2. Proceeding as in step 2 of the proof of Theorem B it suffices to prove the estimate

$$\|\partial\phi_\lambda\|_{L_t^2 L_x^\infty(\mathcal{D})} \leq C\lambda^s \left(\|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})} + \| \Box'_\lambda \phi_\lambda\|_{L_t^1 L_x^2(\mathcal{D})} \right) \tag{0.32}$$

for a metric $h = h^{ij}$ verifying the assumptions A1, A2 in the region $\mathcal{D} = [0,1] \times \mathbf{R}^n$ and whose space-time Fourier transform is supported in the region

$$0 \leq |\tau| + |\xi| \leq \frac{1}{16}\sqrt{\lambda}. \tag{0.33}$$

Step 3. Let $h = h^{ij}$ verify A1, A2 as well as 0.33. Define $\tilde{h}_\lambda(t,x) = h(\frac{t}{\lambda}, \frac{x}{\lambda})$. Clearly,

$$c^{-1}|\xi|^2 \leq \tilde{h}_\lambda^{ij}\xi_i\xi_j \quad \leq \quad c|\xi|^2. \tag{0.34}$$

$$\|\partial^{1+i}\tilde{h}_\lambda\|_{L_t^1 L_x^\infty(\mathcal{D}_\lambda)} \quad \leq \quad C\lambda^{-i} \quad \text{for all } 0 \leq i \leq k \tag{0.35}$$

$$\|\partial^{1+k+j}\tilde{h}_\lambda\|_{L_t^1 L_x^\infty(\mathcal{D}_\lambda)} \quad \leq \quad C\lambda^{-k-\frac{j}{2}} \quad \text{for all } 0 \leq j \tag{0.36}$$

Moreover the space-time Fourier transform of $\tilde{h}_\lambda$ is supported in the region $0 \leq |\tau| + |\xi| \leq \frac{1}{16\sqrt{\lambda}}$. Under these conditions it suffices to prove the Strichartz inequality,

$$\|\partial\psi\|_{L_t^2 L_x^\infty(\mathcal{D}_\lambda)} \leq C\lambda^\epsilon \left(\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_\lambda)} + \| \Box'_{\tilde{h}_\lambda}\psi\|_{L^1 L^2(\mathcal{D}_\lambda)} \right) \tag{0.37}$$

for all ψ whose space-time Fourier transform is supported in the region $0 \leq |\tau| + |\xi| \leq 2$.

Step 4. Define the positive definite metric $g = g_\lambda$ to be the inverse of the matrix $\tilde{h}_\lambda$. Whenever there is no danger of confusion we shall also denote by $g = g_\lambda$ the Lorentzian metric with g_{ij}, $i, j = 1 \ldots n$ as above and $g_{00} = -1, g_{0i} = 0$ for $i = 1 \ldots n$. Denote $|g| = \det(g_{ij})$. Let $\Box_g$ be the associated wave operator,

$$\Box_g = \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}\, g^{\alpha\beta}\partial_\beta) = -\frac{1}{\sqrt{|g|}}\partial_t(\sqrt{|g|}\partial_t) + \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\, g^{ij}\partial_j).$$

Observe that

$$\Box_{g_\lambda}\psi = \Box'_{\tilde{h}_\lambda} + R_\lambda\psi$$

$$R_\lambda\psi = -\frac{\partial_t(\sqrt{|g_\lambda|})}{\sqrt{|g_\lambda|}}\partial_t\phi + \frac{\partial_i(\sqrt{|g_\lambda|}\, g_\lambda^{ij})}{\sqrt{|g_\lambda|}}\partial_j\phi$$

Clearly, $\|R_\lambda \psi\|_{L^1 L^\infty(\mathcal{D}_\lambda)} \le C\|\partial \tilde{h}_\lambda\|_{L^1 L^\infty(\mathcal{D}_\lambda)}\|\partial \psi\|_{L^\infty L^2}$. Therefore 0.37 follows easily from,

$$\|\partial\psi\|_{L^q_t L^\infty_x(\mathcal{D}\lambda)} \le C\left(\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_\lambda)} + \|\Box_{g_\lambda}\psi\|_{L^1 L^2(\mathcal{D}_\lambda)}\right) \tag{0.38}$$

Observe also that the assumptions 0.34–0.36 for $\tilde{h}_\lambda$ remain satisfied for for the inverse metric $g_\lambda = (\tilde{h}_\lambda)^{-1}$, with different constants. This is obvious for 0.34. The others follow multiple applications of the chain rule and a Gagliardo-Nirenberg type inequality:

The proof of Theorem $A(\mu = 1)$ can be thus reduced to the following:

Theorem 0.2 *Let g_λ be a familiy of smooth metrics, $\lambda \ge \lambda_0 > 1$, defined in the region $\mathcal{D}_\lambda = I_\lambda \times \mathbf{R}^n$ with I_λ a time interval of length λ, in which the following assumptions are satisfied,*

$$c^{-1}|\xi|^2 \le g^\lambda_{ij}\xi^i\xi^j \le c|\xi|^2. \tag{0.39}$$

uniformly for all $(x,t) \in \mathcal{D}_\lambda$, $\xi \in \mathbf{R}^n$ and $\lambda \ge \lambda_0$.
   Also,

$$\|\partial^{1+i} g_\lambda\|_{L^1_t L^\infty_x(\mathcal{D}_\lambda)} \quad \le \quad C_i \lambda^{-i} \quad \text{for all } 0 \le i \le k \tag{0.40}$$

$$\|\partial^{1+k+j} g_\lambda\|_{L^1_t L^\infty_x(\mathcal{D}_\lambda)} \quad \le \quad C_{k,j}\lambda^{-k-\frac{j}{2}} \quad \text{for all } 0 \le j \tag{0.41}$$

Under these assumptions we have,

$$\|\partial\psi\|_{L^2_t L^\infty_x(\mathcal{D}\lambda)} \le C\left(\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_\lambda)} + \|\Box_{g_\lambda}\psi\|_{L^1 L^2(\mathcal{D}_\lambda)}\right) \tag{0.42}$$

for all ψ whose Fourier transform supported in the region $\frac{1}{2} \le |\tau| + |\xi| \le 2$.

The next important step is to prove that the Theorem 0.2 can be reduced to the following dispersive inequality:

Theorem 0.3 *Under the same assumptions on the metric g_λ as those of Theorem 0.2, if ϕ is a solution of the homogeneous equation $\Box_{g_\lambda}\phi = 0$, in the domain $\mathcal{D}_\lambda = I_\lambda \times \mathbf{R}^n$, $I_\lambda = [0, t_*]$, $|I_\lambda| \le \lambda$ with the Fourier transform of the data $\phi(t_0), \partial_t\phi(t_0)$ supported in $0 \le |\xi| \le 4$, then;*

$$\|\partial\phi(t)\|_{L^\infty} \le C(1 + |t - t_0|)^{-1+\epsilon}\|\partial\phi(t_0)\|_{L^1} \tag{0.43}$$

Theorem 0.3 implies the following

Theorem 0.4 *Consider the initial value problem*

$$\Box_{g_\lambda}\phi = 0$$
$$\phi(0) = \varphi_0 \quad \partial_t\phi(0) = \varphi_1$$

in the region $\mathcal{D}_\lambda = [0, \lambda] \times \mathbf{R}^n$, in which the assumptions of Theorem 0.2 hold true. Let $q = \frac{2}{1-\epsilon}$, q' the dual exponent. Let P be the operator defined by $P\phi(t,x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi}\chi(\xi)\hat{\phi}(t,\xi)$ with $\hat{\phi}$ the

space Fourier transform of ϕ and χ a compactly supported smooth function, $\chi(\xi) = 1$ for $|\xi| \leq 2$, $\chi(\xi) = 0$ for $|\xi| \geq 4$.

There exists a sufficiently large M , independent on λ, such that,

$$\|\partial P \phi\|_{L^q_t L^\infty_x (\mathcal{D}\lambda)} \leq M \|\partial\phi(0)\|_{L^2}. \tag{0.44}$$

Theorem 0.2 follows easily from the following Corrolary of Theorem 0.4. This is due to the fact that for the ψ of theorem 0.2 we have $P\psi = \psi$.

Corollary 0.5.1 *Under the same assumptions as above consider the inhomogeneous equation,*

$$\square_{g_\lambda} \psi = f$$

Then,

$$\|\partial P \psi\|_{L^q_t L^\infty_x (\mathcal{D}\lambda)} \leq M \Big(\|\partial\psi\|_{L^\infty L^2 (\mathcal{D}\lambda)} + \|f\|_{L^1 L^2 (\mathcal{D}\lambda)} \Big) \tag{0.45}$$

The proof of the Corollary is an immediate consequence of Theorem 0.4 and the standard form of the Duhamel principle.

The proof of Theorem 0.4 is based on the standard TT^*- argument. Yet, because we don't have a parametrix representation for our solutions significantly different from the corresponding results in [S1], [S2], [B-C1],[B-C2], [T1],[T2].

The proof of Theorem 0.2 reduces thus to that of Theorem 0.3. We can perform one more reduction based on the phase space localization described in the proof of Theorem 0.1. Therefore Theorem 0.3 is a consequence of the following:

Theorem 0.5 *Assume that the metric g_λ verifies the same assumptions as those of Theorem 0.2. Consider solutions of the homogeneous wave equation $\square_{g_\lambda}\phi = 0$ in the domain $\mathcal{D}_\lambda$, with initial data $\phi(t_0), \partial_t\phi(t_0)$, $t_0 \in I_\lambda$, supported in a ball of radius 2. Then, for a sufficiently large positive integer N,*

$$\|\partial\phi(t)\|_{L^\infty} \leq C(1 + |t - t_0|)^{-1+\epsilon} \|\partial\phi(t_0)\|_{H^N(\mathbf{R}^n)} \tag{0.46}$$

The hard part of our procedure reduces thus to the proof of the Theorem 0.5. This is done by constructing a vectorfield analogous to K_0 and using it to derive weighted energy estimates. We ouline below the main steps. We start by discussing our main energy estimate.

Let X be an arbitrary timelike vectorfield with deformation tensor ${}^{(X)}\pi = \mathcal{L}_X g$, We write $\pi = {}^{(X)}\pi$ in the form $\tilde{\pi} = \pi - \Omega g$ with Ω a given scalar function. Let $Q_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}(g^{\mu\nu}\partial_\mu\phi\partial_\nu)$ be the energy momentum tensor associate to $\square_g\phi = F$. If ϕ is a solution to the equation we have

$$D^\beta Q_{\alpha\beta} = F\partial_\alpha\phi.$$

Therefore, setting the $X-$ momentum 1-form $P_\alpha = Q_{\alpha\beta}X^\beta$, we have

$$
\begin{aligned}
D^\alpha P_\alpha &= Q^{\alpha\beta}\pi_{\alpha\beta} + FX(\phi) \\
&= \frac{1}{2}(Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta} + \Omega\mathrm{tr}Q) + FX(\phi) \\
&= \frac{1}{2}(Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta} + \Omega\frac{1-n}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) + FX(\phi)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\Omega g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi &= D^\mu(\Omega\phi\partial_\mu\phi) - \partial^\mu(\Omega)\phi\partial_\mu\phi - \Omega\phi\,\Box_g\phi \\
&= D^\mu(\Omega\phi\partial_\mu\phi - \frac{1}{2}\phi^2\partial_\mu\Omega) + \frac{1}{2}\phi^2\,\Box_g\Omega - \Omega\phi F
\end{aligned}
$$

Therefore,

$$
D^\alpha P_\alpha = \frac{1}{2}Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta} - \frac{n-1}{4}\left(D^\mu(\Omega\phi\partial_\mu\phi - \frac{1}{2}\phi^2\partial_\mu\Omega) + \frac{1}{2}\phi^2\,\Box_g\Omega\right) + (X\phi + \frac{n-1}{4}\Omega\phi)F
$$

or, setting

$$
\bar{P}_\alpha = P_\alpha + \frac{n-1}{4}\Omega\phi\partial_\alpha\phi - \frac{n-1}{8}\phi^2\partial_\alpha\Omega \tag{0.47}
$$

we derive

$$
D^\alpha \bar{P}_\alpha = \frac{1}{2}Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta} - \frac{n-1}{8}\phi^2\,\Box_g\Omega + (X\phi + \frac{n-1}{4}\Omega\phi)F \tag{0.48}
$$

Now, integrating on the time slab $[t_0, t] \times \mathbf{R}^n$, and observing that ∂_t is the future unit normal to the hypersurfaces Σ_t we derive,

Proposition 0.6 *Let ϕ verify $\Box\phi = F$ and X an arbitrary vectorfield with deformation tensor $^{(X)}\pi = \pi$. Let Ω be an arbitrary scalar function and $\tilde{\pi} = \pi - \Omega g$. Define, for another vectorfield Y,*

$$
\bar{Q}(X,Y) = Q(X,Y) + \frac{n-1}{4}\Omega\phi Y\phi - \frac{n-1}{8}\phi^2 Y(\Omega)
$$

We have,

$$
\begin{aligned}
\int_{\Sigma_t} \bar{Q}(X,\partial_t)dv_g &= \int_{\Sigma_{t_0}} \bar{Q}(X,\partial_t)dv_g + \frac{1}{2}\int_{[t_0,t]\times\mathbf{R}^n} Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta}dtdv_g \tag{0.49} \\
&\quad - \frac{n-1}{8}\int_{[t_0,t]\times\mathbf{R}^n}\phi^2\,\Box_g\Omega dtdv_g + \int_{[t_0,t]\times\mathbf{R}^n}(X\phi + \frac{n-1}{4}\Omega\phi)Fdtdv_g
\end{aligned}
$$

Observe that 0.49 implies the energy identity 0.15 in the particular case of the Minkowski space, $X = K_0$ and $F = 0$. In that case $\Omega = \frac{1}{n+1}\mathrm{tr}\pi = 4t$, $\mathrm{tr}\pi = g^{\alpha\beta}\pi_{\alpha\beta}$ and $\tilde{\pi} = \hat{\pi} = 0$. In a curved background however π is not zero. In order to control the error terms involving π we need a vectorfield X which is as close as possible to the vectorfield $K_0 = (t^2 + r^2)\partial_t + \sum_i 2tx^i\partial_i$ in flat space. We do this with the help of a special solution u of the Eikonal equation

$$
(\partial_t u)^2 - g^{ij}(t,x)\partial_i u\partial_j u = 0 \tag{0.50}
$$

X–14

whose level hypersurfaces are forward light cones, C_u, with vertices on the time axis G given by the points of coordinates $(t, 0)$. The optical function u can be viewed as the analogue of the function $t - |x|$ in Minkowski space. It corresponds to the interior optical function introduced in section 9.2 of [C-K2]. We also define the null outgoing vectorfield $L = -g^{\mu\nu}\partial_\nu u\partial_\mu = \partial_t u\partial_t - (g^{ij}\partial_i u)\partial_j$, analogous to $\partial_t + \partial_r$, and with the help of L and ∂_t, the null incoming vectorfield $\underline{L}$, analogous to $\partial_t - \partial_r$. Finally we can set $\underline{u} = 2t - u$ and define

$$K_0 = \frac{1}{2}\left(\underline{u}^2 L + u^2 \underline{L}\right).$$

References

[B-C1] H. Bahouri and J.Y.Chemin *"Equations d'ondes quasilineaires et effect dispersif"* American Journal of Mathematics 121(1999), 1337–1377.

[B-C2] H. Bahouri and J.Y.Chemin *"Equations d'ondes quasilineaires et estimations de Strichartz"* International Mathematics Research Notices 21(1999) 1141–1177.

[C-K1] D.Christodoulou, S.Klainerman, *"Asymptotic properties of linear field equations in Minkowski space"*. Comm.Pure Appl.Math. XLIII,(1990), 137-199.

[C-K2] D.Christodoulou, S.Klainerman, *"The global non linear stability of the Minkowski space"*. Princeton Mathematical series, 41 (1993).

[Ho] L.Hormander, *"Lectures on Nonlinear Hyperbolic Equations* Mathematics and Applications 26, Springer-Verlag (1987).

[K-T] Keel-Tao, "Endpoint Strichartz estimates", Amer. J. Math. 120 (1998) 955-980.

[Kl1] S.Klainerman *"Uniform decay estimates and the Lorentz invariance of the classical wave equation"*. Comm.Pure.Appl.Math. 38, (1985), 321-332.

[Kl2] S.Klainerman, *"Remarks on the global Sobolev inequalities in Minkowski Space"*. Comm.Pure.Appl.Math. 40, (1987), 111-117.

[Kl3] S.Klainerman, *"The null condition and global existence to nonlinear wave equations"*. Lect. Appl. Math. 23, (1986), 293-326.

[Kl4] S.Klainerman, *"A Commuting Vectorfields Approach to Strichartz type Inequalities and Applications to Quasilinear Wave Equations"* preprint.

[Kl-R] S. Klainerman, I. Rodniansky *"Improved regularity results for quasilinear wave equations"* in preparation.

[Kl-Si] S.Klainerman, T. Sideris *"On Almost Global Existence for Nonrelativistic Wave Equations in 3D"* Comm.Pure.Appl.Math. 49 1996, 307-321

[L] O. Liess *"Decay Estimates in Crystal Optics"* Assymptotic Analysis 4(1991), 61-95.

[M]   C. Morawetz   "The Limiting Amplitude Principle" Comm.Pure.Appl.Math. 15, 1962, 349-362.

[S1]   H. Smith   *"A parametrix construction for wave equations with $C^{1,1}$ coefficients"* Ann Inst Fourier de Grenoble 48(3) 1994, 797-835.

[S2]   H. Smith   *" Strichartz and Nullform Estimates for Metrics of Bounded Curvature"* preprint.

[T1]   D. Tataru *"Strichartz Estimates for operators with nonsmooth coefficients and the nonlinear wave equation"* To appear in AJM

[T2]   D. Tataru   *'Strichartz Estimates for second order hyperbolic operators with nonsmooth coefficients III"* preprint