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Global Strichartz estimates for variable coefficient second order hyperbolic operators

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1 Introduction

The Strichartz estimates for the wave equation are L^p estimates which contain information related to the dispersion phenomena. In the constant coefficient case such estimates are global and scale invariant. For operators with variable coefficients, however, similar estimates are only known to hold only locally. Nevertheless, it seems reasonable that the global Strichartz estimates should also hold provided that the coefficients are asymptotically flat at infinity and that some nontrapping condition holds. The aim of these notes is to describe some recent results in this direction. Our assumptions on the coefficients, as well as our results, are scale invariant. These results are likely not optimal, but we hope they are a good starting point for further investigations.

Denote by $(t,x) = (t, x_1, \cdots x_n)$ the coordinates in $\mathbb{R} \times \mathbb{R}^n$. The Strichartz estimates for solutions to the homogeneous wave equation in $\mathbb{R} \times \mathbb{R}^n$

$$\Box u = 0 \qquad u(0) = u_0 \quad u_t(0) = u_1$$

have the form

$$\| |D|^{1-\rho} u\|_{L^p(L^q)} \leq \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} \tag{1.1}$$

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Such an estimate holds for all pairs (ρ, p, q) satisfying the relations $2 \leq p \leq \infty$, $2 \leq q \leq \infty$ and

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \rho, \qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \tag{1.2}$$

with the exception of the forbidden endpoint $(1, 2, \infty)$ in dimension $n = 3$. All (ρ, p, q) satisfying these relations are called in the sequel Strichartz pairs. If the equality holds in the second part of (1.2) then the corresponding pair is called a sharp Strichartz pair.

A straightforward consequence of (1.1) is an estimate for solutions to the inhomogeneous problem

$$\Box u = f \qquad u(0) = 0 \quad u_t(0) = 0$$

namely

$$\||D|^{1-\rho} u\|_{L^p(L^q)} \leq \|f\|_{L^1(L^2)} \tag{1.3}$$

The simplest case of (1.3) is the well-known energy estimate

$$\|\nabla u\|_{L^\infty(L^2)} \leq \|f\|_{L^1(L^2)} \tag{1.4}$$

However, there is a larger family of estimates for solutions to the inhomogeneous wave equation where we also vary the norms in the right hand side,

$$\||D|^{1-\rho} u\|_{L^p(0,T;L^q)} \leq \||D|^{\rho_1} f\|_{L^{p_1'}(L^{q_1'})} \tag{1.5}$$

This holds for all Strichartz pairs (ρ, p, q), (ρ_1, p_1, q_1).

Estimates of the above type were first proved in the constant coefficient case in [1], [8]. Further references can be found in a more recent expository article [3]. The endpoint estimate $(p, q) = (2, \frac{2(n-1)}{n-3})$ was only recently obtained in [4] $(n \geq 4)$.

In this article we are interested in the variable coefficient case of these estimates, where we replace $\Box$ by a second order operator of the form

$$\Box_g = \partial_i g^{ij}(t, x) \partial_j$$

which is strongly hyperbolic with respect to time. Then we consider functions u which solve

$$\Box_g u = f \qquad u(0) = 0 \quad u_t(0) = 0$$

and we ask whether the estimates (1.1), (1.5) hold.

If the coefficients g^{ij} are smooth then the estimates hold locally, see [5] (except for the endpoint). For C^2 coefficients, in dimension $n = 2, 3$, the estimates are proved in [6]. On the other hand in [7] they are shown to fail for C^s coefficients, $s < 2$. In [10] we show that the full estimates hold locally in all dimensions for operators with C^2 coefficients. This result is improved in [9], where the assumption on g is relaxed to $\nabla^2 g^{ij} \in L^1(L^\infty)$.

In what follows we assume that the matrices $(g^{ij}(t,x))$, $(g^{ij}(t,x))^{-1}$ are uniformly bounded and of signature $(1, n)$. Furthermore, we also assume that the surfaces $x_0 = const$ are space-like uniformly in x, i.e. that $g^{00} > c > 0$. This assumptions are scale invariant. The assumption that $\nabla^2 g$ is in C^0 or in $L^1(L^\infty)$, however, is not, and is clearly insufficient to guarantee global Strichartz estimates. What we need to add to this is some decay at infinity for the derivatives of the coefficients.

If we are interested in coefficients whose derivatives decay in time then a scale invariant condition is:

$$t^2 |\partial^2 g(t,x)| \leq C$$

Such a condition is, however, insufficient to guarantee that the energy stays bounded. Instead we shall use the slightly stronger condition

$$\sum_{j \in \mathbb{Z}} \sup_{2^j \leq t \leq 2^{j+1}} |t| (|\partial^2 g(t,x)|^{\frac{1}{2}} + |\partial g(t,x)|) \leq \epsilon \tag{1.6}$$

Then our first result is

Theorem 1. *Assume that (1.6) is satisfied. Then the full Strichartz estimates hold.*

A second interesting case is that of spatially decaying coefficients. This includes operators with time independent coefficients. Then a simple scale invariant condition is

$$|x|^2 |\partial^2 g(t,x)| \leq C$$

This is again not sufficient even if the coefficients are time independent, since it does not prevent the existence of trapped rays, or equivalently, of approximate null eigenvalues for the corresponding elliptic operator. A slightly stronger assumption is

$$\sum_{j \in \mathbb{Z}} \sup_{A_j} |x| (|\partial^2 g(t,x)|^{\frac{1}{2}} + |\partial g(t,x)|) \leq \epsilon \tag{1.7}$$

where A_j is the dyadic cylinder

$$A_j = \{2^j \leq |x| \leq 2^{j+1}\}$$

If ϵ is small enough then this precludes the existence of trapped rays, while for arbitrary ϵ it restricts the trapped rays to finitely many dyadic regions. If the coefficients are allowed to depend on time then even prohibiting trapped rays is not sufficient for they might still stay around for an arbitrarily long time. Thus a reasonable assumption is

There exists $M > 0$ such that for each null bicharacteristic ray γ and each $r > 0$ we have $|\gamma \cap \{r \leq |x| \leq 2r\} \leq Mr$. (1.8)

Then one would hope that the following result holds, perhaps after some mild modifications:

Conjecture 2. *Assume that (1.7), (1.8) are satisfied. Then the full Strichartz estimates hold.*

Observe that under the assumptions (1.7), (1.8) not even the energy estimates are straightforward. The difficulty is that the energy can increase by (almost) a fixed factor in an arbitrarily small time so one needs to verify that there is enough scattering so that such energy increases cannot accumulate. The scattering information can be taken into account using L^2 norms in dyadic regions with respect to the spatial distance to the origin. Consequently we introduce an L^2 estimate encapsulating the scattering information, as an intermediate link in the process of obtaining Strichartz estimates.

Our motivation is as follows. Assume for a moment that we can get classical energy estimates for our problem,

$$\|\nabla u\|_{L^\infty(L^2)} \lesssim \|\Box_g u\|_{L^1(L^2)} + \|\nabla u(0)\|_{L^2}$$

If (1.7) holds then one can prove the Strichartz estimates uniformly in cylinders of the form

$$[jr, (j+1)r] \times \{r \leq |x| \leq 2r\}, \quad j \in \mathbb{Z}$$

by rescaling to $r = 1$ and then by applying the local estimates. The next step would be to consider the summation over j. This can be taken care of if the energy of u over such cylinders is square summability with respect to j. In the constant coefficient case and odd dimension this is a consequence of Huygens principle; the argument can also be adapted to even dimension. Thus it is not unreasonable to hope that a similar phenomena happens in the variable coefficient case under suitable assumptions on the coefficients. Note that a similar argument cannot be made for the summation with respect to dyadic values of r.

Based on the above discussion we introduce the space X with norm

$$\|u\|_X = \sup_{r \ dyadic} r^{-\frac{1}{2}} \|u\|_{L^2(\mathbb{R} \times \{r \leq |x| \leq 2r\})}$$

The dual space X' has norm

$$\|f\|_{X'} = \sum_{r \ dyadic} r^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}\times\{r\leq|x|\leq 2r\})}$$

Then we introduce the following energy estimates:

$$\|\nabla u\|_{L^\infty(L^2)\cap X} + \|r^{\frac{n-2}{2}-\frac{n+1}{p}} u\|_{L^p} + \|r^{-\frac{3}{2}}\partial_\theta u\|_{L^2} \lesssim \|Pu\|_{L^1(L^2)+X'} + \|\nabla u(0)\|_{L^2} \qquad (1.9)$$

where $p = 2$ for $n \geq 4$, $p = 2+\epsilon$ for $n = 3$ and $p = 4$ for $n = 2$. In a way these are the hyperbolic counterparts of the local smoothing estimates for the Schroedinger equation.

Then we can prove that

Theorem 3. *Assume that (1.7) holds and that (1.9) is satisfied whenever the right hand side is finite. Then the full Strichartz estimates hold for* $\Box_g$.

This result allows us to reduce the Strichartz estimates to the energy estimates in (1.9). What would be left is to show that (1.7) together with the nontrapping condition (1.8) imply the energy estimates (1.9). Unfortunately for now we have to contend ourselves with a weaker result, namely

Theorem 4. *Let* $n \geq 4$. *Assume that (1.7) holds with an* ϵ *which is sufficiently small. Then the energy estimates (1.9) are satisfied.*

The proof uses only the bounds on the first derivatives of g in (1.7). Obtaining the analogue result in dimensions $n = 2, 3$ remains an open problem for now.

In the rest of the paper we sketch the proofs of the above results. Complete proofs, and, hopefully, more complete results will be published elsewhere. Our plan is as follows. First we explain the energy estimates, which play an essential role in our arguments. Next we reduce the estimates to their dyadic counterparts, in which the coefficients are truncated in frequency in a way which is related to the paradifferential calculus. Then we show how one should adapt the argument in [10], based on the FBI transform, in order to prove Theorem 1. Finally, we indicate how one can use the energy estimates in (1.9) combined with the local Strichartz estimates to reduce the proof of Theorem 3 to the study of high frequency solutions which are microlocalized near outgoing bicharacteristics which are close to the radial direction. Modulo this reduction the proof of Theorem 3 is quite similar to the proof of Theorem 1.

2 Global energy estimates

In this section we prove the global energy estimates which correspond to (1.6), respectively (1.7). We start with the easier case, namely

Lemma 1. *Assume that (1.6) holds. Then*

$$\|\nabla u\|_{L^\infty(L^2)} \lesssim \|\nabla u(0)\|_{L^2} + \|\Box_g u\|_{L^1(L^2)} \tag{2.10}$$

Proof. We define the positive definite energy functional

$$E_0(u(t)) = \frac{1}{2}\int g^{00}|\partial_t u|^2 - \sum_{i,j=1}^n g^{ij}\partial_i u \partial_j u dx$$

Then a simple computation yields

$$\frac{d}{dt}E_0(u(t)) \lesssim \int |\nabla g||\nabla u|^2 + |\partial_t u||\Box_g u| dx \tag{2.11}$$

After using Cauchy-Schwartz and then Gronwall's inequality for $E(u(t))^{\frac{1}{2}}$ this gives

$$E_0(u(t))^{\frac{1}{2}} \leq c(E_0(u(0))^{\frac{1}{2}} + \|\Box_g u\|_{L^1(0,t;L^2)})e^{\int_0^t \|\nabla g(s)\|_{L^\infty} ds}$$

By (1.6) this implies (2.10). □

We continue with the proof of Theorem 4. First we do a change of coordinates which simplifies somewhat the analysis. More precisely, we want to replace the condition (1.7) with the stronger condition

$$\sum_{j\in\mathbb{Z}} \sup_{A_j} |x|(|\partial^2 g(x)|^{\frac{1}{2}} + |\partial g|) + |g - I_n| \leq \epsilon \tag{2.12}$$

This is achieved in the following

Lemma 2. *Let g satisfy (1.7) for a sufficiently small ϵ. Then there exists a change of coordinates χ satisfying the following conditions:*

(i) $\chi(\{x=0\}) = \{x=0\}$.

(ii) $|x|^\alpha|\partial^{\alpha+1}\chi| \lesssim \epsilon$ *fir* $\alpha \geq 1$.

(iii) The function $\chi^* g$ *satisfies (2.12).*

The desired energy estimate (1.9) follows if we prove that

$$\|\nabla u(T)\|_{L^2} + \|\nabla u\|_{X(0,T)} + \||x|^{-\frac{3}{2}}\partial_\theta u\|_{L^2} \lesssim \|Pu\|_{X'(0,T)} + \|\nabla u(0)\|_{L^2} \tag{2.13}$$

holds uniformly with respect to $T > 0$. This is a consequence of the next lemma.

Lemma 3. *Assume that (1.7) holds with ϵ sufficiently small. Fix $C > 0$ sufficiently large. Then for each nonnegative sequence $\{a_j\}_{j\in\mathbb{Z}}$ satisfying*

$$\sum a_j \leq 1$$

there exists a time dependent energy functional E so that

$$C^{-1}\|\nabla u(t)\|_{L^2}^2 \leq E(u(t)) \leq C\|\nabla u(t)\|_{L^2}^2$$

and

$$\frac{d}{dt}E(u(t)) \leq \int_{\mathbb{R}^n} |f(t)||\nabla u(t)|dx - C^{-1}\Big(\int_{\mathbb{R}^n} |x|^{-3}|\partial_\theta u(t)|^2 + \sum_{j\in\mathbb{Z}} a_j \int_{A_j} |x|^{-1}|\nabla u(t)|dx\Big) \tag{2.14}$$

Indeed, if (2.14) holds then integrating it from 0 to T we obtain

$$\|\nabla u(T)\|_{L^2(\mathbb{R}^n)}^2 + \||x|^{-\frac{3}{2}}\partial_\theta u(t)\|_{L^2([0,T]\times\mathbb{R}^n)}^2 + \sum_{j\in\mathbb{Z}} a_j\||x|^{-\frac{1}{2}}\nabla u(t)\|^2 dx_{L^2([0,T]\times A_j)} \leq$$
$$C^2\Big(\|\nabla u(0)\|_{L^2(\mathbb{R}^n)}^2 + \int_0^T \int_{\mathbb{R}^n} |f(t)||\nabla u(t)|dxdt\Big)$$

Replace a_j by $\frac{1}{2}(a_j + b_j)$ and use Cauchy-Schwartz to get

$$\|\nabla u(T)\|_{L^2(\mathbb{R}^n)}^2 + \||x|^{-\frac{3}{2}}\partial_\theta u(t)\|_{L^2([0,T]\times\mathbb{R}^n)}^2 + \frac{1}{2}\sum_{j\in\mathbb{Z}} a_j\||x|^{-\frac{1}{2}}\nabla u(t)\|^2 dx_{L^2([0,T]\times A_j)} \leq$$
$$C^2\Big(\|\nabla u(0)\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2}\sum_{j\in\mathbb{Z}} b_j^{-1}\||x|^{\frac{1}{2}}f(t)\|_{L^2([0,T]\times A_j)}^2\Big)$$

If we maximize the left hand side with respect to $\{a_j\}$ and minimize the right hand side with respect to $\{b_j\}$ then we obtain exactly (2.13).

Proof of Lemma 3. Observe first that without any restriction in generality we can assume that

$$\sup_{A_j} |x||\nabla g| + |g - I_n| \leq 2\epsilon a_j \tag{2.15}$$

In addition we also make the simplifying assumption

$$g^{00} = 1, \quad g^{0j} = 0, \quad j = \overline{1,n}$$

Our argument can be modified to obtain the general case.

We seek an energy functional E of the form

$$E = E_0 + \mu E_1$$

where μ is a small parameter, E_0 is defined as above and

$$E_1(u(t)) \quad = \quad \int_{\mathbb{R}^n} b(r)(\partial_r + \frac{n-1}{2r})u \cdot \partial_t u dx$$

Here the function b is nonnegative, increasing, so that

$$\frac{1}{2} \leq b(r) \leq 1, \qquad 0 \leq b'(r) \leq \frac{1}{r} \tag{2.16}$$

and

$$b'(r) \geq cr^{-1}a_j, \qquad b''(r) \leq 0 \quad \text{ in } [2^j, 2^{j+1}] \tag{2.17}$$

To construct a function b with these properties, start with

$$b_0(r) = \begin{cases} 1 - (1-r^2) & r < 1 \\ 1 & r > 1 \end{cases}$$

Then the function

$$b(r) = \frac{1}{2} + \frac{1}{2}\sum_j a_j b_0(2^{-j-2}r)$$

has the desired properties.

Lemma 4. *Assume that ϵ is sufficiently small. Let b satisfy (2.16), (2.17). Then*

$$\Box_g \frac{b(r)}{r} \geq cr^{-3}, \quad n \geq 4 \tag{2.18}$$

Proof. We can replace $\Box_g$ by $\Box$ at the expense of an error controlled by ϵ times the right hand side in (2.18). But

$$\begin{aligned}\Box\frac{b(r)}{r} &= -(\partial_r^2 + \frac{n-1}{r})\frac{b(r)}{r} \\ &= -\frac{b''(r)}{r} - (n-3)\frac{b'}{r^2} + (n-3)\frac{b}{r^3}\end{aligned}$$

and (2.18) follows.

Next we verify that E is positive definite. Since μ is small, we only need to show that

$$\|r^{-1}u\|_{L^2(\mathbb{R}^n)} \lesssim \|\partial_r u\|_{L^2(\mathbb{R}^n)}$$

But this reduces to the one dimensional estimate

$$\|r^{\frac{n-3}{2}}u\|_{L^2(\mathbb{R})} \lesssim \|r^{\frac{n-1}{2}}\partial_r u\|_{L^2(\mathbb{R})}$$

which in turn follows from the straightforward identity

$$-2 < r^{\frac{n-3}{2}}u, r^{\frac{n-1}{2}}\partial_r u >= (n-2)\|r^{\frac{n-3}{2}}u\|_{L^2(\mathbb{R})}$$

Now compute the time derivative of the energy. For E_0 we combine (2.11) with (2.15) to get

$$\frac{d}{dt}E_0(u(t)) \lesssim \int_{\mathbb{R}^n} |f||\nabla u|dx + \epsilon\sum_{j\in\mathbb{Z}} a_j \int_{A_j} |x|^{-1}|\nabla u|^2 dx) \tag{2.19}$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}E_1(u(t)) &= \int_{\mathbb{R}^n} b(r)(\partial_r\partial_t u\cdot\partial_t u + \partial_r u\cdot\partial_t^2 u + \frac{n-1}{2r}(|\partial_t u|^2 + u\cdot\partial_t^2 u))\ dx \\ &= \int_{\mathbb{R}^n} b(r)(\partial_r\partial_t u\cdot\partial_t u + \partial_r u\cdot\partial_i g^{ij}\partial_j u + \frac{n-1}{2r}(|\partial_t u|^2 + u\cdot\partial_i g^{ij}\partial_j u)) \\ &+ b(r)(\partial_r u + \frac{n-1}{2r}u)f\ dx \\ &= \int_{\mathbb{R}^n} \frac{b}{2}(\partial_r + \frac{n-1}{r})(|\partial_t u|^2 - g^{ij}\partial_i u\cdot\partial_j u) - [\partial_i, b\partial_r]u\cdot g^{ij}\partial_j u \\ &- (\partial_i\frac{(n-1)b}{2r})u\cdot g^{ij}\partial_j u + b(r)(\partial_r u + \frac{n-1}{2r}u)f\ dx \\ &= \int_{\mathbb{R}^n} -\frac{b'(r)}{2}(|\partial_t u|^2 - g^{ij}\partial_i u\cdot\partial_j u) - \frac{b(r)}{r}(\partial_i u\cdot g^{ij}\partial_j u - \partial_r u\cdot\frac{x_i}{r}g^{ij}\partial_j u) \\ &- b'(r)\partial_r u\cdot\frac{x_i}{r}g^{ij}\partial_j u - (\Box_g\frac{(n-1)b}{4r})u^2 + b(r)(\partial_r u + \frac{(n-1)b}{2r}u)f\ dx\end{aligned}$$

Then, using (2.15), (2.16), (2.17) and (2.18) we get

$$\begin{aligned}\frac{d}{dt}E_2(u(t)) &\leq -c(\int_{\mathbb{R}^n} |x|^{-1}|\nabla_\theta u|^2 + |x|^{-3}|u|^2 \, dx + \sum_j a_j \int_{A_j} |x|^{-1}|\nabla u|^2 \, dx) \\ &+ C \int_{\mathbb{R}^n} |f|(|\nabla u| + \frac{|u|}{r}) \, dx\end{aligned}$$

for $n \geq 4$. □

3 Localization in frequency

In this section we describe how to do a Paley Littlewood decomposition of the solution and reduce the problem to appropriate dyadic estimates. Consider first the case when (1.6) is satisfied. Start with a smooth function q in R^+ which is supported in $[0, 2]$ and equals 1 in $[0, 1]$. Then define the symbols

$$s_\lambda(\xi) = q(\frac{|\xi|}{2\lambda}) - q(\frac{|\xi|}{\lambda}), \quad q_\lambda(\xi) = q(\frac{|\xi|}{\lambda})$$

supported in $\{|\xi| \in [\lambda, 4\lambda]\}$ respectively $\{|\xi| < 2\lambda\}$.

Starting with the coefficients g^{ij} we want to define regularized coefficients g^{ij}_λ which are essentially truncated in frequency at the scale $\max\{\lambda, \left(\frac{\lambda}{|t|}\right)^{-\frac{1}{2}}\}$. This can be done as follows. First for $d > 0$ we define the coefficients g^{ij}_d which are regularized at time scales below d by

$$g^{ij}_d(t, x) = (1 - q_d(t))g^{ij}(t, x) + q_d(t)g^{ij}(d, x)$$

Then we set

$$g^{ij}_\lambda = q_{\lambda^{-1}}(t)Q_\lambda g^{ij}_{\lambda^{-1}} + \sum_{k>0} s_{2^k\lambda^{-1}} Q_{2^{-\frac{k}{2}}\lambda} g^{ij}_{2^k\lambda^{-1}}$$

To avoid further difficulties at $t = 0$ we work with the even extension of g^{ij}. It is easy to see that the regularized coefficients satisfy

$$\sup_{|t|\leq\lambda^{-1}} |g^{ij}_\lambda - g^{ij}| + \sum_{d \ dyadic}^{d>\lambda^{-1}} \lambda d \sup_{|t|\in[d,2d]} |g^{ij}_\lambda - g^{ij}| \lesssim \epsilon \tag{3.20}$$

$$\lambda^{-1} \sup_{|t|\leq\lambda^{-1}} |\partial g^{ij}_\lambda| + \sum_{d \ dyadic}^{d>\lambda^{-1}} d \sup_{|t|\in[d,2d]} |\partial g^{ij}_\lambda| \lesssim \epsilon \tag{3.21}$$

and

$$\sup_{|t|\leq \lambda^{-1}} |\partial^\alpha g_\lambda^{ij}| + \sum_{d \ dyadic}^{d>\lambda^{-1}} (\lambda d)^{\frac{|\alpha|+2}{2}} \sup_{|t|\in[d,2d]} |\partial^\alpha g_\lambda^{ij}| \lesssim \epsilon \lambda^d, \quad |\alpha| \geq 2 \tag{3.22}$$

To state the estimates we want to prove introduce the space X with norm

$$\|u\|_X = \sup_{d \ dyadic} d^{-\frac{1}{2}} \|u\|_{L^2(\{d\leq t\leq 2d\})}$$

and the dual space X' with norm

$$\|u\|_{X'} = \sum_{d \ dyadic} d^{\frac{1}{2}} \|u\|_{L^2(\{d\leq t\leq 2d\})}$$

Consider the following form of the Strichartz estimates:

$$\||D|^{1-\rho} u\|_{L^p(L^q)} \lesssim \|\nabla u\|_X + \|\Box_g u\|_{X'} \tag{3.23}$$

We would like to reduce this to its dyadic counterparts,

$$\lambda^{1-\rho} \|S_\lambda u\|_{L^p(L^q)} \lesssim \lambda \|S_\lambda u\|_{X+\lambda^{-\frac{1}{2}}L^2} + \|\Box_{g_\lambda} S_\lambda u\|_{X'\cap\lambda^{\frac{1}{2}}L^2} \tag{3.24}$$

The spaces are modified in the time scale $\{|t| \leq \lambda^{-1}\}$ because the regularized coefficients g_λ have uniform regularity on that scale. What we need is to obtain square summability with respect to dyadic values of λ. The difficulty is that it is impossible to achieve such square summability for the X norms. Fortunately we can truncate u in time and assume without any restriction in generality that u is supported away from, say, $t = -\infty$. Then the energy estimates in Lemma 1 show that the first right hand side term in (3.23), (3.24) is always controlled by the second one. Then one only needs to show that

$$\sum \|\Box_{g_\lambda} S_\lambda u\|^2_{X'\cap\lambda^{\frac{1}{2}}L^2} \lesssim \|\nabla u\|_X^2 + \|\Box_g u\|_{X'}^2$$

The similar norms for $S_\lambda \Box_g u$ are easy to handle, so it remain to estimate the difference,

$$\sum \|(\Box_{g_\lambda} S_\lambda - S_\lambda \Box_g) u\|^2_{X'\cap\lambda^{\frac{1}{2}}L^2} \lesssim \|\nabla u\|_X^2 + \|\Box_g u\|_{X'}^2$$

This can be done using the bounds in (3.20)-(3.22).

A similar argument applies in the case when (1.7) and the energy estimates (1.9) hold, with the obvious change that the role of the time variable t is now played by $r = |x|$.

4 The FBI transform and the proof of the dyadic estimates

The proof of the dyadic estimates is quite similar to the proof of the local estimates in [10], [9]. Consequently, we just outline the main steps and point out the differences.

The essential tool in our proof of the Strichartz estimates is the FBI transform. For a presentation of the FBI transform we refer the reader to Delort's monograph [2]. The technique we use was first developed in [11]. The idea is quite simple, namely to obtain estimates for the FBI transform of the solution. This requires conjugating the operator $\Box_{g_\lambda}$ with respect to the FBI transform. The first part of [11] is devoted to proving L^2 error estimates for this conjugation. The novelty here is that we need to adjust the parameter in the FBI transform as a function of x, t. We cannot use directly the error estimates in [11], but the new estimates we prove are similar in spirit. We first discuss the case when (1.6) holds.

The FBI transform We define the FBI transform of a temperate distribution f in $\mathbb{R}^{n+1}$ is a function in $\mathbb{C}^{n+1}$ defined as

$$(T_\lambda f)(t - i\tau, x - i\xi) = c_n \lambda^{\frac{3(n+1)}{4}} m(t)^{-\frac{n+1}{4}} \int e^{-\frac{\lambda}{2m(t)}[(x-y)^2+(t-s)^2]} e^{i\lambda(x-y)\xi+(t-s)\tau)} f(s,y)\, dsdy \tag{4.25}$$

where m is a smooth function satisfying

$$\sum_{r dyadic} \sup_{r \leq t \leq 2r} \frac{t}{m(t)} < \infty \qquad |\partial^\alpha m(t)| \leq c_\alpha m(t) |t|^{-\alpha}$$

This is a small departure from the usual conventions, which has the advantage that it simplifies somewhat the notations.

If $m = 1$ then the operator T_λ is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{C}^n)$. This in turn implies that T_λ^* is an inverse for T_λ. In our case this is no longer true; however,

Lemma 5. *a) The operator T_λ is bounded from L^2 into $L^2(C^n)$.*

b) T_λ^* is an approximate inverse for T_λ in the sense that

$$\|T_\lambda^* T_\lambda - I\| \leq \lambda^{-\frac{1}{2}} \tag{4.26}$$

If $m = 1$ then the function $e^{\frac{\lambda}{2}\xi^2} T_\lambda u$ is holomorphic,

$$[(\frac{1}{i}\partial_x - \lambda\xi) - \partial_\xi] T_\lambda = 0$$

Instead for $t \geq 0$ we now get

Lemma 6. *We have*

$$\left[m(t)(\frac{1}{i}\partial_x - \lambda\xi) - \partial_\xi\right] T_\lambda = 0,$$

$$\left[m(t)(\frac{1}{i}\partial_t - \lambda\tau) - \partial_\tau\right] T_\lambda = i\left[\lambda^{-1}\frac{m'}{m}(\partial_\tau^2 + \partial_\xi^2) + \frac{(n+1)}{4}m'\right] T_\lambda$$

Furthermore, the following estimate holds:

$$\|\frac{|t|}{m(t)}(m(t)(\frac{1}{i}\partial_t - \lambda\tau) - \partial_\tau)T_\lambda\|_{L^2\to L^2} \lesssim 1 \tag{4.27}$$

The conjugate equations Start now with the equation for $S_\lambda u$,

$$\Box_{g_\lambda} S_\lambda u = f_\lambda$$

with $\nabla u \in X + \lambda^{-\frac{1}{2}} L^2$ and $f \in X' \cap \lambda^{\frac{1}{2}} L^2$. We want to find its counterpart after conjugation with respect to the FBI transform, i.e. an equation for $T_\lambda S_\lambda u$. An analysis similar to the one in [10] yields the conjugate equation

$$\left[\lambda p - (m^{-1}p_\xi + ip_x)\partial_\xi - (m^{-1}p_\tau + ip_t)\partial_\tau\right] w = g \tag{4.28}$$

where $p(x,\xi) = g_\lambda^{00}\tau^2 + 2g_\lambda^{0j}\tau\xi_j + g_\lambda^{ij}\xi_i\xi_j$ is the symbol of the operator $\Box_{g_\lambda}$, $w = \lambda T_\lambda S_\lambda u$ and $g = T_\lambda f_\lambda + R_\lambda S_\lambda u$. The operator R_λ accounts for the error in the conjugation. The error term does not do any harm since it has the same regularity as $T_\lambda f_\lambda$.

If we use Lemma 6 we can change ∂_ξ with $m(t)(\frac{1}{i}\partial_x - \lambda\xi)$, and also ∂_τ with $m(t)(\frac{1}{i}\partial_t - \lambda\tau)$ modulo an additional error term. Thus we get in effect two equations from (4.28). Due to the estimate (4.27) the additional error term is negligible in the first one,

$$\left[\lambda p + m^{-1}(p_\xi\partial_\xi + p_\tau\partial_\tau) + m(p_x\partial_x + p_t\partial_t - ip_x\cdot\xi - ip_t\cdot\tau)\right] w = g_1 \tag{4.29}$$

but is not negligible in the second one,

$$[p_x\partial_\xi + p_t\partial_\tau - p_\xi\partial_x - p_\tau(\partial_t - L)]\, w = g_2 \quad \text{on } p = 0 \tag{4.30}$$

where

$$L = \lambda^{-1}m'((\partial_t - i\lambda\tau)^2 + (\partial_x - i\lambda\xi)^2) - \frac{(n+1)}{4}\frac{m'}{m}$$

The first equation is an ode along the gradient flow of p. The λp term produces a gaussian decay for the fundamental solution away from the characteristic cone $\{p = 0\}$. Solving the equation (4.29) with Cauchy data on the cone yields a good bound for w away from the cone, just as in [10].

To deal with the trace of w on the cone we need to use (4.30). The main component of (4.30) is a transport equation along the Hamilton flow for p. Unlike in [10], here we have an additional second order term. Modulo normal derivatives on the cone which can be estimated from the first equation, this second order term produces a slow diffusion on the characteristic cone forward in time for $t > 0$. Its presence reflects the spreading of the bicharacteristic rays for large time. Due to the λ^{-1} factor, this diffusion term is negligible within each dyadic time interval; its influence becomes visible only across dyadic time intervals. The component of w corresponding to the transport equation on the cone can be again handled as in [10], provided one obtains good bounds for the regularity of the Hamilton flow.

The case of spatially asymptotically flat coefficients Here we discuss the modifications in the above approach in the case when (1.6) is replaced by (1.7) plus the energy estimates (1.9). First observe that if we combine the local results in [10] with (1.9) then we automatically get the Strichartz estimates in any dyadic cylinder $\{d \leq r \leq 2d\}$. The problem is how to sum up these estimates. The summability is, however, taken care of in two important cases, (i) for the low frequencies, because of the bound for u in (1.9), and (ii) along bicharacteristics which are not close to the radial direction, because of the bound for the tangential derivatives of u. This leaves only the (portions of) bicharacteristics which travel spatially close to the radial direction. But along such bicharacteristics the geometry (i.e. the size of the derivatives of the coefficients) resembles precisely the geometry corresponding to (1.6).

References

[1] Philip Brenner. On $L_p - L_{p'}$ estimates for the wave-equation. *Math. Z.*, 145(3):251–254, 1975.

[2] Jean-Marc Delort. *F.B.I. transformation.* Springer-Verlag, Berlin, 1992. Second microlocalization and semilinear caustics.

[3] J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.*, 133(1):50–68, 1995.

[4] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[5] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge. Local smoothing of Fourier integral operators and Carleson-Sjölin estimates. *J. Amer. Math. Soc.*, 6(1):65–130, 1993.

[6] Hart F. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[7] Hart F. Smith and Christopher D. Sogge. On Strichartz and eigenfunction estimates for low regularity metrics. *Math. Res. Lett.*, 1(6):729–737, 1994.

[8] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[9] Daniel Tataru. Strichartz estimates for operators with nonsmooth coefficients iii. preprint, `/http://www.math.nwu/~tataru/nlw`.

[10] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients ii. preprint, `/http://www.math.nwu/~tataru/nlw`.

[11] Daniel Tataru. Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation. *Amer. J. Math.*, 122(2):349–376, 2000.