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Resonances for metric (and other) bottles.

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1 Introduction

In recent years, several results describing upper bounds on the number of resonances have been obtained and in general they are expected to be sharp with respect to the order of magnitude. There has also been considerable progress concerning existence of resonances and corresponding lower bounds, but it is more rare to get the right order of magnitude and even more rare to get close to something like Weyl asymptotics. The methods used so far are either based on straight forward constructions in simple cases, quasi-mode constructions, or some kind of a trace formula. Arguments from scattering theory have also been successfully used.

In this talk, we describe the results of [5], where some of the earlier results are improved and extended, and we get close to Weyl asymptotics for the number of resonances close to the real axis. More precisely we consider fairly massive (possibly long range) perturbations of the Laplacian. The proof involves trace formula techniques (see [4, 5]) to get a reduction to situations reminiscent of that of a potential well in an island, and an adaptation of an argument of G.Vodev [11], to get estimates beyond the real axis.

2 Statement of the results

We work in the black box framework introduced by Zworski and the author in [6], with long range perturbations ([4, 5]). Let

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus_{\perp} L^2(\mathbf{R}^n \setminus B(0, R_0)) \tag{2.1}$$

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be a complex Hilbert space equal to the orthogonal sum of an abstract part $\mathcal{H}_{R_0}$ and L^2 over the complement in $\mathbf{R}^n$ of the open ball of radius R_0 centered at 0. The orthogonal projections will be denoted by $u \mapsto 1_{\mathbf{R}^n\setminus B(0,R_0)}u$, $u \mapsto 1_{B(0,R_0)}u$ and sometimes by the corresponding symbols of restriction.

Let $P : \mathcal{H} \to \mathcal{H}$ be an unbounded self-adjoint operator with domain $\mathcal{D}$, and assume that

$$1_{\mathbf{R}^n\setminus B(0,R_0)}\mathcal{D} = H^2(\mathbf{R}^n \setminus B(0,R_0)) \tag{2.2}$$

$$\text{If } u \in H^2(\mathbf{R}^n) \text{ vanishes near } \overline{B(0,R_0)} \text{ then } u \in \mathcal{D}. \tag{2.3}$$

Here we also consider u as an element of $\mathcal{H}$ in the natural way (with $\mathcal{H}_{R_0}$ component equal to zero).

Outside $B(0,R_0)$ we assume that P coincides with a differential operator Q, in the sense that $(Pu)_{|\mathbf{R}^n\setminus B(0,R_0)} = Q(u_{|\mathbf{R}^n\setminus B(0,R_0)})$, where

$$Q = \sum_{|\alpha|\le 2} a_\alpha(x)D^\alpha, \tag{2.4}$$

and a_α belong to the space $C_b^\infty(\mathbf{R}^n)$ of smooth functions that are bounded together with all their derivatives. We assume that Q is elliptic in the sense that

$$p(x,\xi) := \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \ge \frac{1}{C}|\xi|^2. \tag{2.5}$$

We also assume that

$$P \to -\Delta,\ x \to \infty, \tag{2.6}$$

where $\Delta = \sum \frac{\partial^2}{\partial x_j^2}$, in the sense that the coefficients of P converge to those of $-\Delta$. Let M be a large torus of dimension n so that a neighborhood of $\overline{B(0,R_0)}$ can be viewed as a subset of M. As in [6, 4, 5] we define a reference operator $P^\sharp : \mathcal{H}^\sharp \to \mathcal{H}^\sharp$, where $\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0,R_0))$, such that $P^\sharp$ is equal to P near $\overline{B(0,R_0)}$ in the natural way, and is equal to a positive second order elliptic operator outside this ball. As in the quoted papers, we then know that $P^\sharp$ has a purely discrete spectrum and we let

$$N(P^\sharp, [-\lambda,\lambda]) = \#(\sigma(P^\sharp) \cap [-\lambda,\lambda])$$

denote the number of eigenvalues in the interval $[-\lambda,\lambda]$. Assume

$$N(P^\sharp, [-\lambda,\lambda]) = \mathcal{O}(1)\Phi(\lambda),\ \lambda \ge \mathcal{O}_{bb}(1),\ \Phi = \Phi_{bb}, \tag{2.7}$$

$$\forall C \ge 1,\ \exists \widetilde{C} \ge 1,\ \text{such that } \Phi(Ct) \le \widetilde{C}\Phi(t). \tag{2.8}$$

Here we let P and possibly the black box part $\mathcal{H}_{R_0}$ depend on some additional parameter, while R_0 and the exterior part Q are fixed. Dependence with respect to this additional parameter is indicated by the subscript bb, and estimates and constants are uniform in this parameter when this subscript is absent.

Finally we assume that the a_α extend holomorphically to the set

$$\{r\omega;\ \omega \in \mathbf{C}^n,\ \mathrm{dist}(\omega, S^{n-1}) < \epsilon,\ 0 < \arg r < \theta_0,\ r > R_1\},$$

for some $\epsilon > 0$, $R_1 > R_0$, and that $P \to -\Delta$ when x tends to infinity in this larger set. Here we also assume that

$$\frac{\pi}{n} < \theta_0 < \frac{\pi}{2} \tag{2.9}$$

so that the dimension n is ≥ 3. Under these assumptions we have the following two results, out of which the first one can be viewed as an extension of a result of Vodev [11]. Recall from [6, 4] that the set of resonances of h^2P is a well-defined discrete subset $\mathrm{Res}\,(h^2P)$ of $e^{i]-2\theta_0,0]}]0,\infty[$. Each resonance has a natural multiplicity which is a natural number ≥ 1, and we will count the resonances with their multiplicity.

Theorem 2.1 *If* $K \subset\subset e^{i]-2\theta_0,0[}]0,\infty[$ *is independent of* h*, then* $\sharp(\mathrm{Res}\,(h^2P) \cap K) = \mathcal{O}_K(h^{-n})$, $0 < h \leq h_{bb}$.

Theorem 2.2 *Let* $I \subset\subset J \subset\subset K \subset\subset]0,\infty[$ *be open intervals, and let* $c > 0$ *be small enough depending on* J *and* θ_0*. Then for every* $\epsilon > 0$,

$$\#(\mathrm{Res}\,(h^2P) \cap (J + i] - c, 0])) \begin{cases} \geq \#(\sigma(h^2P^\sharp) \cap I) - \epsilon\Phi(\frac{1}{h^2}) - \mathcal{O}_{I,J,\epsilon,c}(1)h^{-n} \\ \leq \#(\sigma(h^2P^\sharp) \cap K) + \epsilon\Phi(\frac{1}{h^2}) + \mathcal{O}_{I,J,\epsilon,c}(1)h^{-n} \end{cases}$$

Notice that the last result is false in dimension 1. Indeed, we can view $-\Delta$ on $\mathbf{R}$ in such a way that this operator on an arbitrarily long interval $[-R, R]$ is the black box part with some fixed R_0 say $R_0 = 1$. In dimension 2, there is a chance that Theorem 2.2 remains valid, possibly after adding some assumption about the negative spectrum of P.

These results also have a non-semi-classical formulation. It is then practical to consider $\sqrt{\mathrm{Res}\,(P)}$, the set of square roots of resonances, i.e. the poles of the meromorphic extension of $(P - \lambda^2)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$ from the first quadrant across the positive real axis. Theorem 2.1 implies

$$\#(\sqrt{\mathrm{Res}\,(P)} \cap e^{i]-\theta_0+\epsilon,-\epsilon[}]1, r[) \leq \mathcal{O}_\epsilon(1)r^n, \tag{2.10}$$

uniformly with respect to the black box part, for $r > r_{bb} \geq 1$.

To have the corresponding formulation of Theorem 2.2, we assume

$$\forall C \geq 1,\ \exists \widehat{C}(C) > 1,\ t(C) \geq 1,\ \text{such that } \Phi(Ct) \geq \widehat{C}(C)\Phi(t),\ t \geq t(C). \tag{2.11}$$

Again this assumption is uniform with respect to the black box part. With this additional assumption we can get from Theorem 2.2 that for all $\epsilon > 0$, $\theta_1 \in]0, \theta_0[$, there exists $r_{bb} > 0$ such that:

$$\begin{aligned} &\#(\sqrt{\mathrm{Res}\,(P)} \quad \cap e^{i]-\theta_1,0]}]1, r[) \\ &\qquad \begin{cases} \geq \#(\sigma(P^\sharp) \cap]1, (\frac{r}{1+\epsilon})^2[) - \epsilon\Phi(r^2) - \mathcal{O}_{\epsilon,\theta_1}(r^n) \\ \leq \#(\sigma(P^\sharp) \cap]1, ((1+\epsilon)r)^2[) + \epsilon\Phi(r^2) + \mathcal{O}_{\epsilon,\theta_1}(r^n), \end{cases} \end{aligned} \tag{2.12}$$

for $r \geq r_{bb}$, where the estimates are uniform in the black box part. In the case when $\mathcal{H} = L^2(\mathbf{R}^n)$ and P is an elliptic 2nd order differential operator with principal symbol $p(x, \xi)$, the estimates (2.12) tend to Weyl asymptotics when $\int\int_{|x| \leq R_0,\, p(x,\xi) \leq 1} dx d\xi$ tends to infinity.

In [7] we studied the Laplace-Beltrami operator for a metric bottle, obtained by connecting a sphere of large radius to $\mathbf{R}^n$ in such a way that we get the standard Euclidean metric outside the ball $B(0, 1)$, and we got a lower bound on the number of resonances near $\mathbf{R}_+$ of the right order of magnitude. Here and in subsequent improved and generalized results by G. Popov [2], and V. Petkov, M. Zworski [3], the existence of many periodic trajectories with the same period was exploited together with a trace formula or arguments from scattering theory. Periodic trajectories can also be used together with quasi-mode constructions to give lots of resonances near the real axis, see P. Stefanov [8] and earlier works by Stefanov–Vodev [9] and Tang–Zworski [10]. Our results above are of a different nature and they do not use the existence of closed or even trapped classical trajectories. It is not difficult to construct examples of metric perturbations, for which (2.12) gives close to Weyl asymptotics, and for which no classical trapped trajectories exist. Our results also improve existing ones (see [12]) for degenerate operators.

3 Outline of the proof

As in [6], [4, 5], we view the resonances in $e^{i]-2\theta_0,0]}]1, \infty[$ as the eigenvalues in the same sector of

$$P_{\theta_0} = P_{|_{\Gamma_{\theta_0}}}, \tag{3.1}$$

where $\Gamma_{\theta_0} \subset \mathbf{C}^n$ is a suitable contour which coincides with $\mathbf{R}^n$ in $B(0, R_1)$, for some $R_1 > R_0$ and with $e^{i\theta_0}\mathbf{R}^n$ near infinity. This can be done in such a way that the principal symbol p_{θ_0} of P_{θ_0} satisfies

$$|\arg(p_{\theta_0}(x,\xi))+2\theta(x)|<\epsilon_0, \tag{3.2}$$

where $\epsilon_0>0$ is any fixed number and $\theta(x)\in[0,\theta_0]$ depends continuously on x and is equal to 0 for $|x|<R_1$ and is equal to θ_0 for large $|x|$.

In proving a local trace formula for resonances ([4, 5]), we applied a perturbation of trace class norm $\mathcal{O}(\Phi(\frac{1}{h^2}))$ to $h^2P_{\theta_0}$ (or to a more general semiclassical operator) which had the effect of chasing all eigenvalues from some fixed neighborhood of some real positive energy E_0. Here, such a perturbation would be too large, and instead, we make a perturbation supported in a shell around the black box, which chases the eigenvalues from a region outside the real axis and creates an operator which behaves like a Schrödinger operator with a potential well in an island. (See [1].)

The perturbed operator has the form

$$\widetilde{P}(\mu)=P+L(\mu), \tag{3.3}$$

with $P=P_{\theta_0}$ from now on, where

$$L(\mu)=A\mu^2\Phi_1(x)\circ\mathrm{Op}\,(\Phi(x)e^{-2i\theta(x)}\chi(\frac{\xi}{\mu}))\circ\Phi_1(x). \tag{3.4}$$

Here $A>0$ is a sufficiently large fixed constant, $\mu\in\mathbf{C}$ a complex variable satisfying

$$\frac{|\lambda|}{|\mu|}\in[\frac{1}{2},2], \tag{3.5}$$

$$-\theta_0<\arg\lambda,\ \arg\mu<\theta_1, \tag{3.6}$$

where $\theta_1>0$ is small. χ is a suitable function in $\mathcal{S}$, such that $\xi\mapsto\chi(\frac{\xi}{\mu})$ is also in $\mathcal{S}$. $\Phi,\Phi_1\in C_0^\infty(\mathbf{R}^n\setminus\overline{B(0,R_0)})$ are cut-off functions, equal to 1 on a shell $|x|\in[R_1,R_2]$, where R_2 is so large that Γ_{θ_0} coincides with $e^{i\theta_0}\mathbf{R}^n$ for $|x|\geq R_2$. By Op, we denote the standard Weyl quantization of symbols.

Let $\|\ \|$, $\|\ \|_{\mathrm{tr}}$ denote the operator and the trace class norm for operators in $\mathcal{H}$, we have:

Proposition 3.1 *(a) We have* $\|L(\mu)\|=\mathcal{O}(\mu^2)$, $\|L(\mu)\|_{\mathrm{tr}}=\mathcal{O}(\mu^{2+n})$.
(b) If

$$\arg\lambda\notin\{0,-\theta_0\}+[0,\arg\mu]+[-\epsilon_1,\epsilon_1], \tag{3.7}$$

then $(\widetilde{P}-\lambda^2)^{-1}$ *exists and is* $\mathcal{O}(1)|\mu|^{-2}:\mathcal{H}\to\mathcal{H}$*, when* $|\mu|,|\lambda|\gg1$.

Here ϵ_1 can be chosen arbitrarily small, provided that Γ_{θ_0} is chosen conveniently depending on ϵ_1. (When $\arg\mu < 0$, we define $[0, \arg\mu]$ to be $[\arg\mu, 0]$.)

Let first $\mu \gg 1$ and put $\lambda = z/h$, with $z \in \mathbf{C}$, $|z| \sim 1$, $0 < h \ll 1$. Put

$$D(\lambda, \mu) = \det[(\lambda^2 - P)(\lambda^2 - \widetilde{P}(\mu))^{-1}]. \tag{3.8}$$

Then for $\arg z \in]\epsilon_1, \theta_1[$, we get

$$\log|D| \sim h^{-n}, \tag{3.9}$$

while for $\arg z \in] - \theta_0 + \epsilon_1, -\epsilon_1[$:

$$\log|D| \leq \mathcal{O}(1)h^{-n}. \tag{3.10}$$

One can also prove that the distribution of eigenvalues of $\widetilde{P}$ with argument in $[-\epsilon_1, \epsilon_1]$ and module in $[\frac{\mu}{2}, 2\mu]$, is close to the one for $P^\sharp$, and in doing so, one uses ideas from the study of Schrödinger operators with a potential well in an island.

In order to get further, we also need to verify that

$$\begin{aligned}&\exists z_0 = z_0 \text{ with } |z_0| \sim 1,\ \arg z_0 \in] - \theta_0 + \epsilon_1, -\epsilon_1[\text{ such that} \\ &(\text{for the corresponding } \lambda = \lambda_0)\ \log|D| \geq -\mathcal{O}(1)h^{-n}.\end{aligned} \tag{3.11}$$

Assuming (3.11) it is fairly straight forward to estimate the number of zeros of $z \mapsto D$ with $\arg z \in] - \theta_0 + \epsilon_1, -\epsilon_1[$, $|z| \sim 1$, and we get Theorem 2.1. One can also consider the contour integral

$$\frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}(\log D)dz \tag{3.12}$$

over a rectangle shaped contour close to the real axis, with a suitable gaussian function f and deduce that the eigenvalues of $P = P_{\theta_0}$ near the real axis are approximately distributed like those of $\widetilde{P}$. This leads to Theorem 2.2.

For the verification of (3.11), we have been inspired by a work of Vodev [11], who used a theorem of Carleman to obtain estimates as in Theorem 2.1. In our case we also need further control, and it turned out to be convenient to use more direct estimates on subharmonic and harmonic functions.

We fix a small $\epsilon_2 > 0$ and consider

$$D(\lambda) = \det((\lambda^2 - P)(\lambda^2 - \widetilde{P}(e^{i\epsilon_2/2}\lambda))^{-1}), \tag{3.13}$$

for $\epsilon_2 - \theta_0 < \arg\lambda < -\epsilon_2$, $1 \leq |\lambda| \leq r$, $r \gg 1$. Then $\log|D(\lambda)| \leq \mathcal{O}(1)|\lambda|^n$. Making the change of variables,

$$\lambda = \exp(\frac{\theta_0 - 2\epsilon_2}{\pi}w - i(\theta_0 - \epsilon_2)),$$

we get

$$\log|D| \leq e^{\tilde{n}\operatorname{Re} w},\ \tilde{n} = \frac{n(\theta_0 - 2\epsilon_2)}{\pi} > 1, \tag{3.14}$$

when

$$0 \leq \operatorname{Re} w \leq R,\ 0 \leq \operatorname{Im} w \leq \pi, \tag{3.15}$$

where $R \gg 0$. It is easy to see that there exists $w_1 = w_{1,bb}$, with $\frac{\pi}{4} < \operatorname{Im} w_1 < \frac{3\pi}{4}$, $\frac{1}{4} < \operatorname{Re} w_1 < \frac{3}{4}$, for which we have $\log|D| \geq e^{-(\tilde{n}-1)R}$, when R is large enough, $R \geq R_{bb}$. (It suffices to choose w_1 with $\log|D| \neq 0$ and then take R large enough.) Using estimates for the Green and Poisson kernels of the long rectangle (3.15), we can find w_2 with $R-\frac{3}{4} < \operatorname{Re} w_2 < R-\frac{1}{4}$, $\frac{\pi}{4} < \operatorname{Im} w_2 < \frac{3\pi}{4}$, for which $\log|D| \geq -\mathcal{O}(1)e^{-\tilde{n}R}$. For the corresponding $\lambda_2, \mu_2 = e^{i\epsilon_2/2}\lambda_2$, we then get $\log|D| \geq -\mathcal{O}(1)h^{-n}$ as in (3.11), and making similar but easier estimates with λ_2 fixed and μ variable, we find a corresponding *real* $\mu = \mu_3$ for which (3.11) holds with $\lambda_0 = \lambda_2$.

References

[1] B. Helffer, J. Sjöstrand, *Résonances en limite semiclassique,* Bull. de la SMF 114(3), Mémoire 24/25 (1986).

[2] G. Popov, *On the contribution of degenerate periodic trajectories to the wave-trace,* Comm. Math. Phys., 196(2)(1998), 363–383.

[3] V. Petkov, M. Zworski, *Breit-Wigner approximation and the distribution of resonances,* Comm. Math. Phys., 204(2)(1999), 329–351.

[4] J. Sjöstrand, *A trace formula and review of some estimates for resonances,* p. 377–437 in Microlocal analysis and spectral theory, NATO ASI Series C, vol. 490, Kluwer 1997.

[5] J. Sjöstrand, *Resonances for bottles and trace formulae,* Math. Nachr., to appear.

[6] J. Sjöstrand, M. Zworski, *Complex scaling and the distribution of scattering poles,* J. of AMS 4(4)(1992), 1021–1035

[7] J. Sjöstrand, M. Zworski, *Lower bounds on the number of scattering poles,* Comm. PDE 18(5–6)(1993), 847–857.

[8] P. Stefanov, *Quasimodes and resonances: sharp lower bounds,* Duke Math. J., 99(1)(1999), 75–92.

[9] P. Stefanov, G. Vodev, *Neumann resonances in linear elasticity for an arbitrary body,* Comm. Math. Phys. 176(1996), 645–659.

[10] S.U. Tang, M. Zworski, *From quasimodes to resonances,* Math. Res. Lett., 5(3)(1998), 261–272.

[11] G. Vodev, *On the distribution of scattering poles for perturbations of the Laplacian,* Ann. Inst. Fourier 40(1992), 625–635.

[12] G. Vodev, *Asymptotics on the number of scattering poles for degenerate perturbations of the Laplacian,* J. Funct. Anal., 138(2)(1996), 295–310.