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Johannes Sjöstrand

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U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

Resonances for strictly convex obstacles.

(Based on joint work with M.Zworski.)

Johannes Sjöstrand*

Résumé: On considère le problème de Dirichlet à l'éxtérieur d'un obstacle strictement convexe borné à bord C^∞. Sous une hypothèse sur la variation de la courbure, on obtient à un facteur $1+o(1)$ près, le nombre de résonances de module $\le r$, associées à la première racine de la fonction d'Airy.

Let $\mathcal{O} \subset\subset \mathbf{R}^n$, $n\ge 2$ be a convex open bounded set with C^∞ boundary. Assume that $\mathcal{O}$ is strictly convex in the sense that the second fundamental form on the tangent space $T\partial\mathcal{O}$ is positive definite. By Δ we denote the (self-adjoint) Dirichlet realization of the Laplace operator $\sum_1^n \frac{\partial^2}{\partial x_j{}^2}$ on $\mathbf{R}^n\setminus\mathcal{O}$ with domain $(H^2\cap H_0^1)(\mathbf{R}^n\setminus\mathcal{O})$. Then it is well-known that $(\lambda^2+\Delta)^{-1}: L^2\to H^2\cap H_0^1$, holomorphic for $\mathrm{Im}\,\lambda>0$, has a meromorphic extension $R(\lambda): L^2_{\mathrm{comp}}(\mathbf{R}^n\setminus\mathcal{O})\to (H^2\cap H_0^1)_{\mathrm{loc}}(\mathbf{R}^n\setminus\mathcal{O})$ to $\lambda\in\mathbf{C}$ when n is odd and to λ in the logarithmic covering space of $\mathbf{C}\setminus\{0\}$ when n is even. In this talk we only consider this extension in a small angle $e^{-i[0,\theta[}]0,+\infty[$. The poles of the meromorphic extension are called resonances or scattering poles. If λ_0 is a resonance different from 0, we define its multiplicity as the rank (which is finite) of the formal spectral projection $\frac{1}{2\pi i}\int_\gamma R(\lambda)d(\lambda^2)$, where γ is a sufficiently small (to contain no other resonances) positively oriented circle centered at λ_0.

Filipov and Zayev [FZ] have obtained detailed results about the extended resolvent in the case of dimension 2. In a number of works it has been established that there is a constant $C>0$, depending on the geometry of the obstacle and the regularity of the boundary, such that there are only finitely many resonances in a domain of the form

$$\mathrm{Im}\,\lambda\ge -C(\mathrm{Re}\,\lambda)^{\frac13},\ \mathrm{Re}\,\lambda\ge 1. \tag{1}$$

In the case $n=3$, this was obtaind by Babich and Grigoreva [BG], in the case of obstacles with analytic boundary it was obtained as a consequence of Lebeau's results [Le] on the diffraction of Gevrey 3 singularities, by G.Popov [P] and Bardos, Lebeau, Rauch [BaLeR]. For C^∞ boundaries in arbitrary dimension, the result was obtained by Hargé and Lebeau [HLe]. We will comment more about the the best known constants below.

Zworski and the author [SZ2] obtained several upper bounds on the number of resonances in domains of the form

$$\mathrm{Im}\,\lambda\ge -C(\mathrm{Re}\,\lambda)^{\frac13},\ 1\le \mathrm{Re}\,\lambda\le r, \tag{2}$$

when $r\to\infty$ as well as in other domains of the same type with the exponent $1/3$ replaced by other values. One result in this direction says that the number of resonances in the domain (2) for any fixed value of C is $\mathcal{O}(r^{n-1})$. More refined results were also obtained when C in (2) approaches the infimum of the set of best known constants. In the case of analytic boundaries even more refined bounds can be obtained in terms of the dynamics of the boundary geodesics and the curvature [S1].

* Centre de Mathématiques, Ecole Polytechnique, (UMR 7640, CNRS)

As for the existence of infinitely many resonances in some region of the form (1) (or even in any sector away from the imaginary axis) very little has been known, at least in dimension ≥ 3. Bardos-Lebeau-Rauch [BaLeR] showed for generic (strictly convex) obstacles with analytic boundary in odd dimension ≥ 3, that there is constant $C>0$ such that the domain (1) contains infinitely many resonances. By adding a simple Tauberian argument, Zworski and the author [SZ3] showed under the same assumptions that the number of resonances in the domain (2) with $C>0$ large enough, grows at least as fast as $r^{\frac{2}{3}-\epsilon}$ for every $\epsilon>0$.

Let Q be the second fundamental form, defined as a positive definite quadratic form on the tangent space $T\partial\mathcal{O}$ of the boundary. Equip the tangent space with the induced Euclidean norm, and define the tangent sphere bundle $S\partial\mathcal{O}\subset T\partial\mathcal{O}$ of normalized tangent vectors. The geodesic flow on $\partial\mathcal{O}$ as a Riemannian manifold with the induced Euclidean metric is then a group $\Phi_t : T\partial\mathcal{O}\to T\partial\mathcal{O}$, $t\in\mathbf{R}$ which conserves $S\partial\mathcal{O}$. Put

$$C_\infty = 2^{-\frac{1}{3}}\cos\frac{\pi}{6}\min_{S\partial\mathcal{O}} Q^{2/3}\zeta_1, \tag{3}$$

$$C_{\rm a} = 2^{-\frac{1}{3}}\cos\frac{\pi}{6}\lim_{T\to+\infty}(\min_{S\partial\mathcal{O}}\frac{1}{T}\int_0^T Q^{2/3}\circ\Phi_t dt)\zeta_1, \tag{4}$$

where $-\zeta_1,\ -\zeta_2,$ .. are the zeros of the Airy function, ordered so that $0<\zeta_1<\zeta_2<..$. Notice that $C_a\geq C_\infty$. We then know:

There is a constant $C>0$, such that there are at most finitely many resonances in

$$\operatorname{Im}\lambda\geq C-C_\infty(\operatorname{Re}\lambda)^{\frac{1}{3}},\ \operatorname{Re}\lambda\geq 1. \tag{5}$$

When $\partial\mathcal{O}$ is analytic, then for every $\epsilon>0$ there are at most finitely many resonances in

$$\operatorname{Im}\lambda\geq -(C_{\rm a}-\epsilon)(\operatorname{Re}\lambda)^{\frac{1}{3}},\ \operatorname{Re}\lambda\geq 1. \tag{6}$$

These results are established in [SZ2] and [S1] respectively, but could undoubtedly be deduced from [HLe] and [BaLeR] respectively. Recently B. and R. Lascar [LL] have extended the second result to the case of obstacles whose boundary is of Gevrey class of order $s<3$.

We now describe the new result ([SZ4]) and consider an obstacle with C^∞ boundary which is not too far from a ball in the sense that:

$$\frac{\sup_{S\partial\mathcal{O}} Q}{\inf_{S\partial\mathcal{O}} Q} < (\frac{\zeta_2}{\zeta_1})^{\frac{3}{2}} = 2,31186\ . \tag{H}$$

Put $k=2^{-1/3}\cos\frac{\pi}{6}\inf_{S\partial\mathcal{O}}Q^{2/3}$, $K=2^{-1/3}\cos\frac{\pi}{6}\sup_{S\partial\mathcal{O}}Q^{2/3}$, so that $K\zeta_1<k\zeta_2$ and the constant C_∞ in (5) is equal to $k\zeta_1$. We then have

Theorem ([SZ4]).
A) There exists a constant $C>0$ such that there are at most finitely many resonances in $\operatorname{Re}\lambda\geq 1$,

$$K\zeta_1(\operatorname{Re}\lambda)^{1/3}+C\leq -\operatorname{Im}\lambda\leq k\zeta_2(\operatorname{Re}\lambda)^{1/3}-C. \tag{7}$$

B) For every sufficiently large and fixed $C > 0$, the number of resonances with $1 \le \operatorname{Re}\lambda \le r$,

$$k\zeta_1(\operatorname{Re}\lambda)^{1/3} - C \le -\operatorname{Im}\lambda \le K\zeta_1(\operatorname{Re}\lambda)^{1/3} + C, \tag{8}$$

is equal to

$$\frac{1+o(1)}{(2\pi)^{n-1}}\Big(\iint_{\{(x',\xi')\in T^*\partial\mathcal{O};|\xi'|\le 1\}} dx'd\xi'\Big)r^{n-1}, \ r \to +\infty.$$

Here $|\xi'|$ is the induced Euclidean norm on the cotangent space of the boundary.

In the following we give an outline of the proof. We work with the method of complex scaling up to the boundary ([SZ1], [HLe]), which permits to define the resonances in a conic neighborhood of $]0,+\infty[$ as the eigenvalues of $-\Delta_{|\Gamma_\theta}$, where Γ_θ is a maximally totally real submanifold of $\mathbf{C}^n$ which coincides with $e^{i\theta}\mathbf{R}^n$ near infinity, where $\theta > 0$ is small and which has the same boundary as $\partial\mathcal{O}$. We only describe this scaling near the boundary, and it will be convenient to introduce a small semi-classical parameter. Near the boundary we introduce geodesic coordinates (x', y_n) so that y_n is the distance from the point x to the boundary and $x' \in \partial\mathcal{O}$ is the corresponding projection. In these coordinates:

$$-h^2\Delta - 1 - h^{2/3}z = (hD_{y_n})^2 - 2y_nQ(x',y_n,hD_{x'};h) + R(x',hD_{x'};h) - 1 - h^{2/3}z, \tag{9}$$

where Q and R are elliptic semiclassical differential operators. To the leading order R is equal to $-h^2\Delta_{\partial\mathcal{O}}$, the Laplace Beltrami operator of the boundary, and the principal symbol $Q(x',\xi')$ of $Q(x',0,hD_{x'};h)$ can be indentified with the second fundamental form. Following an observation of Hargé and Lebeau [HLe], as in [SZ2], [S1], we put $y_n = e^{i\pi/3}x_n$. Assuming also $Q(x',y_n,hD_{x'};h) = Q(x',hD_{x'};h)$ to simplify the exposition, we get

$$e^{-2\pi i/3}((hD_{x_n})^2 + 2x_nQ(x',hD_{x'};h)) + R(x',hD_{x'}) - 1 - h^{2/3}z. \tag{10}$$

The idea is to try to treat this operator as a degenerate elliptic one. Put $x_n = h^{2/3}t$, divide by $h^{2/3}$ and write x instead of x':

$$e^{-2\pi i/3}(D_t^2 + 2tQ(x,hD_x;h)) + h^{-2/3}(R(x,hD_x;h) - 1) - z. \tag{11}$$

We want to microlocalize in the boundary variables, and the most crucial region is of course the one given by the glancing hypersurface $\Sigma := \{(x,\xi) \in T^*\partial\mathcal{O};\ \lambda = 0\}$, where $\lambda := h^{-2/3}(R(x,\xi) - 1)$ and $R = |\xi|^2$ also denotes the principal symbol of $R(x,hD_x;h)$. The eigenvalues of

$$e^{-2\pi i/3}(D_t^2 + 2tQ(x,\xi)) + \lambda - z$$

on the positive half-line with Dirichlet condition are given by

$$e^{-2\pi i/3}(2Q)^{2/3}\zeta_j + \lambda - z. \tag{12}$$

We therefore expect to be able to reduce problems to the study of h-pseudodifferential operators on the boundary with principal symbols given by (12) or rather by a $N \times N$

system whose principal symbol vanishes precisely when one of the values (12) does, for $j = 1, .., N$. As in the earlier works on diffraction (see for instance [Le]) this leads to second microlocalization. In our setting this is because we first need to apply a Fourier integral operator to reduce $R(x, hD_x; h) - 1$ to hD_{x_1}. The second microlocal calculus then concerns h-quantizations of symbols $a(x, \xi, \lambda; h)$, $\lambda = h^{-2/3}\xi_1$, where $\partial^\alpha_{(x,\xi)}\partial^\ell_\lambda a = \mathcal{O}(\langle\lambda\rangle^{m-\ell})$.

Using such a calculus also with operator valued symbols, we construct auxiliary operators $R_+ : C^\infty(\mathbf{R}^n \setminus \mathcal{O}) \to C^\infty(\partial\mathcal{O})$, $R_- : C^\infty(\partial\mathcal{O}) \to C^\infty(\mathbf{R}^n \setminus \mathcal{O})$ such that the Grushin problem

$$\begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}, \tag{13}$$

is wellposed in natural function spaces for z in a region

$$\Omega =]a_2, a_4[- i]a_1, a_3[, \tag{14}$$

where a_j are independent of h though $a_4 - a_2$ is chosen large, and

$$0 < a_1 < \zeta_1 \cos\frac{\pi}{6} \inf(2Q)^{2/3} < \zeta_1 \cos\frac{\pi}{6} \sup(2Q)^{2/3} < a_3 < \zeta_2 \cos\frac{\pi}{6} \inf(2Q)^{2/3}, \tag{15}$$

where the supremum and the infimum are taken over the glancing hypersurface. For the inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix},$$

the operator E_{-+} has roughly the principal symbol $E^0_{-+} = z - (\lambda + e^{-2\pi i/3}\zeta_1(x, \xi))$, where we put $\zeta_1(x, \xi) = (2Q(x, \xi))^{2/3}\zeta_1$. Let $W =]b_2, b_4[- i]b_1, b_3[$ be relatively compact in Ω with b_1, b_3 satisfying (15). We observe that E^0_{-+} is elliptic (in the 2nd microlocal sense) for $z \in]a_2, a_4[- i(]a_1, b_1[\cup]b_3, a_3[)$ when λ is bounded.

Let $f = f(z)$ be holomorphic in Ω with $|f| \le 1$ in $(]a_2, b_2[\cup]b_4, a_4[) - i]a_1, a_3[$. In the spirit of the local traceformula in [S2], we prove

$$\sum_{j=1,3} \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_j} f(z)(E_{-+}(z)^{-1}\partial_z E_{-+}(z) - \widetilde{E}_{-+}(z)^{-1}\partial_z \widetilde{E}_{-+}(z))dz = \sum_{\{z \in W;\, h^{-1}\sqrt{1+h^{2/3}z} \text{ is a resonance}\}} f(\lambda) + \mathcal{O}(h^{1-n+2/3}). \tag{16}$$

Here γ_1, γ_3 are the horizontal parts of the positively oriented boundary of an intermediate rectangle Γ with $W \subset\subset \Gamma \subset\subset \Omega$, and $\widetilde{E}_{-+}$ is obtained from E_{-+} by a finite rank (of the order $h^{1-n+2/3}$) perturbation of E_{-+}, and is invertible for $z \in \Omega$.

If $a_4 - a_2 > a_3 - a_1$ and b_j are suitably chosen, then we can take f to be a Gaussian, centered at the middle of Ω and independent of h. Using the second microlocal calculus, we see that the LHS of (16) is

$$\frac{h^{1-n+2/3}}{(2\pi)^{n-1}} \iint_{\Sigma\times\mathbf{R}} f(\lambda + e^{-2\pi i/3}\zeta_1(\omega))1_{I(\omega)}(\lambda) L_\Sigma(d\omega)d\lambda + \mathcal{O}(h^{1-n+2/3}), \tag{17}$$

where L_Σ denotes the Liouville measure on Σ, and $I(\omega)$ is the interval of values λ such that $\lambda + e^{-2\pi i/3}\zeta_1(\omega)$ belongs to W. f can be chosen to be large near the middle of Ω and to satisfy the bound prior to (16). We can then arrange so that the integral in (17) dominates over the remainder. Using this in the trace formula (16), we get a lower bound on the number of resonances which is of the right order of magnitude. To get the full asymptotic result, we fix a_1, a_3 and let $a_4 - a_2 =: L$ be very large. Then it is still possible to have (16) (though the choices of Grushin problem and of $\widetilde{E}_{-+}$ will have to depend on L,) now with a remainder $\mathcal{O}(Lh^{1-n+2/3})$, and we notice in this case that we can find Gaussians whose restrictions to Ω take their values in a small sector around the positive half-axis. We are then almost in the situation of sums and integrals of positive quantities and can conclude by standard arguments.

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