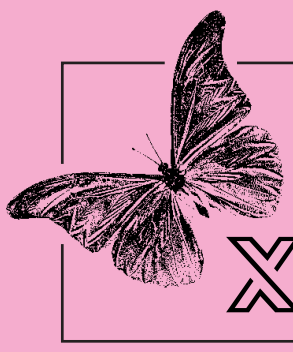

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Two Methods of Solution of the Three-Dimensional Inverse Nodal Problem.

Yu. E. Karpeshina, J. R. McLaughlin

The operator $-\Delta + q$ with the Dirichlet boundary condition is considered in a parallelepiped. The problem of restoring $q(x)$ from positions of nodal surfaces is solved.

1 Introduction.

We consider a rectangular region in three-dimensional space, which is filled with a homogeneous elastic medium, fixed on the sides. There is a force acting on the media, depending linearly on displacement. We will show that the amplitude of the force is uniquely determined by a subset of the nodal surfaces. Thus, we consider the operator

$$Hu = -\Delta u + qu \tag{1}$$

with the Dirichlet boundary condition

$$u\Big|_{\partial K_+} = 0 \tag{2}$$

in the parallelepiped

$$K_+ = \left[0, \frac{\pi}{a_1}\right] \times \left[0, \frac{\pi}{a_2}\right] \times \left[0, \frac{\pi}{a_3}\right]. \tag{3}$$

We represent real potential $q(x)$ in the form:

$$q(x) = \sum_{\beta \in L^*} q_\beta \cos(\beta_1 x_1) \cos(\beta_2 x_2) \cos(\beta_3 x_3), \tag{4}$$

where L^* is the lattice:

$$L^* = \{\beta = (a_1 n_1, a_2 n_2, a_3 n_3),\ n_1, n_2, n_3 = 0, 1, 2, ...\}. \tag{5}$$

We will study eigenfunctions u,

$$Hu = k^2 u, \tag{6}$$

in a high energy region and restore q from information about nodal surfaces, which are zeros of u. In two-dimensional situation the analogous problem was solved by O.H. Hald and J.R. McLaughlin [HM]. In order to develop two-dimensional results, O.H. Hald and J.R. McLaughlin understood how the eigenvalues (the squares of the natural frequencies) and the corresponding eigenfunctions (mode shapes of the membrane) change as q change. Note, that for vibrating membranes arbitrary large eigenvalues can be arbitrarily close together. This presents considerable difficulty as the arbitrarily small differences become small divisors in asymptotic expansion. To overcome this difficulty, O.H. Hald and J.R. McLaughlin began with the $q = 0$ case and establish a dense, well defined set of rectangular membranes for which almost all of the eigenvalues are well separated. The membranes were chosen so that the square of the ratio of the sides is not well approximated by rational numbers. After selecting the "good" rectangles, they gave three conditions that eigenvalues must satisfy: almost all eigenvalues k^2 are at least a given distance (of order $k^{-\delta},\ \delta > 0$) from their nearest neighbor and a large distance from the selected number of neighbors. It was shown that for the "good" rectangles "almost all" eigenvalues satisfy these conditions. Using the $q = 0$ case as the base problem, perturbation results were obtained for the $q \neq 0$ case. The leading terms for "good" eigenvalues and eigenfunctions were established. This made it possible to find asymptotic formulae for significant part of nodal lines. Note that the nodal domains for eigenfunctions corresponding to "good" eigenvalues for $q = 0$ have small diameters when $k \to \infty$, but the nodal domains for the corresponding eigenfunctions for $q \neq 0$ usually do not. However, it was shown that it is possible to cut the large nodal domains into smaller approximate nodal domains. The size of small regions go to zero when $k \to \infty$. The potential can be considered approximately as a constant inside each of them. It is proved that the potential approximately coincides with the first eigenvalue of the Dirichlet problem for the Laplacian in each of this regions. This eigenvalue, and therefore, the potential, are defined completely by geometry of nodal lines, and can be determined numerically. The perturbation formulae established bounds for error in making approximation.

In the three dimensional situation principal difficulties arise on the way of realizing the two-dimensional scheme. The eigenvalues are situated much denser than in the two-dimensional case. For "good" parallelepipeds the average distance between eigenvalues is of order $k^{-1-\delta},\ \delta > 0$ (compare $k^{-\delta}$ for the two-dimensional case). The analogs of two-dimensional conditions of stability are not sufficient for constructing perturbation formulae. To overcome this difficulty we apply the perturbation technic developed by Yu.E. Karpeshina [K] for three-dimensional periodic problems. We use a simple connection between the Dirichlet problem and the periodic problem in the "doubled" region with a symmetric potential. Choosing "good" parallelepipeds, we consider unperturbed eigenvalues ($q = 0$) of the periodic problem in the "doubled" region, which are well spaced (up to a natural degeneration

for a symmetric potential). Then, we choose from them those who satisfy 6 additional conditions [1]. Using counting arguments, we prove that "almost all" eigenvalues in a high energy region satisfy all imposed conditions. To construct perturbation formulae we surround an unperturbed eigenvalue by a small contour on the complex plane, the radius of the contour being half of the distance to the nearest neighbor. We construct perturbation series for the resolvent of the periodic problem on this contour and that enable us to get perturbation series for an eigenvalue and its spectral projection. The spectral projection has dimension 8, which corresponds to natural degeneration of the eigenvalue for a symmetric potential. From this, using antisymmetrization procedure, we easily get an asymptotic formula for the corresponding undegenerated eigenvalue and the eigenfunction of the Dirichlet problem.

Note, that for constructing perturbation series, it should be used some auxiliary operator $\hat{H}$ instead of H_0, corresponding to $q = 0$ (see [K]). This need arises from the fact, that, in the three-dimensional situation, diffraction conditions, which prevent perturbation series from convergence, depend on the potential. Because of the fact, that the diffraction conditions control the convergence of the series, the expressions corresponding to these conditions should be in denominators of terms of the series for the resolvent. The perturbation series constructed with respect to H_0 don't provide this - they, obviously, have denominators independent on q. So, in [K], another initial operator $\hat{H}$ was constructed. It corresponds to all the diffraction conditions. Fortunately, it turns out that with a good accuracy the perturbation series with respect to $\hat{H}$ may be replaced by segments of the perturbation series with respect to H_0.

The perturbation formulae obtained in this way for eigenfunctions of the Dirichlet problem enable us to describe behavior of nodal surfaces.

Another problem is how to get information about q from the nodal surfaces. There are two ways to reconstruct the potential. The first of them is the three-dimensional analog of the method developed by O.H. Hald and J.R. McLaughlin in two-dimensional situation. The nice feature of this approach is that it produce approximation of q in each small region, using information only from this region. However, there is a practical drawback of this approach: doing all those first harmonic computations is quite hard numerically, and it works only for smooth potentials. This motivated us to develop a new approach which would be simpler for practical computations and would work for a wide class of q. In this approach we compare the patterns of nodal surfaces for $q = 0$ and $q \neq 0$. Measuring the distance between them in the middle of every cell, we get a set of dates. With a good accuracy we prove that the dates linearly depend on the Fourier coefficients of $q(x)$, corresponding matrix being known. We get the Fourier coefficients from the deviations of the nodal surfaces from the

[1] For a smooth potential its enough to impose 4 simpler conditions.

initial pattern. The computation of the Fourier coefficients need information about nodal surfaces in every or almost every small region. However, this method is expected to be easier to implement numerically, because essentially includes only summation steps. It demands much less restriction on the smoothness of potential, for the reason that Fourier coefficients are integral characteristics of a function. eigenvalues

2 "Good" Parallelepipeds and Well-Spaced Eigenvalues of the Free Problem.

Without restriction of generality we assume that the lengths of the parallelepiped K_+ are in some interval $[1, a_0]$, $1 < a_0 < \infty$. We write down this formally in the form:

$$\left(\frac{\pi^2}{a_1^2}, \frac{\pi^2}{a_2^2}, \frac{\pi^2}{a_3^2}\right) \in J, \tag{7}$$

$$J = [1, a_0] \times [1, a_0] \times [1, a_0].$$

Let $s \in Z^3 \setminus \{0\}$, $\delta > 0$, $\varepsilon > 0$, and the plane layer $W_s^{\delta,\varepsilon} \subset J$ is defined by the formula:

$$W_s^{\delta,\varepsilon} = \left\{\xi \in J : |(\xi, s)| < \varepsilon |s|^{-2-\delta}\right\}. \tag{8}$$

in J. Let $W^{\delta,\varepsilon}$ be the union of all such layers for different s:

$$W^{\delta,\varepsilon} = \cup_{s \in Z^3 \setminus \{0\}} W_s^{\delta,\varepsilon}. \tag{9}$$

The simple calculation gives:

$$V\left(W^{\delta,\varepsilon}\right) =_{\varepsilon \to 0} O(\varepsilon), \tag{10}$$

here and below $V(\cdot)$ is a volume of a three-dimensional region.

We take the "good" set of parallelepiped J' as following:

$$J'(\delta, \varepsilon) = J \setminus W^{\delta,\varepsilon}. \tag{11}$$

From (11) it follows that $|J'| \approx |J|$ for ε small enough.

To define the notion of well-spaced eigenvalues, let us consider the operator H_0, corresponding to the zero potential:

$$H_0 = -\Delta, \tag{12}$$

$$u\Big|_{\partial K_+} = 0\ , \tag{13}$$

It is easy to see that the eigenvalues and eigenfunctions of H can be parameterized by the points of a lattice L,

$$L = \{\alpha = (a_1 n_1, a_2 n_2, a_3 n_3),\ n_1, n_2, n_3 = 1, 2, ...\}\,. \tag{14}$$

and are given by the formulae:

$$\lambda_{0\alpha} = |\alpha|^2, \tag{15}$$

$$u_{0\alpha}(x) = \frac{8}{V(K_+)} \sin(\alpha_1 x_1) \sin(\alpha_2 x_2) \sin(\alpha_3 x_3). \tag{16}$$

Thus, the spectrum of H_0 consists of the points $|\alpha|^2$, $\alpha \in L$. Let $\delta > 0$ and $M_1(\delta)$ be the set of $\alpha \in L$, such that the corresponding $|\alpha|^2$ are well spaced from their nearest neighbors:

$$M_1(\delta) = \left\{ \alpha \in L : \forall \beta \neq \alpha : ||\beta|^2 - |\alpha|^2| > |\alpha|^{-1-2\delta}. \right\} \tag{17}$$

For "good" parallelepipeds the set $M_1(\delta)$ is rich. We define precisely what does it mean for a set to be reach. Let $B_+(0,r)$ be the positive octant of the ball of radius r centered at the origin. Let $\#\Omega(\delta)$ be the number of lattice points L in a region $\Omega(\delta)$. We say that $\Omega(\delta)$ is rich in L if

$$\lim_{r\to\infty} \frac{\#\left(\Omega \cap B_+(0,r)\right)}{\#B_+(0,r)} = 1 + O(r^{-2\delta}). \tag{18}$$

Lemma 1 *Suppose $\delta > 0$, $\varepsilon > 0$ and the sides of parallelepiped K_+ belongs to $J'(\delta,\varepsilon)$. Then, the set $M_1(\delta)$ is rich in L.*

The lemma, obviously, means, that, if K_+ is a "good" parallelepiped, that there is a reach set of $|\alpha|^2$ in a high energy region, which are well spaced, i.e., their distance to the nearest neighbor is greater than $|\alpha|^{-1-2\delta}$. We use the well-spacing as the first condition in the construction of the perturbation formulae.

3 Perturbation Formulae.

Let us consider $\alpha \in M_1(\delta)$ and a contour $C(\alpha)$ on the complex plane around the point $|\alpha|^2$ with radius $\frac{1}{2}|\alpha|^{-1-2\delta}$. It follows from (17) that all the points $|\beta|^2$, $\beta \neq \alpha$ are outside contour $C(\alpha)$, and $C(\alpha)$ is at the distance greater or equal to $\frac{1}{2}|\alpha|^{-1-2\delta}$ from the spectrum of the free operator. We consider the resolvent $(H-z)^{-1}$, $z \in C(\alpha)$ and construct converging perturbation series for it. First, following [K], we construct the perturbation series for the resolvent of a periodic problem in a doubled region with a symmetric potential and than use antisymmetrization procedure to obtain the corresponding formula for the Dirichlet problem. We prove that the perturbation series for $(H-z)^{-1}$ converges on the rich set $\tilde{M}(q,\delta) \subset M_1(\delta) \subset L$. This set is described by 6 additional conditions (4 for smooth potentials). Integrating the series for $(H-z)^{-1}$ over the contour, we get a perturbation formula for eigenfunctions.

Theorem 1 *Suppose α belongs to the set $\tilde{M}(q,\delta)$. Then there exists a unique ungenerated eigenvalue λ_α of the operator H in the interval $\varepsilon(|\alpha|,\delta) \equiv [|\alpha|^2 + q_0 - |\alpha|^{-1-2\delta}, |\alpha|^2 + q_0 + |\alpha|^{-1-2\delta}]$. The eigenfunction, corresponding to λ_α, admits the representation:*

$$u_\alpha(x) = u_{0\alpha}(x) + \tilde{G}_1(\alpha, x) + \psi(\alpha, x), \tag{19}$$

where

$$\tilde{G}_1(\alpha, x) = \sum_{\beta \in L, 0 < |\beta - \alpha| \le |\alpha|^\varepsilon} \frac{(q u_{0\alpha}, u_{0\beta})}{|\alpha|^2 - |\beta|^2} u_{0\beta}(x),$$

ε being arbitrary chosen from the interval $(0, 1/10)$.

The functions $\tilde{G}_1(\alpha, x)$, $\psi(\alpha, x)$ satisfy the estimates:

$$|\tilde{G}_1(\alpha, x)| \le |\alpha|^{-\xi_1}. \tag{20}$$

$$|\nabla \tilde{G}_1(\alpha, x)| \le |\alpha|^{1-\xi_1}. \tag{21}$$

$$|\psi(\alpha, x)| \le |\alpha|^{-2\xi_1}. \tag{22}$$

$$|\nabla \psi(\alpha, x)| \le |\alpha|^{1-2\xi_1}, \tag{23}$$

ξ_1 being a positive parameter depending on the smoothness of the potential, in particular, $\xi_1 = 1 - 6\delta$ for $q(x)$ whose symmetric extension to the doubled region is infinitely differentiable.

4 Nodal Surfaces.

Let us consider the nodal surfaces, i.e., the surfaces in K_+, defined by the equation: $u_\alpha(x) = 0$. It is easy to see from (16), that the unperturbed nodal surfaces ($q = 0$) are the planes:

$$\begin{aligned} x_1 &= \frac{\pi m_1}{\alpha_1}, \quad m_1 = 0, 1, ... \frac{\alpha_1}{a_1}, \\ x_2 &= \frac{\pi m_2}{\alpha_2}, \quad m_2 = 0, 1, ... \frac{\alpha_2}{a_2}, \\ x_3 &= \frac{\pi m_3}{\alpha_3}, \quad m_3 = 0, 1, ... \frac{\alpha_3}{a_3}. \end{aligned} \tag{24}$$

These planes form a lattice with the elementary cell $\left[0, \frac{\pi}{\alpha_1}\right] \times \left[0, \frac{\pi}{\alpha_2}\right] \times \left[0, \frac{\pi}{\alpha_3}\right]$. If $\alpha \in M(q, \delta)$, then

$$|\alpha_i| > |\alpha|^{1-5\delta}, \quad i = 1, 2, 3. \tag{25}$$

Hence, the elementary cell is quite small. Let us consider an unperturbed nodal surface, say,

$$x_1 = \frac{\pi m_1}{\alpha_1}. \tag{26}$$

Note that for x on this surface

$$\frac{\partial u_{0\alpha}}{\partial x_1}(x) = \frac{8\alpha_1(-1)^{m_1}}{V(K_+)} \sin(\alpha_2 x_2)\sin(\alpha_3 x_3). \tag{27}$$

Let Ω_{m_1} be the set of points on the surface (26), which are far enough from its intersections with another nodal surfaces, namely,

$$\Omega_{m_1} = \left\{ x : x_1 = \frac{\pi m_1}{\alpha_1},\ |\sin(\alpha_2 x_2)| > |\alpha|^{-5\delta},\ |\sin(\alpha_3 x_3)| > |\alpha|^{-5\delta} \right\}. \tag{28}$$

Using (25) it is not difficult to show that Ω_{m_1} has an asymptotically full measure on the plane (26):

$$\frac{|\Omega_{m_1}|}{\pi^2 a_2^{-1} a_3^{-1}} = 1 + O(|\alpha|^{-5\delta}). \tag{29}$$

From (25) and (27) it follows that

$$\left| \frac{\partial u_{0\alpha}}{\partial x_1} \right| > c|\alpha|^{1-15\delta}, \tag{30}$$

when $x \in \Omega_{m_1}$.

Lemma 2 *If α belongs to $\tilde{M}(q,\delta)$, then there is a unique piece of the perturbed nodal surface in the $(k^{-1-10\delta})$-neighborhood of each simply connected component of Ω_{m_1}. In fact, it is in the smaller $(|\alpha|^{-1-\xi_1+16\delta})$-neighborhood and defined by the formula:*

$$x_1(x_2,x_3) = \frac{\pi m_1}{\alpha_1} + \varphi(x_0), \tag{31}$$

$$x_0 \in \Omega_{m_1}, \quad x_0 = \left(\frac{\pi m_1}{\alpha_1}, x_2, x_3 \right),$$

$$\varphi(x_0) = \frac{-u_\alpha(x_0)}{\alpha_1(-1)^{m_1}\sin(\alpha_2 x_2)\sin(\alpha_3 x_3)} + O(|\alpha|^{-1-2\xi_1+30\delta}). \tag{32}$$

The function $\varphi(x_0)$ satisfies the estimate:

$$|\varphi(x_0)| < c|\alpha|^{-1-\xi_1+15\delta}. \tag{33}$$

The lemma follows from formulae (19) – (23), (30) and the Implicit Function Theorem.

Let us consider the points in K_+:

$$\tilde{m} = \left(\frac{\pi m_1}{\alpha_1}, \frac{\pi(m_2+1/2)}{\alpha_2}, \frac{\pi(m_3+1/2)}{\alpha_3} \right),$$

where m_1, m_2, m_3 are integers. It is clear that $\tilde{m} \in \Omega_{m_1}$ for every $\tilde{m}$. Points $\tilde{m}$ form a lattice with the elementary cell: $\left[\frac{\pi}{\alpha_1}, \frac{\pi}{\alpha_2}, \frac{\pi}{\alpha_3}\right]$, which is small (see (25)). when Let us introduce the notation:

$$< f, g >= \frac{\pi^3}{V(K_+)|\alpha_1\alpha_2\alpha_3|} \sum_{\tilde{m}} f(\tilde{m})g(\tilde{m}),$$

f, g being smooth functions in K_+.

Lemma 3 *Suppose* $\alpha \in \tilde{M}(q,\delta)$, $\beta \in L$, $0 < |\alpha - \beta| < |\alpha|^{\varepsilon}$, $0 < \varepsilon < 1/10$. *Then the function* $\varphi(x_0)$, *defined by (32), satisfies the asymptotic formula:*

$$< \varphi, u_{0\beta} >= \frac{(qu_{0\alpha}, u_{0\beta})}{\sigma_{\beta\alpha}} + O(|\alpha|^{-2-\zeta}), \quad \beta \neq \alpha, \tag{34}$$

$$\sigma_{\beta\alpha} = \alpha_1(-1)^{m_1+m_2+m_3+1}(|\alpha|^2 - |\beta|^2), \ sigma_{\beta\alpha} = O(|\alpha|^2),$$

$\zeta > 0$ *and depends on the smoothness of the potential, f.e.,* $\zeta = 1 - \varepsilon - 20\delta$ *for* $q(x)$, *whose symmetric extension to the dubled region is infinitely differentiable.*

5 Two Methods of Solution of the Inverse Nodal Problem.

The method of finding $q(x)$ *from local data.*

The first approach is a three-dimensional analog of the method suggested by O.H. Hald and J.R. McLaughlin for two-dimensions [HM]. The main steps are the following: the nodal domains for $q = 0$ and eigenfunctions corresponding to $\alpha \in \tilde{M}(q,\delta)$ have small diameters when $|\alpha| \to \infty$ (see (24), (25)). From Lemma 2 it follows that perturbed nodal surfaces are close to unperturbed ones outside vicinities of the edges of the small rectangular subregions. This means that the nodal domains for $q \neq 0$ close to unperturbed domains, but not nessasary have small radii, because the intersections (edges) turn, generally speaking, into quasisintersections. We consider the smaller "approximate" nodal domains: the boundary of each of them coincides with the perturbed nodal surfaces in the distance from the subrectangulars edges, while the edges are eliminated by cutting planes. The size of the"approximate" nodal domains goes to zero when $|\alpha| \to \infty$. The potential can be considered approximately as a constant inside each of them. It is proved that the potential approximately coincides with the first eigenvalue of the Dirichlet problem for the Laplacian in each of this regions. This eigenvalue, and therefore, the potential, are defined completely by geometry of nodal lines, and can be determined numerically. The perturbation formulae establish bounds for error in making approximation.

The method of finding Fourier coefficients q_β.

Note, that $\varphi(x_0)$, defined by (31), gives the displacement under perturbation of the nodal surface $x_1 = \frac{\pi m_1}{\alpha_1}$ at points (x_2, x_3) in the direction of x_1. These displacements are measurable in experiments. We will use formula (34) to solve the inverse nodal problem.

Lemma 4 *If* $\alpha \in \tilde{M}(q,\delta)$, $\beta \in L$, $0 < |\beta - \alpha| < |\alpha|^{\varepsilon}$, $0 < \varepsilon < 1/10$, *then the following*

formula holds:

$$(qu_{0\alpha}u_{0\beta}) = \sigma_{\beta\alpha} < \varphi, u_{0\beta} > +O(|\alpha|^{-\zeta}), \quad \beta \neq \alpha. \tag{35}$$

This lemma follows from Lemma 3 and the fact that $< u_\beta, u_\gamma >= \delta_{\beta\gamma} + O(|\alpha|^{-1+\varepsilon+5\delta})$, when $|\beta - \alpha| < |\alpha|^\varepsilon$ and $|\gamma - \alpha| < |\alpha|^\varepsilon$. Note that $(qu_{0\alpha}u_{0\beta})$ coinside with $q_{[\alpha-\beta]}$, $[\alpha - \beta] = (|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|, |\alpha_3 - \beta_3|)$ up to trivial multipliers. Thus, measuring displacement of the nodal lines yields all q_β, $\beta \neq 0$. According to Theorem 1, the square of the frequency (eigenvalue λ_α) corresponding to the eigenfunction u_α satisfies the asymptotic $\lambda_\alpha = |\alpha|^2 + q_0 + O(|\alpha|^{-1-\delta})$. Thus, a measurement of the new frequency gives us q_0.

Note, that it is not nessesary to know the set $\tilde{M}(q, \delta)$ for solving the inverse problem, because of its property of being rich. Formula (35) holds for "almost all" α and has to be used with statistical averaging.

This method works for q whose symmetric extentions to the doubled region belongs to W_2^ν, $\nu > 1/2$.

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