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Ground states of supersymmetric matrix models

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Abstract

We consider supersymmetric matrix Hamiltonians. The existence of a zero-energy bound state, in particular for the $d = 9$ model, is of interest in M-theory. While we do not quite prove its existence, we show that the decay at infinity such a state would have is compatible with normalizability (and hence existence) in $d = 9$. Moreover, it would be unique. Other values of d, where the situation is somewhat different, shall also be addressed. The analysis is based on a Born-Oppenheimer approximation. This seminar is based on joint work with J. Fröhlich, D. Hasler, J. Hoppe and S.-T. Yau.

Bosonic matrix models

Matrix models are Schrödinger operators which first arose [7] in the early 80's as an approximation by finitely many degrees of freedom of relativistic membranes. More recently, supersymmetric matrix models have been interpreted [15] to describe a collection of particles with non–commutative coordinates (D0-branes). In this interpretation, some of the models have been conjectured [2] to describe a strong coupling limit of string theory and, as a result, are believed to have a bound state with energy at the bottom of the essential spectrum.

For expository purposes, let us postpone the definition of supersymmetric matrix models and begin with the simpler *bosonic* matrix models instead. They depend on two integers N, $d \geq 2$ and are as follows. The configuration space is $\mathcal{X} = [\mathrm{i} \cdot \mathrm{su}(N)]^d$, i.e., each configuration is a d–tuple of symmetric, traceless, $N \times N$ matrices:

$$X = (X_1, \dots, X_d) \ , \quad X_s \in \mathrm{i} \cdot \mathrm{su}(N) \ , \quad s = 1, \dots, d$$

Coordinates $x_{sA} \in \mathbb{R}$ can be introduced through $X_s = T_A x_{sA}$ (with sum over A), where T_A, $A = 1, \dots N^2 - 1$ are generators of $\mathrm{i} \cdot \mathrm{su}(N)$ with $\mathrm{tr}(T_A T_B) = \delta_{AB}$. The corresponding momenta are then given as $P_s = -\mathrm{i} T_A \partial / \partial x_{sA}$. The Hamiltonian, acting in the Hilbert space $\mathrm{L}^2(\mathcal{X})$, is

$$H = \sum_{s=1}^{d} \mathrm{tr}(P_s^2) + \sum_{s<t} \mathrm{tr}\big((\mathrm{i}[X_s, X_t])^2\big) \ , \qquad (1)$$

where the trace is w.r.t. $\mathrm{su}(N)$. The group $\mathrm{SU}(N) \times \mathrm{SO}(d) \ni (U, R)$ acts as a group of symmetries of the Hamiltonian: $X_s \mapsto U^* X_s U$, $X_s \mapsto R_{st} X_t$.

Note that the potential in (1) is large at infinity in $\mathcal{X}$ except for some narrowing 'valleys' along its submanifold where all X_s commute. Nevertheless, the quantum mechanical motion of a particle in that potential is confined, because of the increasing zero–point energy associated with the motion transverse to the valley. Indeed, it has been shown [11] that the spectrum of H is purely discrete.

Before indicating the supersymmetric extension of the model we shall illustrate a physical motivation for the bosonic matrix models. We sketch their original derivation [7] as an approximation to 2-dimensional membranes. More generally, but temporarily, consider an M-dimensional closed surface in space $\mathbb{R}^{d+1}$ and view it as an $M+1$-dimensional world sheet in space–time $\mathbb{R}^{d+2}$ parametrized as

$$x^\mu = x^\mu(\lambda_0, \dots, \lambda_M) , \qquad (\mu = 0, \dots, d+1) . \tag{2}$$

The dynamics of the surface is governed by the action

$$S = \int \mathrm{d}\lambda_0 \dots \mathrm{d}\lambda_M \sqrt{|G|} , \quad G = \det\big(\frac{\partial x^\mu}{\partial \lambda_a}\frac{\partial x_\mu}{\partial \lambda_b}\big) , \tag{3}$$

which represents the volume of the world sheet induced by the Minkowski metric $x_0 = x^0$, $x_i = -x^i$, $(i = 1, \dots d{+}1)$ of space–time. We anticipate that, as result of the invariance of the action (3) under a reparametrization of (2), one will obtain the Hamiltonian (1) *restricted* to $\mathrm{SU}(N)$–invariant states. Moreover, the matrices X_s will be traceless because the membrane will be described in its center of mass frame.

A Hamiltonian description of this model is obtained by passing to light cone coordinates

$$\lambda_0 := \frac{1}{2}(x^0 + x^{d+1}) , \quad \xi = \frac{1}{2}(x^0 - x^{d+1}) , \quad \vec{x} = (x^1, \dots, x^d) .$$

The coordinate λ_0 is taken as (fictitious) time; at fixed λ_0 the configuration $\xi, \vec{x}$ is a field in the variables $\lambda = (\lambda_1, \dots, \lambda_M)$. Denoting by $\pi = \pi(\lambda)$, $\vec{p} = \vec{p}(\lambda)$ the canonically conjugate fields, one finds the Hamiltonian

$$H(\vec{x}, \vec{p}; \xi, \pi) = \int \frac{\mathrm{d}^M\lambda}{\pi}(\vec{p}^{\,2} + g) , \quad g = \det\big(\frac{\partial \vec{x}}{\partial \lambda_a} \cdot \frac{\partial \vec{x}}{\partial \lambda_b}\big) \tag{4}$$

with constraints

$$p \cdot \frac{\partial \vec{x}}{\partial \lambda_a} + \pi \frac{\partial \xi}{\partial \lambda_a} = 0 , \quad (a = 1, \dots M)$$

resulting from reparametrization invariance. As $\pi = \pi(\lambda)$ is a cyclic coordinate, one is left with a field $\vec{x}(\lambda)$ with d components. The Gram determinant g can be expressed through Lagrange's identity as a sum over $M \times M$ submatrices of $(\partial x_i / \partial \lambda_a)_{i=1,\dots d; a=1,\dots M}$:

$$g = \sum_{i_1 < \dots < i_M} \big(\det \frac{\partial x_{i_a}}{\partial \lambda_b}\big)^2 .$$

This expression is particularly simple in the case of membranes $M = 2$:

$$g = \sum_{s<t} \big(\frac{\partial x_s}{\partial \lambda_1}\frac{\partial x_t}{\partial \lambda_2} - \frac{\partial x_t}{\partial \lambda_1}\frac{\partial x_s}{\partial \lambda_2}\big)^2 \equiv \sum_{s<t} \big\{x_s, x_t\big\}^2 .$$

It should at this point appear plausible that an approximation of the membrane by means of finitely many degrees of fredoom results in the replacement of the Hamiltonian (4) by (1).

Supersymmetric matrix models

Let us now indicate the supersymmetric extension [3, 13] of the model (1). Consider first Clifford generators γ^i, $(i = 1, \dots, d)$, i.e., $\{\gamma^s, \gamma^t\} = 2\delta^{st}$, realized as *real* matrices

$$\gamma^i = (\gamma^i_{\alpha\beta})_{\alpha,\beta=1,\dots,s_d} ,$$

where s_d is the dimension of the irreducible (real) representation. Furthermore, consider Clifford generators $\Theta_{\alpha A}$ $(\alpha = 1, \dots, s_d;\ A = 1, \dots, N^2 - 1)$ (with relations $\{\Theta_{\alpha A}, \Theta_{\beta B}\} = \delta_{\alpha\beta}\, \delta_{AB}$) irreducibly realized on some representation space $\mathcal{C}$. We shall set $\Theta_\alpha = T_A \Theta_{\alpha A}$, a Clifford algebra valued $\mathrm{su}(N)$ -matrix.

The Hilbert space is now $\mathrm{L}^2(\mathcal{X}, \mathcal{C})$ (instead of $\mathrm{L}^2(\mathcal{X})$) and a further term is added to the Hamiltonian (1):

$$H = \sum_{s=1}^{d} \mathrm{tr}(P_s^2) + \sum_{s<t} \mathrm{tr}\big((\mathrm{i}[X_s, X_t])^2\big) - \mathrm{tr}(\Theta_\alpha \gamma^s_{\alpha\beta}[X_s, \Theta_\beta]) . \tag{5}$$

The symmetry group of the model is as in the bosonic case, except that $\mathrm{SO}(d)$ is replaced by its covering group $\mathrm{Spin}(d)$. In addition the model admits supersymmetry [1] in dimensions $d = 2, 3, 5, 9$: On $\mathrm{SU}(N)$ invariant states,

$$\big\{Q_\alpha, Q_\beta\big\} = \delta_{\alpha\beta} H ,$$

where the Q_α's are the supercharges

$$Q_\alpha = \gamma^s_{\alpha\beta}\, \mathrm{tr}(\Theta_\beta P_s) - \frac{\mathrm{i}}{4}[\gamma^s, \gamma^t]_{\alpha\beta}\, \mathrm{tr}(\Theta_\beta [X_s, X_t]) .$$

It should be noted that the spectrum of (5) is no longer discrete. In fact [14], $\sigma(H) = [0, \infty)$.

According to recent developments in string theory and M-theory [15, 2] the $d = 9$ model is conjectured to describe N D0-branes. In line with this conjecture the following question about the existence of zero–modes is expected to be answered in the affermative:

Is 0 an eigenvalue of H? More precisely: Does there exist $\psi \in \mathrm{L}^2(\mathcal{X}, \mathcal{C})$ with $H\psi = 0$, which is $\mathrm{SU}(N) \times \mathrm{Spin}(d)$ invariant? Is it unique?

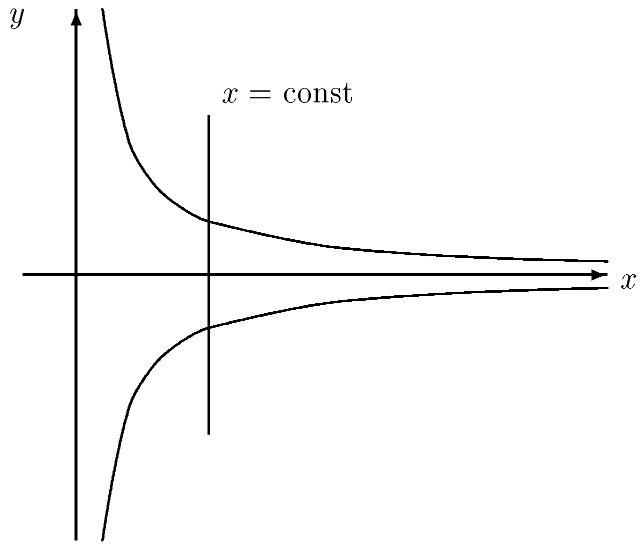
Among the various works predating ours we mention for $N = 2$ [16, 10], indicating the existence of such states, and [8], where an asymptotic analysis of such states (related to ours) is made. See also [9] for general N. On the other hand, the expected answer is no for $d = 2, 3, 5$ (see [5] for $d = N = 2$).

A simple model

Before stating our result concerning (5), let us illustrate the issue and the method by discussing a simpler model. Consider the Hamiltonian

$$H = p_x^2 + p_y^2 + x^2 y^2 + x\sigma_3 + y\sigma_1 , \tag{6}$$

acting on $\mathrm{L}^2(\mathbb{R}^2, \mathbb{C}^2)$, with σ_i, $(i = 1, 2, 3)$ being the Pauli matrices. The potential x^2y^2 exhibits 4 valleys, one of which, indicated by an equipotential line, is seen here:



The Hamiltonian can be written as

$$H = p_x^2 + \int^{\oplus} \mathrm{d}x\, H(x) \ ,$$

with $H(x)$ acting on the fiber $F = \mathrm{L}^2(\mathbb{R}, \mathbb{C}^2)$. Treating $y\sigma_1$ as a perturbation, the ground state of the latter is

$$\varphi(x; y) = \pi^{-1/4}|x|^{1/4}\mathrm{e}^{-xy^2/2}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{y}{4x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \ldots\right]$$

with ground state energy

$$E(x) = x - x - \frac{1}{8x^2} + \ldots \ .$$

Note that the contribution x of the harmonic oscillator $p_y^2 + x^2y^2$ is cancelled by an equal and opposite contribution from $x\sigma_3$. This parallels the fact that bosonic matrix models have purely discrete spectrum, whereas their supersymmetric counterparts have only essential spectrum. It is now natural to postulate that the ground state of H is given, asymptotically along the valley, by a Born-Oppenheimer ansatz:

$$\psi(x, y) = f(x)\varphi(x; y) \ . \tag{7}$$

The effective Hamiltonian for f is computed as

$$h = p_x^2 + E(x) + \|A\varphi(x)\|_F^2 + \ldots = p_x^2 - \frac{1}{8x^2} + \frac{1}{8x^2} + \ldots = p_x^2 + \mathrm{O}(x^{-5}) \ .$$

Besides of $E(x)$ a term of the same order arises because the infinitesimal translation p_x is dilating the state along the valley ($A = (p_y y + y p_y)/2$ is the generator of dilations in the fiber). A possible zero–mode, i.e., a state satisfying $hf = 0$, should then behave as $f(x) = 1$ or $f(x) = x$ at infinity (actually: the second solution is spurious) and is thus not expected to occur, as it would not be square integrable. Let us mention that this approach, being purely asymptotic, does rule out zero-modes of the form (7) with $\varphi(x;\cdot)$ an excited state of $H(x)$.

The Hamiltonian (6) is supersymmetric:

$$H = Q^2 \ , \qquad Q = p_x\sigma_3 - p_y\sigma_1 - xy\sigma_2 \ .$$

One may thus just look for zero–modes of Q, which we again analyze asymptotically. We anticipate the length scale $x^{-1/2}$ of the ground state by a change of variable, $\widetilde{y} = x^{1/2}y$, as this quantity is effectively of order 1. Then

$$Q = Q_0 x^{1/2} + \widehat{Q}\frac{\mathrm{d}}{\mathrm{d}x} + Q_1 x^{-1} \ ,$$

where the coefficients are operators on F:

$$Q_0 = \mathrm{i}\sigma_1\frac{\partial}{\partial\widetilde{y}} \ , \qquad \widehat{Q} = -\mathrm{i}\sigma_3 \ , \qquad Q_1 = -(\mathrm{i}/2)\sigma_3\widetilde{y}\frac{\partial}{\partial\widetilde{y}} \ .$$

The equation $Q\psi = 0$ is therefore an ordinary differential equation in x for $\psi(x,\cdot) \in F$, with $x = \infty$ being a singular point of the second kind [4]. The generalized power series ansatz corresponding to the eigenvalue 0 of Q_0 is

$$\psi(x,\widetilde{y}) = x^{-\kappa}\sum_{k=0}^{\infty} x^{-\frac{3}{2}k}\psi_k(\widetilde{y}) \ ,$$

which yields

$$\begin{cases} Q_0\psi_0 = 0 \\ \kappa\widehat{Q}\psi_0 = Q_1\psi_0 + Q_0\psi_1 \\ \ldots \end{cases} .$$

The solution of the first equation is $\psi_0(\widetilde{y}) = \mathrm{e}^{-\widetilde{y}^2/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the projection of the second onto ψ_0 is $\kappa(\psi_0, \widehat{Q}\psi_0)_F = (\psi_0, Q_1\psi_0)_F$, implying $\kappa = -1/4$ (corresponding to $f(x) = 1$ above).

$N = 2$ supersymmetric matrix models

The above analysis can be carried over to $N = 2$ supersymmetric matrix models. Writing $X \in \mathrm{i}\,\mathrm{su}(N = 2)$ as $X = \vec{q}\cdot\vec{\sigma}$, $\vec{q} \in \mathbb{R}^3$, the configuration spaces becomes $\mathcal{X} = \mathbb{R}^{3d}$, and the Hamiltonian

$$H = \sum_{s=1}^{9} \vec{p}_s^{\ 2} + \sum_{s<t}\left(\vec{q}_s \times \vec{q}_t\right)^2 + \mathrm{i}\vec{q}_s\cdot\left(\vec{\Theta}_\alpha \times \vec{\Theta}_\beta\right)\gamma^s_{\alpha\beta} \ . \tag{8}$$

The potential $\sum_{s<t}(\vec{q}_s \times \vec{q}_t)^2$ vanishes on the $(d+1)$–dimensional manifold

$$\{q = (\vec{q}_1, \ldots, \vec{q}_d) \mid \vec{q}_s \parallel \vec{q}_t \text{ for all } s,\, t\} \ .$$

Its points can thus be expressed as $\vec{q}_s = r\vec{e}E_s$ with $r > 0$ and $\vec{e} \in S^{3-1}$, $E \in S^{d-1}$; points in a conical neighborhood of the manifold can be expressed in terms of tubular coordinates

$$\vec{q}_s = r\vec{e}E_s + r^{-1/2}\vec{y}_s$$

with $\vec{y}_s \cdot \vec{e} = 0$, $\vec{y}_s E_s = \vec{0}$. A prefactor has been put explicitely in front of the transversal coordinates $\vec{y}_s$, so as to account for the length scale $r^{-1/2}$ of the ground state. Also note that the change

$$(\vec{e}, E, y) \mapsto (-\vec{e}, -E, y) \tag{9}$$

does not affect $\vec{q}_s$. Hence only states which are even under the antipode map (9) lift to $\mathcal{X}$.

We can now describe the structure of a putative ground state.

Theorem. *Consider the equations $Q_\beta \psi = 0$ for a formal power series solution near $r = \infty$ of the form*

$$\psi = r^{-\kappa} \sum_{k=0}^{\infty} r^{-\frac{3}{2}k} \psi_k \ , \tag{10}$$

where: $\psi_k = \psi_k(\vec{e}, E, y)$ is square integrable w.r.t. $\mathrm{d}e\, \mathrm{d}E\, \mathrm{d}y$;
ψ_k is $\mathrm{SU}(2) \times \mathrm{Spin}(d)$ invariant;
$\psi_0 \neq 0$.
Then, up to linear combinations,

- *d=9: The solution is unique, and $\kappa = 6$;*
- *d=5: There are three solutions with $\kappa = -1$ and one with $\kappa = 3$;*
- *d=3: There are two solutions with $\kappa = 0$;*
- *d=2: There are no solutions.*

All solutions are even under the antipode map (9),

$$\psi_k(\vec{e}, E, y) = \psi_k(-\vec{e}, -E, y) \ ,$$

except for the state $d = 5$, $\kappa = 3$, which is odd.

The integration measure is $\mathrm{d}q = \mathrm{d}r \cdot r^2 \mathrm{d}e \cdot r^{d-1} \mathrm{d}E \cdot r^{-\frac{1}{2} \cdot 2(d-1)} \mathrm{d}y = r^2 \mathrm{d}r\, \mathrm{d}e\, \mathrm{d}E\, \mathrm{d}y$. The wave function (10) is square integrable at infinity if $\int^\infty \mathrm{d}r\, r^2 (r^{-\kappa})^2 < \infty$, i.e., if $\kappa > 3/2$. The theorem is consistent with the statement according to which **only** for $d = 9$ a (unique) normalizable ground state for (8) (which would have to be even) is possible.

We refer to [6] for the proof of the theorem. Here we merely sketch the argument in the $d = 9$ case for uniqueness of the $\mathrm{SU}(2) \times \mathrm{Spin}(9)$ invariant ground state. As in the simple model described above, the equation to be solved at lowest order is $Q^0_\alpha \psi_0 = 0$. Ignoring invariance, this equation admits a large space of solutions, namely

$$\psi_0(\vec{e}, E, y) = \mathrm{e}^{-\sum_s \vec{y}_s^2/2} |F(E, \vec{e})\rangle \ ,$$

with $|F(E, \vec{e})\rangle \in N(E, \vec{e})$, a 2^8–dimensional subspace of $\mathcal{C}$. While $\mathrm{SU}(2)$ acts trivially on these solutions, $\mathrm{Spin}(9) \ni R$ does not:

$$(\mathcal{R}(R)\psi_0)(\vec{e}, E, y) = \mathrm{e}^{-\sum_s \vec{y}_s^2/2} \mathcal{R}_{\mathrm{F}}(R) |F(R^{-1}E, \vec{e})\rangle \ ,$$

where $\mathcal{R}_{\mathrm{F}}$ denotes the 'fermionic part' of the representation, i.e., it acts on $\mathcal{C}$ only. Invariant states, $\mathcal{R}(R)\psi_0 = \psi_0$ are thus in bijective correspondence to states invariant under the 'little group' Spin(8), i.e., to states $|F(E,\vec{e})\rangle \in N(E,\vec{e})$ satisfying

$$\mathcal{R}_F(R)|F(E,\vec{e})\rangle = |F(E,\vec{e})\rangle$$

for some arbitrary but fixed E and all R with $RE = E$. Infinitesimally, such rotations take place in a plane (with vectors U, $V \in \mathbb{R}^9$) orthogonal to E: $U_s E_s = V_s E_s = 0$. The generators of the little group are represented on $N(E,\vec{e})$ as $M^{\|}_{st} U_s V_t$ with

$$M^{\|}_{st} = -(\mathrm{i}/4)(\vec{\Theta}_\alpha \cdot \vec{e}\,)\gamma^{st}_{\alpha\beta}(\vec{\Theta}_\beta \cdot \vec{e}\,)\ , \tag{11}$$

where $\gamma^{st} = (1/2)(\gamma^s\gamma^t - \gamma^t\gamma^s)$. We need to decompose the Spin(8) representation on $N(E,\vec{e})$ into irreducibles. To begin with, the Clifford algebra of the operators $\vec{\Theta}_\alpha \cdot \vec{e}$, $\alpha = 1,\ldots,s_9 = 16$ acts irreducibly on $N(E,\vec{e})$, but the representation decomposes (see e.g. [12]) by passing to the subalgebra of even elements, resp. to Spin(16):

$$N(E,\vec{e}) = 128_- \oplus 128_+\ .$$

The further branching under the embedding Spin(16) $\hookleftarrow$ Spin(9) given by (11) is

$$N(E,\vec{e}) = (44 \oplus 84) \oplus 128\ ,$$

followed by Spin(9) $\hookleftarrow$ Spin(8):

$$N(E,\vec{e}) = (1 \oplus 8_{\mathrm{v}} \oplus 35_{\mathrm{v}}) \oplus (28 \oplus 56_{\mathrm{v}}) \oplus (8_{\mathrm{s}} \oplus 8_{\mathrm{c}} \oplus 56_{\mathrm{s}} \oplus 56_{\mathrm{c}})\ .$$

This shows that exactly one 1–dimensional representation occurs.

References

[1] M. Baake, P. Reinicke, V. Rittenberg, Fierz identities for real Clifford algebras and the number of supercharges, J. Math. Phys. **26**, 1070-1071 (1985).

[2] T. Banks, W. Fischler, S.H. Shenker, L. Susskind, M theory as a matrix model: a conjecture, Phys. Rev. D**55**, 5112-5128 (1997); `hep-th/9610043`.

[3] M. Claudson, M. Halpern, Supersymmetric ground state wave functions, Nucl. Phys. B**250**, 689-715 (1985).

[4] E.A. Coddington, N. Levinson, *Theory of ordinary differential equations*, Krieger (1987).

[5] J. Fröhlich, J. Hoppe, On zero–mass ground states in super–membrane matrix models. Comm. Math. Phys. **191**, 613-626 (1998); `hep-th/9701119`.

[6] J. Fröhlich, G.M. Graf, D. Hasler, J. Hoppe, S.-T. Yau, Asymptotic form of zero energy wave functions in supersymmetric matrix models, `hep-th/9904182`.

[7] J. Goldstone, unpublished.
J. Hoppe, *Quantum theory of a massless relativistic surface*, MIT Ph.D. Thesis (1982); Proceedings of the workshop *Constraints theory and relativistic dynamics*, World Scientific (1987).

[8] M.B. Halpern, C. Schwartz, Asymptotic search for ground states of SU(2) matrix theory, Int. J. Mod. Phys. A**13**, 4367-4408 (1998); `hep-th/9712133`.

[9] A. Konechny, On asymptotic Hamiltonian for SU(N) matrix theory, JHEP 9810 (1998); `hep-th/9805046`.

[10] S. Sethi, M. Stern, D–Brane bound state redux. Comm. Math. Phys. **194**, 675-705 (1998); `hep-th/9705046`.

[11] B. Simon, Some quantum operators with discrete spectrum but classically continuous spectrum, Ann. Phys. **146**, 209-220 (1983).

[12] B. Simon, *Representations of finite and compact groups*, American Mathematical Society (1996).

[13] B. de Wit, J. Hoppe, H. Nicolai, On the quantum mechanics of supermembranes, Nucl. Phys. B**305**, 545-581 (1988).

[14] B. de Wit, M. Lüscher, H. Nicolai, The supermembrane is unstable, Nucl. Phys. B**320**, 135-159 (1989).

[15] E. Witten, Bound states of strings and p–branes. Nuclear Phys. B**460**, 335-350 (1996); `hep-th/9510135`.

[16] P. Yi, Witten index and threshold bound states of D-branes, Nucl. Phys. B**505**, 307-318 (1997); `hep-th/9704098`.