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The Role of Green's Functions in Inverse Scattering at Fixed Energy

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In this talk I will discuss some key steps in the recovery of electric and magnetic fields from the scattering amplitude at fixed energy. This is part of my joint work with G. Eskin [ER], but I hope that the version here will be a more transparent. The Hamiltonian for a particle in external electric and magnetic fields with potentials $A(x)$ and $V(x)$ in R^n is given by

$$H = (i\partial_x + A(x))^2 + V(x) = -\Delta + P(x, D),$$

so that $P(x, D)$ is a symmetric operator of order one. Since we wish to solve the inverse problem at fixed energy, it is natural to assume that the coefficients decay exponentially, and we assume

$$|\partial^\alpha V(x)| < Ce^{-\delta|x|}, |\partial^\alpha A(x)| < Ce^{-\delta|x|},$$

for $|\alpha| < n + 6$. To define the scattering amplitude, we introduce the distorted plane waves at energy k^2 for this problem. These are the solutions of $(H - k^2)\phi = 0$ of the form $\phi = exp(i\zeta \cdot x) + v$ with $|\zeta| = k$, where $v = \lim_{\epsilon \to 0_+} v_\epsilon$ and v_ϵ is the L^2 solution of

$$(-\Delta - (k^2 + i\epsilon))v_\epsilon = -P(x, D)(e^{i\zeta \cdot x} + v_\epsilon).$$

This is, of course, the construction of distorted plane waves by the limiting amplitude principle, and with the strong hypotheses on the potentials used here one can show that the limit v exists in the appropriate spaces for any $k > 0$. This is essentially a consequence of [A] and [H], see [ER, p. 200]. The scattering amplitude is defined in terms of the asymptotics of v as $|x|$ goes to infinity. Setting $r = |x|$ and $\theta = x/|x|$, we have

$$v(r\theta, \zeta) = \frac{e^{ikr}}{r^{n-1/2}}(a(\theta, \zeta) + O(1/r)),$$

where $a(\theta, \zeta)$ is the scattering amplitude.

Theorem [ER]: When $n > 2$, $V(x)$ and $rotA(x)$, the magnetic field, can be recovered from $a(\theta, k\omega)$, $(\theta, \omega) \in S^{n-1} \times S^{n-1}$.

Remark: One cannot expect to recover $A(x)$, because the conjugating H with multiplication by $exp(if)$, f of compact support, does not change the scattering amplitude, but it replaces $A(x)$ by $A(x) + grad(f)$.

In the case that V and A have compact support this theorem is a simple consequence of the result of Nakamura, Sun and Uhlmann [NSU] which is based on their earlier work [S] and [NU]. The case $A = 0$, i.e. no magnetic potential but exponentially decaying electric potential, was proven by Novikov in [N]. Isozaki discusses the method of [ER] from different point of in [I].

1. The Green's Functions.

If we define $g = P(x, D)(exp(i\zeta \cdot x) + v)$ and let h denote the Fourier transform of g, then we have

$$v = (2\pi)^{-n} lim_{\epsilon\to 0_+} \int_{R^n} \frac{h(\eta)e^{i\eta\cdot x}}{\eta^2 - k^2 - i\epsilon} d\eta$$

$$= (2\pi)^{-n} \int_{R^n} \frac{h(\eta)e^{i\eta\cdot x}}{\eta^2 - k^2 - i0} d\eta$$

.

Using this representation for v, we can write the Fourier transform of the equation

$$(-\Delta - k^2)v + P(x, D)v = -P(x, D)e^{i\zeta\cdot x}$$

as

$$Eq.(1) \qquad h(\xi) + (2\pi)^{-n} \int_{R^n} \frac{\hat{p}(\xi - \eta, \eta)h(\eta)}{\eta^2 - k^2 - i0} d\eta = -\hat{p}(\xi - \zeta, \zeta),$$

where $\hat{p}(\xi, \zeta)$ is the Fourier transform of the symbol $p(x, \zeta)$ in x. Note that $h = h(\eta, \zeta)$, though we will often suppress the second variable as in (1). Equation (1) is just a version of the Lipman-Schwinger equation, but it is particularly well suited for inverse scattering. If one computes the asymptotics of v from the integral representation above, one finds $a(\theta, \zeta) = c(n, k)h(k\theta, \zeta)$. The coefficient $c(3, k) = 1/(4\pi)$, but it is more complicated in other dimensions. Thus (1) expresses the relation between the coefficients and the scattering amplitude in a particularly compact form.

Equation (1) is a Lipman-Schwinger equation in terms of the outgoing Green's function for $-\Delta - k^2$, i.e. the inverse Fourier transform of the distribution $(\eta^2 - k^2 - i0)^{-1}$. As it stands, (1) is not useful for the inverse problem because the integral term in it involves values of h off the sphere $|\xi| = k$, and it contains no parameters that could be used to suppress that term. A method for overcoming this was introduced by Faddeev, [F]. One can replace the Green's function in (1) by a new Green's function depending on several parameters. Moreover, this can be done in such a way that the solution of the new equation for all parameter values will still be determined on the sphere $|\xi| = k$ by the scattering amplitude at energy k^2. The new parameters will provide the freedom needed to solve the inverse problem. Faddeev's method in this problem is as follows. Define the distribution

$$g_{\nu,\sigma}(f) = \lim_{\epsilon\to 0_+} \int_{R^n} \frac{f(\xi)}{\xi^2 + i\epsilon(\nu\cdot\xi - \sigma)} d\xi$$

$$= \int_{R^n} \frac{f(\xi)}{\xi^2 + i0(\nu\cdot\xi - \sigma)} d\xi,$$

where $|v| = 1$ and σ is real. We let $h^*(\xi, \sigma)$ denote the solution, assuming it exists, of the equation obtained by replacing $(\eta^2 - k^2 - i0)^{-1}$ in (1) by $(\eta^2 - k^2 + i0(\nu\cdot\xi - \sigma)^{-1}$. The function $h^* = h^*_\nu(\xi, \zeta, \sigma)$, but we will often suppress the dependence on ν and ζ. The distributions $g_{\nu,\sigma}$ and

$$g_o(f) = \int_{R^n} \frac{f(\xi)}{\xi^2 - k^2 - i0} d\xi$$

only differ by a measure supported on the sphere $|\xi| = k$. One computes

$$g_{\nu,\sigma}(f) = g_o(f) - d(k,n)\int_{k\omega\cdot\nu>\sigma} f(k\omega)d\omega,$$

and this relation leads to

$$Eq.(2) \qquad h^*(\xi,\zeta,\sigma) = h(\xi,\zeta) - d(k,n)\int_{k\omega\cdot\nu<\sigma} h(\xi,k\omega)h^*(h(k\omega,\zeta,\sigma)d\omega$$

Hence, assuming that (2) is uniquely solvable for h^*, which turns out to be a consequence of the unique solvability of the defining equation for h^*, the restriction of h^* to $|\xi| = k$ is determined by the restriction of h to $|\xi| = k$, i.e. the scattering amplitude.

It is convenient here to change variables in h^* and introduce h_* defined by $h_*(\xi,\zeta,\sigma) = h^*(\xi+\sigma\nu,\zeta+\sigma\nu,\sigma)$. Then h_* is a solution of

$$Eq.(3) \qquad h_*(\xi) + (2\pi)^{-n}\int_{R^n}\frac{\hat{p}(\xi-\eta,\eta+\sigma\nu)h_*(\eta)}{(\eta+\sigma\nu)^2-k^2+i0\eta\cdot\nu}d\eta = -\hat{p}(\xi-\zeta,\zeta+\sigma\nu)$$

.

At this point we would like to extend the variable σ into the upper half plane. Since the function $((\eta+i\tau\nu)^2-k^2)^{-1}$ is locally integrable for $\tau > 0$ and has the limit $(\eta^2-k^2+i0\eta\cdot\nu)^{-1}$ as τ goes to 0_+, it is natural to define $h_*(\xi,\zeta,i\tau)$ by the equation

$$Eq.(4) \qquad h_*(\xi,\zeta,i\tau) + (2\pi)^{-1}\int_{R^n}\frac{\hat{p}(\xi-\eta,\eta+i\tau\nu)h_*(\eta,\zeta,i\tau)}{(\eta+i\tau\nu)^2-k^2}d\eta = -\hat{p}(\eta-\zeta,\eta+i\tau\nu)$$

This is the final version of Lipman-Schwinger equation that we will need, and it involves the last of the fundamental solutions (Green's functions) that we will use. To see the potential usefulness of $h_*(\xi,\zeta,i\tau)$ consider its restriction to the curves $\gamma_d(s) = (\xi(s),\zeta(s),i\tau(s))$ where $d \in R^n$ and $|d| < 2k$, and

$$Eq.(5) \qquad \xi(s) = d/2 + s\mu, \zeta(s) = -d/2 + s\mu, \tau(s) = \sqrt{s^2 + d^2/4 - k^2}$$

Here μ and ν are chosen so that $|\mu| = |\nu| = 1$, $d\cdot\mu = d\dot{\nu} = \mu\cdot\nu = 0$ – this is why we need $n > 2$. If one assumes that the integral term in (4) goes to zero as τ goes to infinity – which is not hard to show when $A = 0$ – then the asymptotic behavior of $h_*(\xi(s),\zeta,i\tau(s))$ will determine the asymptotic behavior of $\hat{p}(d,\zeta(s)+i\tau(s)\nu)$ which brings us very close to recovering A and V. Note that, in the case $A = 0$, $\hat{p}(d,\zeta(s+i\tau(s)\nu)$ is just the Fourier transform of V on the ball $|d| < 2k$ which does determine V, since V is exponentially decreasing. Thus we need to show that h_*, at least on the curves γ_d is determined by the scattering amplitude. Since as s goes to zero (γ_d) crosses the the set of (ξ,ζ,σ) where $h_*(\xi,\zeta,\sigma)$ is determined by the scattering amplitude, we are faced with the following problem:

Can we define $h_*(\xi,\zeta,z)$ on an open subset of $Imz > 0$, $|Rez| < \epsilon$, so that it will be an analytic continuation of $h_*(\xi,\zeta,\sigma)$ into the upper half plane?

2. The Analytic Continuation.

Now we need to introduce the Banach space in which the equations will be solved. The natural choice is a space which reflects the properties of the inhomogeneous term in Eqs. 1,3 and 5, i.e. $\hat{p}$. Our hypotheses coefficients imply that, $\hat{p}(\xi-\zeta,\zeta+z\nu)$ is analytic in (ξ,ζ) for $|Im\xi|, |Im\zeta| < \delta/2$ and decreases like $|\xi|^{-n-6}$ as $|\xi|$ goes to infinity. Hence we introduce

$$B = \{f \in C(|Im\xi| \leq \delta/3) : f \text{ is analytic in } |Im\xi| < \delta/3 \text{and } sup(1+|\xi|)^{n+1}|f(\xi)| \text{ is finite}\}.$$

On this space we have the operator

$$Eq.(6) \qquad [A(i\tau)f](\xi) = \int_{R^n} \frac{\hat{p}(\xi-\eta,\eta+i\tau\nu)f(\eta)}{(\eta+i\tau\nu)^2-k^2} d\eta$$

which we wish to continue analytically to an operator $A(z)$ for z in a set of the form $Imz > 0, |Rez| < \epsilon$. To study $A(i\tau)$ we introduce the coordinates $\eta_\nu = \eta\cdot\nu$, $\eta' = \eta - \eta_\nu\nu$, $r = |\eta'|$ and $\omega = \eta'/|\eta'|$. Since for $z = \sigma + i\tau$,

$$(\eta+z\nu)\cdot(\eta+z\nu) = k^2 = r^2 + (\eta_\nu+\sigma)^2 - \tau^2 - k^2 2i\tau(\eta_\nu+\sigma),$$

we see

a) for $|\sigma| < \epsilon$ and $|\eta_\nu| > 2\epsilon$ the denominator in the integrand in (6) does not vanish.

b) for $|\sigma| < \epsilon$ and $|\eta_\nu| < 3\epsilon$ and ϵ sufficiently small compared to k the denominator of the integrand in (6) will not vanish for $r < k/2$ or $r > 2(\tau^2+k^2)^{1/2}$, and

c) for $|\sigma| < \epsilon$ and $|\eta_\nu| < 3\epsilon$ and ϵ sufficiently small, if we write

$$(\eta+z\nu)\cdot(\eta+z\nu) = (r-\sqrt{B})(r+\sqrt{B}),$$

we may assume that $Re\sqrt{B}$ is positive and the absolute value of $Im\sqrt{B}$ is bounded by 6ϵ uniformly for $\tau > 0$.

Introducing $\rho(s) = 1$ for $|s| > 2\epsilon$ with support $1 = \rho(s)$ contained in $|s| < 3\epsilon$, we can can multiply the integrand in (6) by $\rho(\eta_\nu)$ and by $1 = \rho(\ eta_\nu)$ producing $A_1(i\tau)$ and $A_2(i\tau)$ with $A(i\tau) = A_1(i\tau) + A_2(i\tau)$. In view of a) the analytic continuation of A_1 is immediate. Using the analyticity of f and $\hat{p}$, we will deform the integration in r from 0 to infinity in the definition of A_2 into the upper half-plane for $\eta_\nu > 0$ and into the lower half-plane for $\eta_\nu < 0$. In view of b) and c) we can do this so that the contour stays on the real axis near $r = 0$ and for r near infinity, stays within the domain of analyticity of f and $\hat{p}$, and the denominator of the integrand in A_2 does not vanish when $i\tau$ is replaced by $\sigma + i\tau$ with $|\sigma| < \epsilon$. This makes the analytic continuation of A_2 possible, and from the form of the contour deformation one sees that $A(z)$ extends continuously to $A(\sigma)$ on the real axis, where $A(\sigma)$ is the operator in (3).

3. Remarks on the rest of the proof.

It is not difficult to show that $A(z)$ is compact on B. Thus by standard results on analytic compact operator valued funcions, if $I + A(z)$ is invertible at one point in

$Imz > 0$, $|Rez| < \epsilon$, it will be invertible everywhere outside a discrete set and its extension to $Imz = 0$ will be invertible outside a closed set of measure zero. When $A = 0$, one can show that the norm of $A(i\tau)$ goes to zero as τ goes to infinity. Hence, in that case $h_*(\xi, \zeta, z)$ is analytic in (ξ, ζ, z) on $\{|Im\xi| < \epsilon\} \times \{|Im\zeta| < \epsilon\} \times \{D\}$ and $h_*(\gamma_d(s))$ is analytic in s. One computes $\lim_{s\to\infty} h_*(\gamma_d(s)) = \hat{V}(d)$ and completes the proof. When the magnetic field is present, the argument is more complicated because we could not prove that $A(i\tau)$ goes to zero as τ goes to infinity, and it probably does not. This required a rather long detour that is really the bulk of [ER]. In the end we found that

$$Eq.(7) \qquad \lim_{s\to\infty} h_*(\gamma_d(s))/s = 2\hat{A}(d) \cdot (\mu + i\nu)$$

.

Varying μ and ν, one sees that this determines $\hat{A}(d)$ for $|d| < 2k$ modulo scalar multiples of d. Thus we recover $A(x)$ modulo a gradient and hence $rotA$. Curiously, (7) is exactly what one would obtain from (4) if $A(i\tau(s))$ went to zero.

Bibliography

[A] Agmon,S. "Spectral properties of Schrödinger operators", Annali di Pisa, Serie IV, 2, 151-218(1975)

[ER] Eskin, G., Ralston, J. "Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy",Commun. Math. Phys. 173, 199-224(1995)

[F] Faddeev, L.D. "The inverse problem of quantum scattering II", J. Sov. Math. 5, 334-396(1976)

[H] Hörmander, L. "Uniqueness theorems for second order differential equations", C.P.D.E.8, 21-64(1983)

[I] Isozaki, H. "Multi-dimensional inverse scattering theory for Schrödinger operators", Reviews in Math. Phys. 8, 591-622(1996)

[NU] Nakamura, G., Uhlmann, G. "Global uniqueness for an inverse boundary value problem arising in elasticity", Invent. Math. 118, 457-474(1994)

[NSU] Nakamura, G., Sun, Z., Uhlmann, G. "Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field", Math. Ann. 303,377-388(1995)

[N] Novikov, R,G, "The inverse scattering problem at fixed energy for the three dimensional Schrödinger operator with an exponentially decreasing potential", Commun Math, Phys. 161 569-595(1994)

[S] Sun, Z. "An inverse boundary value problem for Schrödinger operator with vector potentials, Trans. AMS 338(2), 953-969(1993)