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1. Statement of the problem

Consider the spectral problem

$$Av = \nu v\,, \tag{1.1}$$

$$\left(B^{(j)}v\right)\Big|_{\partial M} = 0\,, \qquad j = 1, 2, \dots, m, \tag{1.2}$$

where $\nu > 0$ is the spectral parameter and A is a positive selfadjoint elliptic linear differential operator of order $2m$ ($m \in \mathbb{N}$) acting on half-densities on a compact n-dimensional ($n \geqslant 2$) manifold M with boundary $\partial M \neq \varnothing$. The $B^{(j)}$ are linear differential operators describing the boundary conditions. The manifold, its boundary and the coefficients of A, $B^{(j)}$ are assumed to be infinitely smooth, and the problem (1.1), (1.2) is assumed to be regularly elliptic. By $\mathcal{A}$ we shall denote the abstract operator in $L^2(M)$ associated with the eigenvalue problem (1.1), (1.2).

Consider the time-dependent unitary operator $\mathbf{U}(t) : L^2(M) \to L^2(M)$ defined as

$$\mathbf{U}(t) \overset{\text{def}}{=} \exp(-it\mathcal{A}^{1/(2m)}) = \sum_{k=1}^{+\infty} \exp(-it\lambda_k)\, v_k(x) \int\limits_{M_y} (\,\cdot\,)\, \overline{v_k(y)}\, dy\,. \tag{1.3}$$

Here $\lambda_k = \nu_k^{1/(2m)} > 0$, ν_k are the eigenvalues and v_k are the orthonormalized eigenfunctions of the problem (1.1), (1.2), and $t \in (-\infty\,, +\infty)$ is a parameter. The operator (1.3) is called the *wave group*. The wave group plays an important role in the spectral theory of partial differential operators: knowledge of the singularities of the Schwartz kernel of the operator $\mathbf{U}(t)$ allows one to derive with high accuracy (by use of Fourier Tauberian theorems) asymptotics of the counting function $N(\lambda)$ and of the spectral function $e(\lambda, x, y)$.

The aim of this paper is to outline a scheme for the effective (modulo C^∞) construction of the Schwartz kernel of the wave group.

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As the manifold M has a boundary it is difficult to achieve this aim fully. So we will be forced to restrict ourselves by introducing some microlocalization in T^*M and some localization in t. Microlocalization in T^*M will be introduced by studying the operator

$$\mathbf{U}_P(t) \stackrel{\text{def}}{=} \mathbf{U}(t)\,P \tag{1.4}$$

instead of the original operator $\mathbf{U}(t)$. Here P is a pseudodifferential operator of order $\mathbf{p}$ satisfying certain acceptability conditions (see Section 2). Localization in t will be introduced by performing our constructions on the time interval (T_-, T_+) which will be smaller than the original time interval $(-\infty, +\infty)$.

Sections 2–6 prepare the tools necessary for our construction, and Section 7 contains the main result.

2. Acceptable pseudodifferential operators

Let us introduce first the necessary notation.

By $\overset{\circ}{M} \stackrel{\text{def}}{=} M \setminus \partial M$ we denote the interior of M.

By T^*M, $T^*\overset{\circ}{M}$, $T^*\partial M$ we denote the cotangent bundles on M, $\overset{\circ}{M}$, ∂M respectively. By $T'M$, $T'\overset{\circ}{M}$, $T'\partial M$ we denote the cotangent bundles T^*M, $T^*\overset{\circ}{M}$, $T^*\partial M$ with the zero section ($\xi = 0$, $\xi' = 0$, see below) excluded.

Let us denote local coordinates (and points) on M, ∂M by $x = (x_1, x_2, \ldots, x_n)$, $x' = (x_1, x_2, \ldots, x_{n-1})$ respectively, and their dual coordinates on the fibers T_x^*M, $T_x^*\partial M$ — by $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1})$. Thus (x, ξ), (x', ξ') are local coordinates (or points) on the cotangent bundles T^*M, $T^*\partial M$.

Near ∂M we always use special local coordinates $x = (x', x_n)$ such that $\partial M = \{x_n = 0\}$, and $x_n > 0$ for points in $\overset{\circ}{M}$; consequently $\xi = (\xi', \xi_n)$. Moreover, we fix (specify) once and for all the choice of the coordinate x_n near ∂M.

By A, $B^{(j)}$ we denote linear differential operators of orders $2m$, $0 \leqslant m_j < 2m$ respectively the coefficients of which are complex-valued infinitely differentiable functions of x, x'; these operators appear in (1.1), (1.2). By $A_{2m}(x, \xi)$, $B^{(j)}_{m_j}(x', \xi)$ we denote the principal symbols of the operators A, $(B^{(j)}\,\cdot\,)\big|_{\partial M}$, i.e. homogeneous polynomials of degrees $2m$, m_j in ξ obtained by leaving only the leading (of orders $2m$, m_j) derivatives in A, $B^{(j)}$, replacing each $D_{x_k} \stackrel{\text{def}}{=} -i\partial/\partial x_k$ by ξ_k, $k = 1, 2, \ldots, n$.

We assume, without loss of generality, that the operators $(B^{(j)}\,\cdot\,)\big|_{\partial M}$ act from the space of M-half-densities to the space of ∂M-half-densities. Note that the restriction of a M-half-density to ∂M is a ∂M-half-density because we have fixed the coordinate x_n.

The first condition on the pseudodifferential operator P is that the support of its Schwartz kernel $p(x, y)$ is separated from the boundary of the manifold $M \times M$.

In order to formulate the other conditions (see Definition 2.4 below) we have to consider the following Hamiltonian system of equations on $T'M$:

$$\dot{x}^* = h_\xi(x^*, \xi^*), \qquad \dot{\xi}^* = -h_x(x^*, \xi^*)\,. \tag{2.1}$$

Here $h(x,\xi) = (A_{2m}(x,\xi))^{1/(2m)} > 0$ is our Hamiltonian, the dot $\dot{}$ denotes differentiation with respect to time t, the subscripts denote partial derivatives, and $(x^*,\xi^*) = (x^*(t;y,\eta),\xi^*(t;y,\eta))$ is the solution (trajectory) satisfying the initial condition

$$(x^*(0;y,\eta),\xi^*(0;y,\eta)) = (y,\eta) \in T'\overset{\circ}{M}.$$

As our manifold has a boundary it may happen that at some moment of time $t = \tau$ the ray x^* hits the boundary. In this case we shall reflect the trajectory (x^*,ξ^*) in accordance with the following reflection law: $x^*(t)$ and $\xi^{*\prime}(t)$ are continuous continuous at the moment of reflection and the value of the Hamiltonian is preserved. It is easy to see that this reflection law may give up to m reflected trajectories corresponding to a given incident trajectory. When writing $(x^*(t;y,\eta),\xi^*(t;y,\eta))$ we shall assume that this trajectory depends continuosly on $(t;y,\eta)$, which means that we have specified a particular type of reflection.

Let $T_- < 0 < T_+$ be some fixed numbers.

DEFINITION 2.1. We shall say that the trajectory $(x^*(t;y,\eta),\xi^*(t;y,\eta))$ is *well-defined* on the interval (T_-, T_+) if it experiences on this time interval a finite number of reflections, and at each moment of reflection $\tau \in (T_-, T_+)$ we have $\xi^{*\prime}(\tau;y,\eta) \neq 0$, and all the ξ_n-roots of the algebraic equation

$$A_{2m}(x^{*\prime}(\tau;y,\eta),0,\xi^{*\prime}(\tau;y,\eta),\xi_n) = A_{2m}(y,\eta) \tag{2.2}$$

are simple.

The last condition in Definition 2.1 implies, in particular, that the reflections are transversal. However this condition has wider implications. Note that when we say "the ξ_n-roots of the algebraic equation are simple" we mean *all the roots in the complex plane*, not only the real ones. The role of the complex ξ_n-roots will become clear in Section 4 when we introduce the concept of a boundary layer oscillatory integral associated with the canonical transformation. In a sense, the complex ξ_n-roots correspond to Hamiltonian trajectories which leave after reflection the real space and become complex (in the analytic situation this statement has a precise meaning).

DEFINITION 2.2. We shall say that the point $(y,\eta) \subset T'\overset{\circ}{M}$ is *acceptable* if it satisfies the following three conditions:

(1) the Hamiltonian trajectory originating from this point does not reach the boundary on the time interval $(T_-, 0)$;
(2) any billiard trajectory originating from this point is well-defined on the time interval (T_-, T_+) (in the sense of Definition 2.1) and does not reach the boundary at the moment $t = T_+$;
(3) for any billiard trajectory $(x^*(t;y,\eta),\xi^*(t;y,\eta))$ originating from this point at any moment of reflection $t = \tau \in (0, T_+)$ the number $\nu = A_{2m}(y,\eta)$ is not an eigenvalue of the auxiliary one-dimensional spectral problem

$$A_{2m}(x^{*\prime}(\tau;y,\eta),0,\xi^{*\prime}(\tau;y,\eta),D_{x_n})v = \nu v, \tag{2.3}$$

$$(2.4) \qquad \left(B^{(j)}_{m_j}(x^{*\prime}(\tau;y,\eta),\xi^{*\prime}(\tau;y,\eta),D_{x_n})v\right)\Big|_{x_n=0} = 0\,, \quad j=1,2,\ldots,m,$$

on the half-line $0 \leqslant x_n < +\infty$.

Definition 2.2 (as well as all other definitions in the remainder of this section) obviously depends on our choice of the numbers T_- and T_+ .

DEFINITION 2.3. We shall say that the open conic set $O \subset T'\overset{\circ}{M}$ is acceptable if it is connected, simply connected and all the points $(y,\eta) \in O$ are acceptable.

DEFINITION 2.4. We shall say that the pseudodifferential operator P is acceptable if there exists an acceptable open conic set $O \subset T'\overset{\circ}{M}$ containing $\operatorname{cone\,supp} P$.

In order to understand the properties of acceptable points and sets let us now introduce the notion of *type* of a billiard trajectory.

DEFINITION 2.5. Consider a billiard trajectory $(x^*(t;y,\eta),\xi^*(t;y,\eta))$ which originates from an acceptable point $(y,\eta) \in T'\overset{\circ}{M}$. Let

$$0 < t^*_1(y,\eta) < t^*_2(y,\eta) < \ldots < t^*_{\mathbf{r}}(y,\eta) < T_+$$

be the moments of reflection of this billiard trajectory; $\mathbf{r} \geqslant 0$. (Note that in view of Definitions 2.1, 2.2 the number of reflections $\mathbf{r}$ is finite.) For $i = 1,2,\ldots \mathbf{r}$ consider the equation

$$(2.5) \qquad h(x^{*\prime}(t^*_i(y,\eta);y,\eta),0,\xi^{*\prime}(t^*_i(y,\eta);y,\eta),\xi_n) \; = \; h(y,\eta)$$

(this is the conservation of the Hamiltonian condition which appears in our definition of the reflection law). Having numerated the real ξ_n-roots of the equation (2.5) in order of growth, let us denote by $2\mathfrak{n}_i$ the sequential number of the root $\xi_n = \xi^*_n(t^*_i(y,\eta)+0;y,\eta)$. Then the multiindex $\mathfrak{n} = \mathfrak{n}_1|\mathfrak{n}_2|\ldots|\mathfrak{n}_{\mathbf{r}}$ is called the *type* of the billiard trajectory.

REMARK 2.6. For real ξ_n the equation (2.5) is equivalent to

$$(2.6) \qquad A_{2m}(x^{*\prime}(t^*_i(y,\eta);y,\eta),0,\xi^{*\prime}(t^*_i(y,\eta);y,\eta),\xi_n) \; = \; A_{2m}(y,\eta)\,.$$

The equation (2.6) is an algebraic equation of order $2m$, so it can have not more that $2m$ real ξ_n-roots. Consequently each of the natural numbers $\mathfrak{n}_i$ appearing in the type of a billiard trajectory $\mathfrak{n} = \mathfrak{n}_1|\mathfrak{n}_2|\ldots|\mathfrak{n}_i|\ldots|\mathfrak{n}_{\mathbf{r}}$ can take not more than m possible values: $1 \leqslant \mathfrak{n}_i \leqslant m$.

Definition 2.5 basically means that a billiard trajectory is uniquely determined on our time interval by its starting point and type.

Further on in this paper for the sake of convenience we always deal with the equation (2.6) instead of (2.5).

LEMMA 2.7. *Let (y,η) be an acceptable point. Then there exists only a finite number of types of billiard trajectories originating from this point.*

LEMMA 2.8. *Let $O \subset T'\overset{\circ}{M}$ be an acceptable open conic set. Then the set of types of billiard trajectories originating from a point $(y,\eta) \in O$ does not depend on the choice of this point.*

Further on in this paper the operator P will be assumed to be acceptable (in the sense of Definition 2.4), and $O \subset T'\overset{\circ}{M}$ will denote some acceptable open conic set containing $\operatorname{cone\,supp} P$. By $\mathfrak{N}(O)$ we shall denote the set of types of billiard trajectories originating from points $(y,\eta) \in O$; the notation $\mathfrak{N}(O)$ is justified in view of Lemma 2.8. By (y,η) we shall denote points from O.

The Schwartz kernel of the operator (1.4) admits an effective construction for any acceptable pseudodifferential operator P. However the representation of the result in the general case requires the use of graph theory notation. So in order to avoid technical complications we shall assume further on in this paper that we have only one reflection on the time interval $(0, T_+)$, i.e. that the set $\mathfrak{N}(O)$ consists only of multiindices of length 1. The construction for an arbitrary set of types $\mathfrak{N}(O)$ is the same as for one reflection, only the notation is more complicated.

3. Standard oscillatory integrals

Let us denote by $0 < t_1^*(y,\eta) < T_+$ the moment of reflection, and by $2q$ the number of real ξ_n-roots of the equation (2.6) ($i = 1$). Obviously, the set $\mathfrak{N}(O)$ contains in our case (case of one reflection) exactly q elements and these elemets are the numbers $1, 2, \ldots, q$.

By

$$(3.1) \qquad (x^*(t;y,\eta), \xi^*(t;y,\eta)), \qquad t \in (T_-, t_1^*(y,\eta)], \quad (y,\eta) \in O,$$

we shall denote the trajectory before reflection. By

$$(3.2) \quad (x^{(l)*}(t;y,\eta), \xi^{(l)*}(t;y,\eta)), \qquad t \in [t_1^*(y,\eta), T_+), \quad (y,\eta) \in O, \quad l = 1, 2, \ldots, q,$$

we shall denote the real reflected trajectories.

Set

$$\mathfrak{D} \overset{\text{def}}{=} \{ (t;y,\eta) \,:\, (y,\eta) \in O,\, t \in (T_-, t_1^*(y,\eta)] \},$$

$$\mathfrak{C} \overset{\text{def}}{=} \{ (t,x;y,\eta) \,:\, (t;y,\eta) \in \mathfrak{D},\, x = x^*(t;y,\eta) \},$$

and

$$\mathfrak{D}^{(l)} \overset{\text{def}}{=} \{ (t;y,\eta) \,:\, (y,\eta) \in O,\, t \in [t_1^*(y,\eta), T_+) \},$$

$$\mathfrak{C}^{(l)} \overset{\text{def}}{=} \{ (t,x;y,\eta) \,:\, (t;y,\eta) \in \mathfrak{D}^{(l)},\, x = x^{(l)*}(t;y,\eta) \},$$

$l = 1, 2, \ldots, q$. Under this notation the sets $\mathfrak{D}^{(l)}$ are identical for all $l = 1, 2, \ldots, q$, however we will stick to this notation because these sets might become different when one considers the case of an arbitrary number of reflections.

Set also

$$\mathfrak{C}' \overset{\text{def}}{=} \{ (t,x';y,\eta) \,:\, t = t_1^*(y,\eta),\, x' = x^{*\prime}(t_1^*(y,\eta), y, \eta),\, (y,\eta) \in O \}.$$

According to this definition the set $\mathfrak{C}'$ is a subset of $(T_-\,,\,T_+)\times\partial M_x\times O$. However we shall often view the set $\mathfrak{C}'$ as a subset of $(T_-\,,\,T_+)\times M_x\times O$, using the obvious inclusion $\partial M_x=\{x_n=0\}\subset M_x$.

All the phase functions appearing in this paper, including the ones introduced in the following definition, are assumed to be positively homogeneous in η of degree 1 and with non-negative imaginary part. The fact that we allow our phase functions to be complex-valued is crucial, because otherwise we would not be able to produce a global construction.

DEFINITION 3.1. We say that the phase function φ is a standard phase function associated with the canonical transformation (3.1) if it is defined on a connected simply connected conic neighborhood $\mathcal{O}\subset(T_-\,,\,T_+)\times M_x\times O$ of the set $\mathfrak{C}$ and satisfies the following conditions:

$$\varphi_\eta(t,x;y,\eta)\ =\ 0\quad\text{if and only if}\quad(t,x;y,\eta)\in\mathfrak{C}\,,$$
$$\varphi_x(t,x^*;y,\eta)\ =\ \xi^*\,,\quad\forall\,(t;y,\eta)\in\dot{\mathfrak{D}}\,,$$
$$\det\varphi_{x\eta}(t,x;y,\eta)\ \neq\ 0\,,\quad\forall\,(t,x;y,\eta)\in\mathcal{O}\,.$$

By $\mathfrak{F}$ we denote the set of all standard phase functions associated with the canonical transformation (3.1).

DEFINITION 3.1$^{(l)}$. We say that the phase function $\varphi^{(l)}$ is a standard phase function associated with the canonical transformation (3.2) if it is defined on a connected simply connected conic neighborhood $\mathcal{O}^{(l)}\subset(T_-\,,\,T_+)\times M_x\times O$ of the set $\mathfrak{C}^{(l)}$ and satisfies the following conditions:

$$\varphi^{(l)}_\eta(t,x;y,\eta)\ =\ 0\quad\text{if and only if}\quad(t,x;y,\eta)\in\mathfrak{C}^{(l)}\,,$$
$$\varphi^{(l)}_x(t,x^{(l)*};y,\eta)\ =\ \xi^{(l)*}\,,\quad\forall\,(t;y,\eta)\in\mathfrak{D}\,,$$
$$\det\varphi^{(l)}_{x\eta}(t,x;y,\eta)\ \neq\ 0\,,\quad\forall\,(t,x;y,\eta)\in\mathcal{O}^{(l)}\,.$$

By $\mathfrak{F}_l$ we denote the set of all standard phase functions associated with the canonical transformation (3.2).

Properites of phase functions of the type described above were studied in [1]. Let us mention briefly some of these properties:

(1) the classes $\mathfrak{F}$, $\mathfrak{F}_l$ are non-empty;
(2) the classes $\mathfrak{F}$, $\mathfrak{F}_l$ are contractible as topological spaces;
(3) any phase function which is defined locally and satisfies locally the conditions of Definition 2.1 can be extended up to a phase function of the class $\mathfrak{F}$ or $\mathfrak{F}_l$;
(4) it is possible to choose a phase function of the class $\mathfrak{F}$ or $\mathfrak{F}_l$ which is locally linear with respect to x in some local coordinates.

Consider now the expression $\det^2\varphi_{x\eta}$. It is easy to see that it is a a (1/2)-density with respect to x and a $(-1/2)$-density with respect to y. Consequently the argument of this expression does not change under changes of local coordinates. Let us choose a particular continuous branch $\arg_0(\det^2\varphi_{x\eta})$ of the argument $\arg(\det^2\varphi_{x\eta})$, specified by the following condition: $\arg_0(\det^2\varphi_{x\eta})\big|_{t=0,x=y}=0$.

For any phase function $\varphi^{(l)} \in \mathfrak{F}_l$ there exists a phase function $\varphi \in \mathfrak{F}$ such that

$$\varphi^{(l)}\big|_{x\in\partial M} = \varphi\big|_{x\in\partial M}. \tag{3.3}$$

It can be shown that formula (3.3) implies

$$\begin{aligned} h_{\xi_n}\left(x^*|_{t=t_1^*+0}\,,\,\xi^*|_{t=t_1^*+0}\right) \det \varphi^{(l)}_{x\eta}\big|_{\mathfrak{C}'} \\ = h_{\xi_n}\left(x^{(l)*}\Big|_{t=t_1^*}\,,\,\xi^{(l)*}\Big|_{t=t_1^*}\right) \det \varphi_{x\eta}\big|_{\mathfrak{C}'}. \end{aligned} \tag{3.4}$$

Formula (3.4) allows us to choose a particular continuous branch $\arg_0(\det^2\varphi^{(l)}_{x\eta})$ of the argument $\arg(\det^2\varphi^{(l)}_{x\eta})$, specified by the following condition:

$$\arg_0(\det^2\varphi^{(l)}_{x\eta})\Big|_{\mathfrak{C}'} = \arg_0(\det^2\varphi_{x\eta})\big|_{\mathfrak{C}'}.$$

We denote by $S^{\mathbf{p}}$ the class of complex-valued C^∞-functions $a(t,x;y,\eta)$ which admit an asymptotic expansion

$$a(t,x;y,\eta) \sim \sum_{k=0}^{\infty} a_{\mathbf{p}-k}(t,x;y,\eta), \qquad |\eta| \to \infty,$$

with $a_{\mathbf{p}-k}(t,x;y,\eta)$ positively homogeneous in η of degree $\mathbf{p}-k$. We also use the notation $đ\eta = (2\pi)^{-n}\,d\eta_1\,d\eta_2\,\ldots\,d\eta_n$.

DEFINITION 3.2. We say that

$$\mathcal{I}_{\varphi,a}(t,x,y) = \int e^{i\varphi(t,x;y,\eta)}\,a(t,x;y,\eta)\,\varsigma(t,x;y,\eta)\,d_\varphi(t,x;y,\eta)\,đ\eta \tag{3.5}$$

is a standard oscillatory integral associated with the canonical transformation (3.1) if

(1) the phase function φ and the amplitude $a \in S^{\mathbf{p}}$ are defined in a connected simply connected conic neighborhood $\mathcal{O} \subset (T_-\,,\,T_+) \times M_x \times O$ of the set $\mathfrak{C}$;
(2) φ is a standard phase function associated with the canonical transformation (3.1), i.e., $\varphi \in \mathfrak{F}_i$;
(3) $\operatorname{supp} a \subset (T_-\,,\,T_+) \times M_x \times \mathbf{O}$, where $\mathbf{O}$ is some conically compact conic subset of O.

In formula (3.5)

$$d_\varphi(t,x;y,\eta) = (\det{}^2\varphi_{x\eta})^{1/4} = |\det\varphi_{x\eta}|^{1/2}\,e^{i\arg_0(\det^2\varphi_{x\eta})/4}.$$

The function ς in formula (3.5) is a cut-off around the set $\mathfrak{C}$.

A distribution $\mathcal{I}(t,x,y)$ which can be written modulo $C^\infty((T_-\,,\,T_+)\times M_x\times M_y)$ as a standard oscillatory integral (3.5) associated with the canonical transformation (3.1) is called a *standard Lagrangian distribution* of order $\mathbf{p}$ associated with this transformation.

DEFINITION 3.2$^{(l)}$. We say that

$$\mathcal{I}_{\varphi^{(l)},a^{(l)}}(t,x,y) = \int e^{i\varphi^{(l)}(t,x;y,\eta)}\, a^{(l)}(t,x;y,\eta)\, \varsigma^{(l)}(t,x;y,\eta)\, d_{\varphi^{(l)}}(t,x;y,\eta)\, d\!\!\bar{}\,\eta \tag{3.6}$$

is a standard oscillatory integral associated with the canonical transformation (3.2) if

(1) the phase function $\varphi^{(l)}$ and the amplitude $a^{(l)} \in S^{\mathbf{p}}$ are defined in a connected simply connected conic neighborhood $\mathcal{O}^{(l)} \subset (T_-, T_+) \times M_x \times O$ of the set $\mathfrak{C}^{(l)}$;
(2) $\varphi^{(l)}$ is a standard phase function associated with the canonical transformation (3.2), i.e., $\varphi^{(l)} \in \mathfrak{F}_l$;
(3) $\operatorname{supp} a^{(l)} \subset (T_-, T_+) \times M_x \times \mathbf{O}$, where $\mathbf{O}$ is some conically compact conic subset of O.

In formula (3.6)

$$d_{\varphi^{(l)}(t,x;y,\eta)} = (\det{}^2\varphi^{(l)}_{x\eta})^{1/4} = |\det\varphi^{(l)}_{x\eta}|^{1/2}\, e^{i\arg_0(\det^2\varphi^{(l)}_{x\eta})/4}.$$

The function $\varsigma^{(l)}$ in formula (3.6) is a cut-off around the set $\mathfrak{C}^{(l)}$.

A distribution $\mathcal{I}^{(l)}(t,x,y)$ which can be written modulo $C^\infty((T_-, T_+) \times M_x \times M_y)$ as a standard oscillatory integral (3.6) associated with the canonical transformation (3.2) is called a *standard Lagrangian distribution* of order $\mathbf{p}$ associated with this transformation.

Any standard oscillatory integral (3.5) can be rewritten (modulo C^∞) with an amplitude $\mathfrak{a}(t;y,\eta)$ independent of x. The restriction of this amplitude $\mathfrak{a}$ to the set $\mathfrak{O}$ is called the *(full) symbol* of our standard Lagrangian distribution. For a given Lagrangian distribution and a given phase function φ the (full) symbol is defined uniquely modulo $S^{-\infty}$. The leading homogeneous term $\mathfrak{a}_{\mathbf{p}}$ (of degree $\mathbf{p}$) of the symbol $\mathfrak{a}$ is called the *principal symbol.* The principal symbol does not depend on the choice of a particular phase function, and is determined by the standard Lagrangian distribution itself.

It will be convenient for us to introduce the linear operator $\mathfrak{S}$ mapping the original amplitude $a(t,x;y,\eta)$ of an oscillatory integral into the corresponding symbol $\mathfrak{a}(t;y,\eta)$. The operator $\mathfrak{S}$ depends, of course, on the phase function φ. This operator admits an asymptotic expansion into a series of positively homogeneous in η terms:

$$\mathfrak{S} \sim \sum_{r=0}^{\infty} \mathfrak{S}_{-r},$$

where the operators $\mathfrak{S}_{-r}$ are positively homogeneous in η of degree $-r$. The explicit formulae for the operators $\mathfrak{S}_{-r}$ are

$$\mathfrak{S}_0 = (\,\cdot\,)|_{x=x^*},$$

and

$$\mathfrak{S}_{-r} =$$

$$\left\{ \left[i\, d_\varphi^{-1}\, \partial_\eta^T\, d_\varphi \left(\sum_{k=1}^{2r} \sum_{|\alpha|=k-1} \frac{c_k\, (\varphi_\eta)^\alpha}{\alpha!\, k} \left((\varphi_{x\eta})^{-1} \partial_x \right)^\alpha \right) (\varphi_{x\eta})^{-1} \partial_x \right]^r (\,\cdot\,) \right\} \Bigg|_{x=x^*}$$

for $r \geqslant 1$. Here the coefficients c_k are determined from the following recursive system of linear algebraic equations:

$$\sum_{k=1}^{q} \frac{q!}{k!\,(q-k)!}\, c_k = 1\,, \qquad q = 1, 2, \ldots$$

Similar formulae hold for standard Lagrangian distributions $\mathcal{I}^{(l)}(t,x,y)$ associated with canonical transformations (3.2).

4. Boundary layer oscillatory integrals

When we constructed the reflected trajectories of the Hamiltonian system of equations (2.1) (see Section 2) we used as the initial values of ξ_n^* at $t = t_1^*$ the real ξ_n-roots of the algebraic equation (2.6) ($i = 1$). However, we can also choose as the initial value of ξ_n^* at $t = t_1^*$ one of the $m - q$ complex ξ_n-roots with poitive imaginary part. This gives us $m - q$ complex reflected trajectories

(4.1) $\qquad (x^{(l)*}(t; y, \eta), \xi^{(l)*}(t; y, \eta)), \qquad (y, \eta) \in O\,, \quad l = q+1, q+2, \ldots, m\,.$

These complex trajectories are understood as formal Taylor expansions in powers of $t - t_1^*$. We shall use the sign $\simeq$ to decribe the equality of two formal Taylor expansions in powers of $t - t_1^*$.

Similarly to Definition $3.1^{(l)}$ we introduce

DEFINITION $4.1^{(l)}$. We say that the phase function $\varphi^{(l)}$ is a boundary layer phase function associated with the canonical transformation (4.1) if it is defined on a connected simply connected conic neighborhood $\mathcal{O}^{(l)} \subset (T_-\,, T_+) \times M_x \times O$ of the set $\mathfrak{C}'$ and satisfies the following conditions:

$$\varphi_\eta^{(l)}(t, x; y, \eta) = 0 \quad \text{if and only if} \quad (t, x; y, \eta) \in \mathfrak{C}'\,,$$

$$\varphi(l)(t, x^{(l)*}; y, \eta) \simeq 0\,,$$

$$\varphi_x^{(l)}(t, x^{(l)*}; y, \eta) \simeq \xi^{(l)*}\,,$$

$$\det \varphi_{x\eta}^{(l)}(t, x; y, \eta) \neq 0\,, \quad \forall\, (t, x; y, \eta) \in \mathcal{O}^{(l)}\,.$$

By $\mathfrak{F}_l^{\mathrm{bl}}$ we denote the set of all boundary layer phase functions associated with the canonical transformation (4.1).

The properties of boundary layer phase functions are similar to those of standard phase functions. In particular one can specify the choice of a particular continuous branch $\arg_0(\det^2 \varphi_{x\eta}^{(l)})$ of the argument $\arg(\det^2 \varphi_{x\eta}^{(l)})$,

Similarly to Definition $3.2^{(l)}$ we introduce

DEFINITION $4.2^{(l)}$. We say that

$$\mathcal{I}_{\varphi^{(l)},a^{(l)}}(t,x,y) = \int e^{i\varphi^{(l)}(t,x;y,\eta)}\, a^{(l)}(t,x;y,\eta)\, \varsigma^{(l)}(t,x;y,\eta)\, d_{\varphi^{(l)}}(t,x;y,\eta)\, đ\eta \tag{4.2}$$

is a boundary layer oscillatory integral associated with the canonical transformation (4.1) if

(1) the phase function $\varphi^{(l)}$ and the amplitude $a^{(l)} \in S^{\mathbf{p}}$ are defined in a connected simply connected conic neighborhood $\mathcal{O}^{(l)} \subset (T_-\,,\,T_+)\times M_x \times O$ of the set $\mathfrak{C}'$;

(2) $\varphi^{(l)}$ is a boundary layer phase function associated with the canonical transformation (3.2), i.e., $\varphi^{(l)} \in \mathfrak{F}_l^{\mathrm{bl}}$;

(3) $\operatorname{supp} a^{(l)} \subset (T_-\,,\,T_+)\times M_x \times \mathbf{O}$, where $\mathbf{O}$ is some conically compact conic subset of O.

In formula (4.2)

$$d_{\varphi^{(l)}(t,x;y,\eta)} = (\det{}^2 \varphi^{(l)}_{x\eta})^{1/4} = |\det\varphi^{(l)}_{x\eta}|^{1/2}\, e^{i\arg_0(\det^2\varphi^{(l)}_{x\eta})/4}\,.$$

The function $\varsigma^{(l)}$ in formula (4.2) is a cut-off around the set $\mathfrak{C}'$.

A distribution $\mathcal{I}^{(l)}(t,x,y)$ which can be written modulo $C^\infty((T_-\,,\,T_+)\times M_x \times M_y)$ as a boundary layer oscillatory integral (4.2) associated with the canonical transformation (4.1) is called a *boundary Lagrangian distribution* of order $\mathbf{p}$ associated with this transformation.

Any boundary layer oscillatory integral (4.2) can be rewritten (modulo C^∞) with an amplitude $\mathfrak{a}(t;y,\eta)$ independent of x. The jet of this amplitude $\mathfrak{a}$ is called the *(full) symbol* of our boundary layer Lagrangian distribution. Here by a *jet* we understand the equivalence class of all amplitudes $\mathfrak{a}(t;y,\eta) \in S^{\mathbf{p}}$ defined in some conic neighborhoods of the set $\{t = t_1^*(t;y,\eta)\} \subset (T_-\,,\,T_+)\times O$ which differ by amplitudes with an infinite order zero at $t = t_1^*(t;y,\eta)$. In other words, the jet of the function $\mathfrak{a}$ at $t = t_1^*$ is uniquely determined by the set of Taylor coefficients $\frac{1}{k!}\,\partial_t^k \mathfrak{a}\big|_{t=t_1^*}$, $k = 0,1,2,\ldots$

The leading homogeneous term $\mathfrak{a}_{\mathbf{p}}$ (of degree $\mathbf{p}$) of the symbol $\mathfrak{a}$ is called the *principal symbol*. The principal symbol does not depend (as a jet) on the choice of a particular phase function, and is determined by the boundary layer Lagrangian distribution itself.

The (full) symbol of a boundary layer Lagrangian distribution can be computed by the same formulae as the (full) symbol of a standard Lagrangian distribution, see end of Section 3.

5. Boundary oscillatory integrals

Boundary oscillatory integrals are the oscillatory integrals which appear when one takes the restriction $|_{x\in\partial M}$ of a standard or boundary layer oscillatory integral.

Thus, the phase function φ', amplitude a, and cut-off ς in a boundary oscillatory integral do not depend on the variable x_n.

The theory of boundary oscillatory integrals is similar to that of standard oscillatory integrals (see Section 3); in fact it is slightly simpler because we have one variable less and because ∂M is a manifold without boundary. The only substantial difference with standard oscillatory integrals is that in the case of boundary oscillatory integrals the weight factor should be taken as

$$d_{\varphi'}(t,x';y,\eta) = \left(\det{}^2\begin{pmatrix}\varphi'_{x'\eta}\\ \varphi'_{t\eta}\end{pmatrix}\right)^{1/4}$$
$$= \left|\det\begin{pmatrix}\varphi'_{x'\eta}\\ \varphi'_{t\eta}\end{pmatrix}\right|^{1/2} \exp\left(\frac{i}{4}\arg_0\left(\det{}^2\begin{pmatrix}\varphi'_{x'\eta}\\ \varphi'_{t\eta}\end{pmatrix}\right)\right).$$

This ensures that the boundary oscillatory integral is a half-density with respect to $x' \in \partial M$ and with respect to $y \in M$.

Any boundary oscillatory integral with amplitude $a(t,x';y,\eta)$ can be rewritten (modulo C^∞) with an amplitude $\mathfrak{a}(y,\eta)$ independent of x' and t. This amplitude $\mathfrak{a}$ is called the *(full) symbol* of our boundary Lagrangian distribution. The leading homogeneous term $\mathfrak{a}_{\mathbf{p}}$ (of degree $\mathbf{p}$) of the symbol $\mathfrak{a}$ is called the *principal symbol.* The principal symbol does not depend on the choice of a particular phase function, and is determined by the boundary Lagrangian distribution itself.

Consider now a differential operator $B(x',D_x)$ of order $\mathbf{b}$, and suppose that $(B\,\cdot)|_{\partial M_x}$ acts from the space of half-densities on M_x into the space of half-densities on ∂M_x. In particular, B can be the identity operator, it acts into the space of half-densities on ∂M_x because in Section 2 we agreed to specify once and for all the choice of the "normal" coordinate. Denote by $B_{\mathbf{b}}(x',\xi)$ the principal symbol of $(B\,\cdot)|_{\partial M_x}$.

LEMMA 5.1. *Let $\mathcal{I}(t,x,y)$ be a standard Lagrangian distribution of order $\mathbf{p}$ associated with the canonical transformation (3.1), and let $\mathfrak{a}_{\mathbf{p}}(t;y,\eta)$ be its principal symbol. Then $(B\mathcal{I})|_{\partial M_x}$ is a boundary Lagrangian distribution of order $\mathbf{p}+\mathbf{b}$ with principal symbol*

$$\frac{B_{\mathbf{b}}\big(x^{*\prime}|_{t=t_1^*}\,,\,\xi^*|_{t=t_1^*}\big)\ \mathfrak{a}_{\mathbf{p}}|_{t=t_1^*}}{\sqrt{-h_{\xi_n}\big(x^*|_{t=t_1^*}\,,\,\xi^*|_{t=t_1^*}\big)}}\,.$$

LEMMA 5.2. *Let $\mathcal{I}^{(l)}(t,x,y)$ be a standard Lagrangian distribution of order $\mathbf{p}$ associated with the canonical transformation (3.2), and let $\mathfrak{a}^{(l)}_{\mathbf{p}}(t;y,\eta)$ be its principal symbol. Then $(B\mathcal{I}^{(l)})\big|_{\partial M_x}$ is a boundary Lagrangian distribution of order $\mathbf{p}+\mathbf{b}$ with principal symbol*

$$\frac{B_{\mathbf{b}}\big(x^{(l)*\prime}\big|_{t=t_1^*}\,,\,\xi^{(l)*}\big|_{t=t_1^*}\big)\ \mathfrak{a}^{(l)}_{\mathbf{p}}\Big|_{t=t_1^*}}{\sqrt{h_{\xi_n}\big(x^{(l)*}\big|_{t=t_1^*}\,,\,\xi^{(l)*}\big|_{t=t_1^*}\big)}}\,. \tag{5.1}$$

LEMMA 5.3. *Let $\mathcal{I}^{(l)}(t,x,y)$ be a boundary layer Lagrangian distribution of order $\mathbf{p}$ associated with the canonical transformation (4.1), and let $\mathfrak{a}_{\mathbf{p}}^{(l)}(t;y,\eta)$ be its principal symbol. Then $(B\mathcal{I}^{(l)})\big|_{\partial M_x}$ is a boundary Lagrangian distribution of order $\mathbf{p}+\mathbf{b}$ with principal symbol (5.1).*

6. Characteristic properties of distributions associated with the wave group

Denote by $\mathbf{u}_P(t,x,y)$ the Schwartz kernel of the operator (1.4), that is

$$\mathbf{U}_P(t) = \int_M \mathbf{u}_P(t,x,y)\,(\,\cdot\,)\,dx\,.$$

We will attempt to approximate the distribution $\mathbf{u}_P(t,x,y)$ by the distribution

$$u_P(t,x,y) = \mathcal{I}_{\varphi,\mathfrak{a}} + \sum_{l=1}^{m} \mathcal{I}_{\varphi^{(l)},\mathfrak{a}^{(l)}}\,, \tag{6.1}$$

where

$$\mathcal{I}_{\varphi,\mathfrak{a}}(t,x,y) = \int e^{i\varphi(t,x;y,\eta)}\,\mathfrak{a}(t,x;y,\eta)\,\varsigma(t,x;y,\eta)\,d_\varphi(t,x;y,\eta)\,đ\eta \tag{6.2}$$

is a standard oscillatory integral associated with a canonical transformation (3.1) (see Definition 3.1), and

$$\mathcal{I}_{\varphi^{(l)},\mathfrak{a}^{(l)}}(t,x,y) = \int e^{i\varphi^{(l)}(t,x;y,\eta)}\,\mathfrak{a}^{(l)}(t,x;y,\eta)\,\varsigma^{(l)}(t,x;y,\eta)\,d_{\varphi^{(l)}}(t,x;y,\eta)\,đ\eta \tag{6.3}$$

are standard oscillatory integrals associated with the canonical transformations (3.2) (see Definition $3.2^{(l)}$, $l \leqslant q$) and boundary layer oscillatory integrals associated with the canonical transformations (4.1) (see Definition $4.2^{(l)}$, $l > q$) respectively. The amplitudes $\mathfrak{a}, \mathfrak{a}^{(l)} \in S^{\mathbf{p}}$ are to be determined is such a way that $u_P(t,x,y)$ approximates $\mathbf{u}_P(t,x,y)$ modulo C^∞. Note that these amplitudes should be independent of x (we can always search for oscillatory integrals with amplitudes independent of x in view of the results of Sections 3, 4).

LEMMA 6.1. *Let the oscillatory integrals (6.2), (6.3) be such that*

$$D_t^{2m}\mathcal{I}_{\varphi,\mathfrak{a}} - A_x\mathcal{I}_{\varphi,\mathfrak{a}} \in C^\infty((T_-\,,T_+)\times M_x\times M_y)\,, \tag{6.4}$$

$$D_t^{2m}\mathcal{I}_{\varphi^{(l)},\mathfrak{a}^{(l)}} - A_x\mathcal{I}_{\varphi^{(l)},\mathfrak{a}^{(l)}} \in C^\infty((T_-\,,T_+)\times M_x\times M_y)\,,\quad l=1,2,\ldots,m\,, \tag{6.5}$$

$$(6.6) \qquad (B_x^{(j)} u_P)\Big|_{\partial M_x} \in C^\infty((T_-\,,\,T_+)\times \partial M_x \times M_y)\,, \qquad j = 1,2,\dots,m,$$

$$(6.7) \qquad u_P(0,x,y) - p(x,y) \in C_0^\infty(\mathring{M}_x \times \mathring{M}_y)\,.$$

Then

$$u_P(t,x,y) - \mathbf{u}_P(t,x,y) \in C^\infty((T_-\,,\,T_+)\times M_x \times M_y)\,.$$

Recall that by $p(x,y)$ we denote the Schwartz kernel of the (acceptable) pseudodifferential operator P, that is

$$P = \int_M p(x,y)\,(\,\cdot\,)\,dx\,.$$

Lemma 6.1 is a version of Lemma 2.2.2 from [2] and Theorem 1 from [3].

7. Main result

THEOREM 7.1. *In the case of one reflection for any set of phase functions* $\varphi \in \mathfrak{F}$, $\varphi^{(l)} \in \mathfrak{F}_l$, $l = 1,2,\dots,q$, $\varphi^{(l)} \in \mathfrak{F}_l^{\mathrm{bl}}$, $l = q+1, q+2, \dots, m$, *associated with our Hamiltonian billiards there exist amplitudes* $\mathfrak{a}, \mathfrak{a}^{(l)} \in S^{\mathrm{P}}$, $l = 1,2,\dots,m$, *such that the distribution (6.1) coincides with the Schwartz kernel* $\mathbf{u}_P(t,x,y)$ *of the operator* $\exp(-it\mathcal{A}^{(1/2m)})P$ *modulo* $C^\infty((T_-\,,\,T_+)\times M_x \times M_y)$.

Theorem 7.1 is proved by substituting (6.1)–(6.3) into (6.4)–(6.7), computing the symbols of all the oscillatory integrals (in accordance with the formulae described in Sections 3–5), and integrating the resulting ordinary differential equations in the variable t.

The procedure outlined above is a version of the procedure suggested initially in [2]. The difference is that in this paper we used the technique of global oscillatory integrals with complex phase [1], which significantly simplifies the construction (we do not have to match local oscillatory integrals in time and in space) and makes it invariant with respect to changes of local coordinates.

Our construction is described in greater detail in Chapters 2, 3 of [4].

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