J. Sjöstrand

Wiener type algebras of pseudodifferential operators

Séminaire Équations aux dérivées partielles (Polytechnique) (1994-1995), exp. nº 4, p. 1-19

<http://www.numdam.org/item?id=SEDP_1994-1995____A4_0>

CENTRE
DE
MATHEMATIQUES

Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE

F-91128 PALAISEAU Cedex (FRANCE)
Tél. (1) 69 33 40 91
Fax (1) 69 33 30 19 ; Télex 601.596 F

Séminaire 1994-1995

EQUATIONS AUX DERIVEES PARTIELLES

WIENER TYPE ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS

J. SJÖSTRAND

Exposé n° IV

20 décembre 1994

Wiener type algebras of pseudodifferential operators

Johannes Sjöstrand
Centre de Mathématiques, Ecole Polytechnique
F-91128 Palaiseau cedex, France
URA 169, CNRS

0. Introduction.

In [S1] we introduced a Wiener type algebra of pseudors (short word for "pseudodifferential operators" to be used below), which is an "explicit" Banach algebra containing $\mathrm{Op}(S^0_{0,0})$. In particular there is no loss of derivatives in the calculus. (For less explicit semi-norms allowing a "no loss" calculus, see [GraUeW], [GraSch] and further references given there.) We hope that these types of algebras may find applications in non-linear problems or for problems in high dimension.

Here we review most of [S1] and simplify some of the arguments there. We also establish a non-commutative Wiener lemma: If A is an operator of our class which has an L^2 bounded left inverse B, then B is also in the class. The proof is an extension of a (possibly new) proof of the classical Wiener lemma, which makes no use of Fourier transform or of abstract theory, and which might apply to other classes of operators as well. (We do not exclude the possibility of an abstract proof.) We also develop a calculus of h-pseudors, based on a pretty sharp version of the stationary phase method, and give a version of the sharp Gårding inequality.

It is a pleasure to acknowledge stimulating discussions with J.-M. Bony and N. Lerner. In particular, Bony pointed out to me the confinement characterization of $\mathrm{Op}(S_{\mathrm{w}})$, which is used in the proof of the inversion result mentioned above. I also have had instructive conversions about the classical Wiener lemma with J.-P. Kahane, J. Peyrière and M. Zworski, as well as indirectly with P. Gérard.

1. The symbol space S_{w} and oscillatory convolutions.

If $e_1,..,e_m$ is a basis in $\mathbf{R}^m$, we say that $\Gamma=\oplus_1^m\mathbf{Z}e_j$ is a lattice. Let Γ be such a lattice and let $\chi_0\in\mathcal{S}(\mathbf{R}^m)$ have the property that $1=\sum_{j\in\Gamma}\chi_j$, where $\chi_j(x)=(\tau_j\chi_0)(x)=\chi_0(x-j)$. We let S_{w} be the space of $u\in\mathcal{S}'(\mathbf{R}^m)$ such that

$$U(\xi)=_{\mathrm{def}}\sup\nolimits_{j\in\Gamma}|\mathcal{F}\chi_ju(\xi)|\in L^1(\mathbf{R}^m). \tag{1.1}$$

Here $\mathcal{F}$ denotes the standard Fourier transformation: $\mathcal{F}u(\xi)=\widehat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$. S_{w} is a Banach space with the norm

$$\|u\|_{\Gamma,\chi_0}=\|\sup_{j\in\Gamma}|\mathcal{F}\chi_ju|\|_{L^1}.$$

Lemma 1.1. *The definition of S_w does not depend on the choice of Γ,χ_0.*

Proof. Let Γ', χ_0' be another choice. Put $|\chi|^2=\sum_{k\in\Gamma}|\chi_k|^2$, $|\chi|^{-2}=1/|\chi|^2$ which are smooth and Γ-periodic functions. Let $U(\xi)$ be the function in (1.1). We have,

$$\mathcal{F}\chi_j'u=\sum_{k\in\Gamma}\mathcal{F}\chi_j'|\chi|^{-2}\overline{\chi}_k\chi_ku,$$

so

$$\mathcal{F}\chi_j' u(\xi) = \sum_{k\in\Gamma} \int\int e^{-ix\cdot\xi}\chi_j'(x)(|\chi|^{-2}\overline{\chi}_k)(x)e^{ix\cdot\eta}\widehat{\chi_k u}(\eta)dx d\eta/(2\pi)^m.$$

Here we make N integrations by parts by means of $\langle\eta-\xi\rangle^{-2}(1+(\eta-\xi)\cdot D_x)$ and observe that

$$|\chi_j'(x)|\chi|^{-2}\overline{\chi}_k(x)| \leq C_N\langle j-k\rangle^{-N}\langle x-\frac{j+k}{2}\rangle^{-N},$$

and similarly for the derivatives. We then get,

$$|\mathcal{F}\chi_j' u(\xi)| \leq C_N \sum_{k\in\Gamma}\langle j-k\rangle^{-N}\int\langle\xi-\eta\rangle^{-N}U(\eta)d\eta \leq \widetilde{C}_N U_N,$$

where $U_N = \langle\cdot\rangle^{-N} * U$ and $*$ indicates convolution. ♯

We also observe that the definition of S_{w} does not change if we replace Γ, χ_0, by $\mathbf{R}^m$, $\chi_0' \in \mathcal{S}$, where now $\chi_t' = \tau_t\chi_0'$, $t\in\mathbf{R}^m$, $\chi_0'\in\mathcal{S}$, $\int\chi_t' dt = 1$.

Example. The Hörmander space $S^0_{0,0}$ of smooth functions on $\mathbf{R}^m$ which are bounded with all their derivatives, is a subspace of S_{w}, and we can take $U = C_N\langle\cdot\rangle^{-N}$ for any $N > m$.

Clearly, $\mathcal{S}$ is not dense in S_{w} for convergence in norm, so we need another notion of convergence:

Definition 1.2. Let $u, u_\nu \in S_{\mathrm{w}}$, $\nu = 1,2,3,\ldots$. We say that $u_\nu \to u$ narrowly when $\nu\to\infty$, if $u_\nu \to u$ in $\mathcal{S}'$ (weakly) and if there is a $U\in L^1(\mathbf{R}^m)$ such that $|\widehat{\chi_j u_\nu}(\xi)| \leq U(\xi)$ for all $j\in\Gamma$, $\nu = 1,2,\ldots$.

It is easy to check that this definition does not depend on the choice of Γ, χ_0 (or $\mathbf{R}^m, \chi_0$). Moreover, $\mathcal{S}$ is dense in S_{w} for narrow convergence. In fact, it was noticed in [S] that if $u\in S_{\mathrm{w}}$, then we can find $u_\nu\in\mathcal{S}$ converging to u narrowly, such that

$$|\widehat{\chi_j u_\nu}| \leq C_N\langle\cdot\rangle^{-N} * (\sup_j|\widehat{\chi_j u}|), \tag{1.2}$$

for every N, where C_N is independent of u.

If $u\in S_{\mathrm{w}}(\mathbf{R}^m)$, $v\in S_{\mathrm{w}}(\mathbf{R}^k)$, then $u\otimes v\in S_{\mathrm{w}}(\mathbf{R}^{m+k})$. More precisely, if $\Gamma\subset\mathbf{R}^m$, $\Gamma'\subset\mathbf{R}^k$ are lattices and χ_j, $j\in\Gamma$, χ_k', $k\in\Gamma'$ are corresponding partitions of unity, then $\chi_j\otimes\chi_k'$, $(j,k)\in\Gamma\times\Gamma'$ is a partition of unity, and

$$|\mathcal{F}((\chi_j\otimes\chi_k')(u\otimes v))(\xi,\eta)| \leq (\sup_j|\widehat{\chi_j u}(\xi)|)(\sup_k|\widehat{\chi_k' v}(\eta)|).$$

It is also clear that if $u_\nu\to u$, $v_\nu\to v$ narrowly, then $u_\nu\otimes v_\nu\to u\otimes v$ narrowly.

Let $L\subset\mathbf{R}^m$ be a linear subspace. Then we can define $S_{\mathrm{w}}(L)$, and if $u\in S_{\mathrm{w}}(\mathbf{R}^m)$, it follows that $u_{|L}\in S_{\mathrm{w}}(L)$. Moreover, $u\mapsto u_{|L}$ is narrowly continuous. To see this, we may assume that L: $x''=0$, where $x=(x',x'')$, $x''\in\mathbf{R}^d$, $x'\in\mathbf{R}^{n-d}$. Choose $\Gamma=\mathbf{Z}^n=\Gamma'\times\Gamma''$, $\Gamma'=\mathbf{Z}^{n-d}$, $\Gamma''=\mathbf{Z}^d$, and choose χ_0 such that $j''\neq 0 \Rightarrow \mathrm{supp}\chi_j\cap\mathbf{R}^{n-d}=\emptyset$. Then $\sum_{j'\in\Gamma'}\chi_{(j',0)}(x',0)=1$, and

$$|\mathcal{F}(\chi_{(j',0)}(x',0)u(x',0))(\xi')| = |\frac{1}{(2\pi)^d}\int\mathcal{F}(\chi_{(j',0)}u)(\xi)d\xi''| \leq U'(\xi'),$$

if $|\mathcal{F}(\chi_j u)(\xi)| \leq U(\xi)$ and $U'(\xi') = \int U(\xi)d\xi''/(2\pi)^d$. Clearly $U' \in L^1$ if $U \in L^1$.

If $u, v \in S_{\mathrm{w}}(\mathbf{R}^m)$, then $uv \in S_{\mathrm{w}}(\mathbf{R}^m)$ and $(u,v) \mapsto uv$ is narrowly continuous as in the statement for tensorproducts. In fact, uv can be identified with $u \otimes v_{|\mathrm{diag}(\mathbf{R}^m \times \mathbf{R}^m)}$.

Theorem 1.3. *Let $\Phi(x)$ be a non-degenerate quadratic form on $\mathbf{R}^m$. Then the convolution operator $u \mapsto e^{i\Phi} * u$ is bounded from S_{w} to S_{w}, and is continuous in the sense of narrow convergence.*

Proof. We shall work with partitions of unity with compact support. By our density remarks, it is enough to consider $e^{i\Phi} * u$ in the case when $u \in \mathcal{S}$. Let Γ' be a second lattice and let $\chi_0' \in C_0^\infty$ have the property: $1 = \sum_{j\in\Gamma'} \chi_j'$, with $\chi_j' = \tau_j \chi_0'$. Let $\widetilde{\chi}_0 \in C_0^\infty$ satisfy: $\widetilde{\chi}_0 \chi_0 = \chi_0$ and put $\widetilde{\chi}_k = \tau_k \widetilde{\chi}_0$. Then for $j \in \Gamma'$, $k \in \Gamma$, we have

$$\mathcal{F}(\chi_j'(e^{i\Phi} * \chi_k u))(\xi) = \int\int\int e^{i(-x\cdot\xi + \Phi(x-y) + y\cdot\eta)} \chi_j'(x)\widetilde{\chi}_k(y)\widehat{\chi_k u}(\eta)d\eta dy dx/(2\pi)^m$$
$$= \frac{e^{iF(j,k)}}{(2\pi)^m}\int\int\int e^{i(-x\cdot(\xi - \partial_x\Phi(j-k)) + y\cdot(\eta - \partial_x\Phi(j-k)))}\chi_{j,k}(x,y)\widehat{\chi_k u}(\eta)dy dx d\eta,$$

where F is real-valued and where

$$\chi_{j,k}(x,y) = \chi_j(x)\widetilde{\chi}_k(y)e^{i\Phi((x-j)-(y-k))}.$$

Here we have also used the Taylor sum formula:

$$\Phi(x-y) = \Phi(j-k) + \partial_x\Phi(j-k)\cdot(x-j) - \partial_y\Phi(j-k)\cdot(y-k) + \Phi((x-y)-(j-k)).$$

Notice that the modulus of any derivative of $\chi_{j,k}(x,y)$ can be bounded by a constant which is independent of x, y, j, k.

We make $2N$ integrations by parts, using the operators,

$$\frac{1 - (\xi - \partial_x\Phi(j-k))\cdot D_x}{\langle \xi - \partial_x\Phi(j-k)\rangle^2}, \qquad \frac{1 + (\eta - \partial_x\Phi(j-k))\cdot D_y}{\langle \eta - \partial_x\Phi(j-k)\rangle^2},$$

with the notation $D_x = \frac{1}{i}\partial_x$, $\langle x\rangle = \sqrt{1+x^2}$. After estimating the resulting x, y integrals in a straight forward way, we get:

$$\begin{aligned}(1.3)\qquad &\mathcal{F}(\chi_j'(e^{i\Phi} * \chi_k u))(\xi)\\ &= \mathcal{O}_N(1)\int \langle \xi - \partial_x\Phi(j-k)\rangle^{-N}\langle \eta - \partial_x\Phi(j-k)\rangle^{-N}|\widehat{\chi_k u}(\eta)|d\eta.\end{aligned}$$

Since Φ is non-degenerate, $\langle \xi - \partial_x\Phi(j-k)\rangle$ is of the same order of magnitude as $\langle \Phi''^{-1}\xi - j + k\rangle$ and similarly for $\langle \eta - \partial_x\Phi(j-k)\rangle$. With $N > m$, we then get:

$$(1.4)\qquad \sum_{k\in\Gamma}\langle \xi - \partial_x\Phi(j-k)\rangle^{-N}\langle \eta - \partial_x\Phi(j-k)\rangle^{-N} \leq C_N\langle \xi - \eta\rangle^{-N}.$$

Summing over k in (1.3) we get

$$(1.5) \qquad |\mathcal{F}(\chi'(e^{i\Phi} * u))(\xi)| \leq C_N \int \langle \xi - \eta \rangle^{-N} \sup_k |\widehat{\chi_k u}(\eta)| d\eta$$

and consequently $\sup_j |\mathcal{F}(\chi'(e^{i\Phi} * u))(\xi)|$ is also bounded by the left hand hand side of (1.5) (which is an L^1 function).

The narrow continuity follows from the estimates above and from the fact that if $u_\nu \to u$ in $\mathcal{S}'$, then $e^{i\Phi} * u_\nu \to e^{i\Phi} * u$ in $\mathcal{S}'$. ♯

We now turn to the preparation of the calculus of h-pseudors. We let Φ be a non-degenerate quadratic form on $\mathbf{R}^m$ and choose $C_m \neq 0$ such that $\mathcal{F}(C_m h^{-m/2} e^{i\Phi/h}) = e^{-ih\Phi^{-1}(\xi)}$, where Φ^{-1} is the inverse quadratic form. In the following we shall assume that $0 < h \leq 1$.

Theorem 1.4. *The map $S_{\mathrm{w}} \ni u \mapsto C_m h^{-m/2} e^{i\Phi/h} * u \in S_{\mathrm{w}}$ is uniformly bounded with respect to h. Moreover, if $u \in S_{\mathrm{w}}$, then $C_m h^{-m/2} e^{i\Phi/h} * u \to u$ in S_{w} norm, when $h \to 0$.*

Proof. We prove only the last statement, since the first one can easily be obtained from the same arguments. Let $\chi_0 \in C_0^\infty$, Γ be as above. Choose $C_0 > 0$ so large that $|j-k| > C_0 \Rightarrow \mathrm{dist}(\mathrm{supp}\chi_j, \mathrm{supp}\chi_k) \sim |j-k|$. Let $u \in S_{\mathrm{w}}$ and let $U = \sup_j |\mathcal{F}\chi_j u| \in L^1$. For $|j-k| \leq C_0$, consider,

$$\mathcal{F}\chi_j(C_m h^{-m/2} e^{i\Phi/h} * (\chi_k u)) = \frac{1}{(2\pi)^m} \widehat{\chi}_j * (e^{-ih\Phi^{-1}} \widehat{\chi_k u}).$$

Here $|e^{-ih\Phi^{-1}} \widehat{\chi_k u} - \widehat{\chi_k u}| \leq |e^{-ih\Phi^{-1}} - 1| U \to 0$ in L^1, when $h \to 0$, so

$$\begin{aligned} |\mathcal{F}(\chi_j(C_m h^{-m/2} e^{i\Phi/h} * (\chi_k u)) - \chi_j \chi_k u)| &\leq \frac{1}{(2\pi)^m} |\widehat{\chi}_j| * |e^{-ih\Phi^{-1}} - 1| U \\ &= \frac{1}{(2\pi)^m} |\widehat{\chi}_0| * |e^{-ih\Phi^{-1}} - 1| U \to 0 \text{ in } L^1 \text{ when } h \to 0, \end{aligned}$$

It follows that

$$(1.6) \qquad |\mathcal{F}(\chi_j u) - \mathcal{F} \sum_{|k-j| \leq C_0} \chi_j(C_m h^{-m/2} e^{i\Phi/h} * (\chi_k u))| \leq C_1 |\widehat{\chi}_0| * |e^{-ih\Phi^{-1}} - 1| U.$$

Now consider for $|j-k| > C_0$:

$$\begin{aligned} &\mathcal{F}\chi_j(e^{i\Phi/h} * \chi_k u)(\xi) = \\ &\frac{C_m h^{-m/2}}{(2\pi)^m} \int\int\int e^{i(-x\cdot\xi + \Phi(x-y)/h + y\cdot\eta)} \chi_j(x) \widetilde{\chi}_k(y) \widehat{\chi_k u}(\eta) dy dx d\eta \\ &= \int K_{j,k}(\xi, \eta) \widehat{\chi_k u}(\eta) d\eta, \end{aligned}$$

with $\widetilde{\chi}_k$ as in the proof of Theorem 1.3 and where

$$K_{j,k}(\xi,\eta)=\frac{C_m h^{-m/2}}{(2\pi)^m}\int\int e^{i(-x\cdot\xi+\Phi(x-y)/h+y\cdot\eta)}\chi_j(x)\widetilde{\chi}_k(y)dxdy. \tag{1.7}$$

Here

$$\frac{1}{h}\Phi(x-y)-x\cdot\xi+y\cdot\eta=\frac{1}{h}\Phi(x-y)-(x-y)\cdot\frac{(\xi+\eta)}{2}-\frac{(x+y)}{2}\cdot(\xi-\eta),$$

so we put $\widetilde{x}=x-y$, $\widetilde{y}=(x+y)/2$, $\chi_j(x)\widetilde{\chi}_k(y)=\psi_{j,k}(\widetilde{x},\widetilde{y})$ and we get

$$K_{j,k}(\xi,\eta)=\frac{C_m h^{-m/2}}{(2\pi)^m}\int\int e^{i(\Phi(\widetilde{x})/h-\widetilde{x}\cdot(\xi+\eta)/2-\widetilde{y}\cdot(\xi-\eta))}\psi_{j,k}(\widetilde{x},\widetilde{y})d\widetilde{x}d\widetilde{y}. \tag{1.8}$$

Here we notice that $\pi_{\widetilde{x}}\mathrm{supp}\psi_{j,k}=\mathrm{supp}\chi_j-\mathrm{supp}\widetilde{\chi}_k$ (with $\pi_{\widetilde{x}}$ denoting the projection $(\widetilde{x},\widetilde{y})\mapsto\widetilde{x}$), and we may assume that C_0 has been chosen so large that $\mathrm{dist}((\pi_{\widetilde{x}}\mathrm{supp}\psi_{j,k})+B(0,1),0)\sim|j-k|$. In (1.8), we can always make integrations by parts in $\widetilde{y}$ and gain negative powers of $\langle\xi-\eta\rangle$. As for the integration by parts in $\widetilde{x}$, it depends on wether the critical point of $\widetilde{x}\mapsto\frac{1}{h}\Phi(\widetilde{x})-\widetilde{x}\cdot\frac{\xi+\eta}{2}$ is close to $\pi_{\widetilde{x}}\mathrm{supp}\psi_{j,k}$ or not.

If $\frac{\xi+\eta}{2}\in\frac{1}{h}\Phi'(\pi_{\widetilde{x}}\mathrm{supp}\psi_{j,k}+B(0,1))$, we get,

$$K_{j,k}(\xi,\eta)=\mathcal{O}_N(1)\langle\xi-\eta\rangle^{-N},\forall N. \tag{1.9}$$

If $\frac{\xi+\eta}{2}\notin\frac{1}{h}\Phi'(\pi_{\widetilde{x}}\mathrm{supp}\psi_{j,k}+B(0,1))$, we get

$$K_{j,k}(\xi,\eta)=\mathcal{O}_N(1)h^N\mathrm{dist}(\frac{\xi+\eta}{2},\frac{1}{h}\Phi'(\pi_{\widetilde{x}}\mathrm{supp}\psi_{j,k}))^{-N}\langle\xi-\eta\rangle^{-N}. \tag{1.10}$$

In terms of the original coordinates, we get for all $N>0$, that $K_{j,k}(\xi,\eta)=$

$$\mathcal{O}_N(1)\times\begin{cases}\langle\xi-\eta\rangle^{-N}, \text{ if } \dfrac{\xi+\eta}{2}\in\dfrac{1}{h}\Phi'(\mathrm{supp}\chi_j-\mathrm{supp}\widetilde{\chi}_k+B(0,1)),\\ h^N\langle\xi-\eta\rangle^{-N}\mathrm{dist}(\dfrac{\xi+\eta}{2},\dfrac{1}{h}\Phi'(\mathrm{supp}\chi_j-\mathrm{supp}\widetilde{\chi}_k+B(0,1)))^{-N}, \text{ if not.}\end{cases}$$

We shall estimate

$$\mathcal{K}_j(\xi,\eta)=\sum_{|k-j|>C_0}|K_{j,k}(\xi,\eta)| \tag{1.11}$$

uniformly with respect to j. Since Φ is non-degenerate, there exists $N_0\in\mathbf{N}$ such that for every (j,ξ,η), we have $\frac{\xi+\eta}{2}\in\frac{1}{h}\Phi'(\mathrm{supp}\chi_j-\mathrm{supp}\widetilde{\chi}_k+B(0,1))$ for at most N_0 values of k, and for some $k=k(j,\xi,\eta)$, we have $\mathrm{dist}(\frac{\xi+\eta}{2},\frac{1}{h}\Phi'(\mathrm{supp}\chi_j-\mathrm{supp}\widetilde{\chi}_k))\sim|k-k(j,\xi,\eta)|$ for the remaining values of k. It is then clear that

$$\mathcal{K}_j(\xi,\eta)\leq\mathcal{O}_N(1)\langle\xi-\eta\rangle^{-N}. \tag{1.12}$$

Moreover, there exists a $C_1 > 0$ such that if $|\xi|, |\eta| \leq 1/C_1 h$, then $\frac{\xi+\eta}{2} \notin \frac{1}{h}\Phi'(\mathrm{supp}\chi_j - \mathrm{supp}\widetilde{\chi}_k + B(0,1))$ for all j, k (with $|j-k| > C_0$) and consequently,

$$\mathcal{K}_j(\xi,\eta) \leq \mathcal{O}_N(1) h^N \langle \xi - \eta \rangle^{-N}, \quad |\xi|, |\eta| \leq 1/C_1 h. \tag{1.13}$$

We have

$$| \sum_{|k-j|>C_0} \mathcal{F}\chi_j(C_m h^{-m/2} e^{i\Phi/h} * (\chi_k u))| \leq$$
$$\sum_{|k-j|>C_0} \int |K_{j,k}(\xi,\eta)| U(\eta) d\eta = \int \mathcal{K}_j(\xi,\eta) U(\eta) d\eta,$$

and the estimates (1.12,13) are uniform w.r.t. j. Consequently,

$$| \sum_{|k-j|>C_0} \mathcal{F}(\chi_j(C_m h^{-m/2} e^{i\Phi/h} * (\chi_k u)))(\xi)| \leq C_N U_{N,h}, \tag{1.14}$$

where,

$$U_{N,h} = \int \langle \xi - \eta \rangle^{-N} (h^N 1_{\{|\xi|,|\eta|\leq 1/C_0 h\}}(\xi,\eta) + (1 - 1_{\{|\xi|,|\eta|\leq 1/C_0 h\}}))(\xi,\eta)) U(\eta) d\eta, \tag{1.15}$$

which tends to 0 in L^1, when $h \to 0$. Combining this with (1.6), we get

$$|\mathcal{F}\chi_j(C_m h^{-m/2} e^{i\Phi/h} * u - u)| \leq V_{N,h}, \tag{1.16}$$

where

$$V_{N,h} = C_N(\langle \cdot \rangle^{-N} * (|e^{-i\Phi^{-1}/h} - 1| U) + U_{N,h}). \tag{1.17}$$

The theorem follows. $\sharp$

From the estimates above, it also follows that

$$|\mathcal{F}\chi_j(C_m h^{-m/2} e^{i\Phi/h} * u)| \leq C_N \langle \cdot \rangle^{-N} * U. \tag{1.18}$$

Let Φ be as in the preceding theorem and put $\Psi = -\Phi^{-1}$, so that $C_m e^{i\Phi/h} * = e^{ih\Psi(D)}$. Let $u \in \mathcal{S}'(\mathbf{R}^m)$ and assume that for some $N \in \mathbf{N}$, we have

$$\Psi(D)^k u \in S_{\mathrm{w}}, \quad 0 \leq k \leq N. \tag{1.19}$$

Applying Taylor's formula,

$$f(1) = \sum_0^{N-1} \frac{f^{(k)}(0)}{k!} + \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} f^{(N)}(t) dt$$

to the function $f(t) = e^{ith\Psi(D)}u$, we get

$$e^{ih\Psi(D)}u = \sum_0^{N-1} \frac{(ih)^k \Psi(D)^k}{k!} u + h^N i^N \int \frac{(1-t)^{N-1}}{(N-1)!} e^{ith\Psi(D)} \Psi(D)^N u dt, \tag{1.20}$$

where according to Theorem 1.4, we have

$$i^N \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} e^{ith\Psi(D)} \Psi(D)^N u dt \to \frac{i^N \Psi(D)^N}{N!} u \text{ in } S_{\mathrm{w}},\ h \to 0. \tag{1.21}$$

2. Pseudor algebras.

For $t \in [0,1]$, $a \in \mathcal{S}'(\mathbf{R}^{2n})$, $u \in \mathcal{S}(\mathbf{R}^n)$, it is well known that we can define $\mathrm{Op}_t(a)u \in \mathcal{S}'$ (formally) by:

$$\mathrm{Op}_t(a)u(x) = \frac{1}{(2\pi)^n} \int\int e^{i(x-y)\cdot\xi} a(tx+(1-t)y,\xi)u(y)dyd\xi, \tag{2.1}$$

where for $t = 1/2$ (Weyl quantization) we drop the subscript t. For $0 < h \leq 1$, we also define

$$\mathrm{Op}_{t,h}(a) = \mathrm{Op}_t(a(\cdot, h(\cdot\cdot))). \tag{2.2}$$

The operator space $\mathrm{Op}_t(S_{\mathrm{w}})$ is independent of t:

Proposition 2.1. *Let $t, s \in [0,1]$, $a_t, a_s \in \mathcal{S}'(\mathbf{R}^{2n})$, and assume that $\mathrm{Op}_t(a_t) = \mathrm{Op}_s(a_s)$. Then $a_t \in S_{\mathrm{w}}$ iff $a_s \in S_{\mathrm{w}}$. Moreover the correspondence $S_{\mathrm{w}} \ni a_s \mapsto a_t \in S_{\mathrm{w}}$ is bounded and narrowly continuous.*

Proof. Since $a_t = e^{i(t-s)D_x\cdot D_\xi} a_s$, it suffices to apply Theorem 1.3. $\sharp$

The proposition immediately extends to $\mathrm{Op}_{t,h}(S_{\mathrm{w}})$. From now on, we work with the Weyl quantization. If $a, b \in \mathcal{S}(\mathbf{R}^{2n})$, we recall that the Weyl composition $c = a\sharp b$, defined by $\mathrm{Op}(c) = \mathrm{Op}(a) \circ \mathrm{Op}(b)$, is given by:

$$a\sharp b(x,\xi) = (e^{\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}(a(x,\xi)\otimes b(y,\eta)))_{(y,\eta)=(x,\xi)}, \tag{2.3}$$

where $\sigma(D_x, D_\xi; D_y, D_\eta) = D_\xi \cdot D_y - D_x \cdot D_\eta$ is the standard symplectic form.

Theorem 2.2. *The Weyl composition extends (uniquely) to a bilinear map $S_{\mathrm{w}} \times S_{\mathrm{w}} \to S_{\mathrm{w}}$ which is norm continuous and preserves narrow convergence of sequences.*

Proof. $e^{\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}$ is a convolution operator as in Theorem 1.3, so it suffices to apply that theorem together with the remarks of section 1 about tensorproducts and restriction to subspaces. $\sharp$

It is then clear that $\mathrm{Op}(S_{\mathrm{w}})$ is a Banach algebra. Since S_{w} is closed under complex conjugation, we have $A \in \mathrm{Op}(S_{\mathrm{w}}) \Rightarrow A^* \in \mathrm{Op}(S_{\mathrm{w}})$, where A^* denotes the complex adjoint of A. Following the idea of the proof of Kohn-Nirenberg [KN] for L^2 boundedness, we see that $a \in S_{\mathrm{w}} \Rightarrow \mathrm{Op}(a) \in \mathcal{L}(L^2(\mathbf{R}^n), L^2(\mathbf{R}^n))$, and that $\|\mathrm{Op}(a)\|_{\mathcal{L}(..)} \leq C\|a\|_{S_{\mathrm{w}}}$. Here we only recall the basic idea and refer to [S1] for further details: We may equip S_{w} with a norm $|||\cdot|||$, such that $|||a\sharp b||| \leq |||a|||\,|||b|||$, $|||\overline{a}||| = |||a|||$. If $A = \mathrm{Op}(a)$, $|||a||| < 1$ we can construct $B = (I - A^*A)^{1/2} = I - \frac{1}{2}A^*A + ...$ as a convergent power series in $\mathrm{Op}(S_{\mathrm{w}})$, and $A^*A + B^2 = I$, so $\|A\|_{\mathcal{L}(L^2,L^2)} \leq 1$.

For h-pseudors, we have the composition formula

$$c = a\sharp_h b = (e^{\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}a \otimes b)_{(y,\eta)=(x,\xi)},$$

when $\mathrm{Op}_h(c) = \mathrm{Op}_h(a) \circ \mathrm{Op}_h(b)$, and the map $S_{\mathrm{w}} \otimes S_{\mathrm{w}} \ni (a,b) \mapsto a\sharp_h b \in S_{\mathrm{w}}$ is uniformly continuous wih respect to h. Moreover, if $a, b \in S_{\mathrm{w}}$ are independent of h, then $a\sharp_h b \to ab$ in S_{w} when $h \to 0$. These facts follow from Theorem 1.4.

If a, b are independent of h and $\partial^\alpha_{(x,\xi)}a$, $\partial^\alpha_{(x,\xi)}b$ belong to S_{w} for $|\alpha| \leq N \in \mathbf{N}$, then $\sigma(D_x,D_\xi;D_y,D_\eta)^k(a\otimes b) \in S_{\mathrm{w}}$ for $k \leq N$, and from the observation after the proof of Theorem 1.4, we infer that

$$a\sharp_h b = \sum_{k=0}^{N-1} \frac{h^k}{k!}(\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta))^k(a\otimes b)_{(y,\eta)=(x,\xi)} + h^N r_N, \tag{2.4}$$

where,

$$r_N \to \frac{1}{N!}(\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta))^N(a\otimes b)_{(y,\eta)=(x,\xi)} \text{ in } S_{\mathrm{w}},\ h \to 0. \tag{2.5}$$

Let $S_{\mathrm{w},N}$ be the space of symbols of the form,

$$a(x,\xi;h) = a_0(x,\xi) + ha_1(x,\xi) + .. + h^{N-1}a_{N-1}(x,\xi) + h^N r_N(x,\xi;h), \tag{2.6}$$

with $\partial^\alpha_{(x,\xi)}a_j \in S_{\mathrm{w}}$ for $|\alpha| + j \leq N$ and $r_N(\cdot;h)$ uniformly bounded in S_{w} for $0 < h \leq 1$. Then, if $a, b \in S_{\mathrm{w}}$, we have $a\sharp_h b \in S_{\mathrm{w},N}$. Moreover, if we define $\widetilde{S}_{\mathrm{w},N}$ as the space of all $a \in S_{\mathrm{w},N}$ of the form (2.6) with $r_N(\cdot;h) \to a_N$ in S_{w}, when $h \to 0$, then for $a, b \in \widetilde{S}_{\mathrm{w},N}$, we have $a\sharp_h b \in \widetilde{S}_{\mathrm{w},N}$.

3. The confinement point of view.

In this section, we work with $h = 1$. Let $\Gamma \subset \mathbf{R}^{2n}$ be a lattice and let $\chi_0 \in \mathcal{S}(\mathbf{R}^{2n})$ with $\sum_{j\in\Gamma}\chi_j = 1$, where $\chi_j = \tau_j\chi_0$. Put $\chi_j^{\mathrm{w}} = \mathrm{Op}(\chi_j)$. The following result was pointed out to me by J.M.Bony:

Proposition 3.1. *Let $A : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$. Then the following two statements are equivalent:*

$$A \in \mathrm{Op}(S_{\mathrm{w}}), \tag{3.1}$$

and

$$\exists U \in \ell^1(\Gamma); \|\chi_j^{\mathrm{w}} A \chi_k^{\mathrm{w}}\|_{\mathcal{L}(L^2,L^2)} \leq U(j-k),\ j,k \in \Gamma. \tag{3.2}$$

The property (3.2) defines a special case of classes of operators considered by A. Unterberger [U] and by J.M. Bony [B].

Proof. Using properties like: $\|\chi_j^{\mathrm{w}} \chi_k^{\mathrm{w}}\|_{\mathcal{L}(L^2,L^2)} \leq C_N \langle j-k\rangle^{-N}$ and operators like χ^2, χ^{-2} of the next section, it is easy to prove (cf section 1) that the property (3.2) only depends on A and not on the choice of the lattice and on χ_0. We may even replace Γ by $\mathbf{R}^{2n}$ and get the following property equivalent to (3.2): Let $\chi_0 \in \mathcal{S}(\mathbf{R}^{2n})$ have the property that $\int \chi_t(x)dt = \int \chi_0(x-t)dt \neq 0$:

$$\exists U \in L^1(\mathbf{R}^{2n});\ \|\chi_t^{\mathrm{w}} A \chi_s^{\mathrm{w}}\|_{\mathcal{L}(L^2,L^2)} \leq U(t-s). \tag{3.3}$$

For $\alpha = (\alpha_x, \alpha_\xi) \in \mathbf{R}^{2n}$, let $e_\alpha(x) = Ce^{i((x-\alpha_x)\cdot\alpha_\xi + \frac{i}{2}(x-\alpha_x)^2)}$, where $C>0$ is chosen so that $\|e_\alpha\| = 1$. Here the norm is that of L^2, if nothing else is specified and the corresponding scalar product will be denoted by $(\cdot|\cdot\cdot)$. Let $\chi_\alpha \in \mathcal{S}(\mathbf{R}^{2n})$ be the Weyl symbol of the orthogonal projection onto $\mathbf{C}e_\alpha$, so that

$$\chi_\alpha^{\mathrm{w}} u = (u|e_\alpha)e_\alpha. \tag{3.4}$$

A straight forward computation shows that,

$$\chi_\alpha = \tau_\alpha \chi_0,\ \chi_0(x,\xi) = (2\sqrt{\pi})^n C^2 e^{-(x^2+\xi^2)}, \tag{3.5}$$

so the requirement prior to (3.3) is satisfied. Since e_α is L^2-normalized, we have

$$\|\chi_\alpha^{\mathrm{w}} A \chi_\beta^{\mathrm{w}}\|_{\mathcal{L}(L^2,L^2)} = |(Ae_\beta|e_\alpha)|. \tag{3.6}$$

Another straight forward computation shows that

$$|(Ae_\beta|e_\alpha)| = |\mathcal{F}((\tau_{\frac{\alpha+\beta}{2}}\psi)a)(\alpha_\xi - \beta_\xi, -(\alpha_x - \beta_x))|, \tag{3.7}$$

where $\psi = \widetilde{C}\chi_0$ for some $\widetilde{C} > 0$ and with χ_0 given in (3.5). Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

$$\|\chi_\alpha A \chi_\beta\|_{\mathcal{L}(L^2,L^2)} = \mathcal{F}((\tau_{\frac{\alpha+\beta}{2}}\psi)a)(J(\alpha-\beta)), \tag{3.8}$$

so we get the equivalence,

$$\|\chi_\alpha A \chi_\beta\| \leq U(\alpha-\beta),\ \alpha,\beta \in \mathbf{R}^{2n} \Leftrightarrow |\mathcal{F}((\tau_\alpha\psi)a)(\beta)| \leq (U \circ J^{-1})(\beta),\ \alpha,\beta \in \mathbf{R}^{2n}.\ \sharp$$

4. A non-commutative Wiener lemma.

As a warm up exercise we shall first give a proof of the classical Wiener lemma, which does not use any kind of Fourier transform. (See [Lo] for an abstract proof and [Z], [LKa] for a short direct proof by Calderon.) Then by generalizing this proof, we establish that if an operator in $\mathrm{Op}(S_{\mathrm{w}})$ has an L^2 bounded inverse, then the inverse belongs to the same class. Let us recall that the corresponding statement with S_{w} replaced by $S^0_{0,0}$ is wellknown and follows from the so called Beals lemma [Be]. (It is less well-known that Shubin [Sh] independently obtained a similar result for the classes $S^m_{1,0}$.)

Let $0 \le \psi_0 \in C_0^\infty(\mathbf{R}^m)$, with $\sum_{j\in\mathbf{Z}^m} \tau_j\psi_0 = 1$, $\tau_j f(x) = f(x-j)$. (In this first part we shall work systematically on $\mathbf{Z}^m$.) Put $\psi_0^\epsilon(x) = \psi_0(\epsilon x)$, $\psi_k^\epsilon(x) = (\tau_{k/\epsilon}\psi_0^\epsilon)(x) = \psi_0(\epsilon x - k)$. Notice that $\sum_{k\in\mathbf{Z}^m}\psi_k^\epsilon = 1$. Also notice that an equivalent (ϵ dependent) norm on $\ell^1(\mathbf{Z}^m)$ is given by

$$|||a||| = \sum_k \|\psi_k^\epsilon a\|, \tag{4.1}$$

where in this section $\|\cdot\|$ will always denote either the ℓ^2 or the L^2 norm depending on the context.

Let $a \in \ell^1$ and assume that the convolution operator $a*$ is invertible in the ℓ^2 sense:

$$\|u\| \le \|a*u\|. \tag{4.2}$$

It is then clear that the inverse of $a*$ is of the form $b*$ for some $b \in \ell^2$ and the Wiener lemma states that b is actually in ℓ^1. We shall now prove this:

First notice that

$$\begin{aligned}\|\psi_j^\epsilon u\| &\le \|a*\psi_j^\epsilon u\| \le \|\psi_j^\epsilon(a*u)\| + \|[a*,\psi_j^\epsilon]u\| \\ &\le \|\psi_j^\epsilon(a*u)\| + \sum_{k\in\mathbf{Z}^m}\|[a*,\psi_j^\epsilon]\psi_k^\epsilon(\psi^\epsilon)^{-2}\psi_k^\epsilon u\|,\end{aligned}$$

where we write $(\psi^\epsilon)^2 = \sum(\psi_j^\epsilon)^2$, $(\psi^\epsilon)^{-2} = ((\psi^\epsilon)^2)^{-1}$, and notice that these functions are bounded from above and from below by positive constants independent of ϵ. Hence,

$$\|\psi_j^\epsilon u\| \le \|\psi_j^\epsilon(a*u)\| + \mathcal{O}(1)\sum_k \|[a*,\psi_j^\epsilon]\psi_k^\epsilon\|\,\|\psi_k^\epsilon u\|. \tag{4.3}$$

Here $[a*,\psi_j^\epsilon]\psi_k^\epsilon$ has the matrix $-(\psi_j^\epsilon(x)-\psi_j^\epsilon(y))a(x-y)\psi_k^\epsilon(y)$, $x,y\in\mathbf{Z}^d$. Since $\psi_k^\epsilon \le 1$, we have

$$|(\psi_j^\epsilon(x)-\psi_j^\epsilon(y))a(x-y)\psi_k(y)| \le |\psi_j^\epsilon(x)-\psi_j^\epsilon(y)|\,|a(x-y)| \le \mathcal{O}(1)\min(1,\epsilon|x-y|)|a(x-y)|.$$

It is then clear that

$$\|[a*,\psi_j^\epsilon]\psi_k^\epsilon\| \to 0,\ \epsilon\to 0,\ \text{uniformly w.r.t. } j,k. \tag{4.4}$$

For $|j-k| \geq C_0$, sufficiently large, we know that

$$\text{dist}(\text{supp}\psi_j^\epsilon, \text{supp}\psi_k^\epsilon) \sim \frac{1}{\epsilon}|j-k|, \tag{4.5}$$

and the matrix of $[a*, \psi_j^\epsilon]\psi_k^\epsilon$ simplifies to,

$$-\psi_j^\epsilon(x)a(x-y)\psi_k^\epsilon(y). \tag{4.6}$$

we estimate the corresponding operator norm by means of Shur's lemma:

$$\begin{aligned}
\sup_x \sum_y |\psi_j^\epsilon(x)a(x-y)\psi_k^\epsilon(y)| &\leq \sup_{x\in \text{supp}\psi_j^\epsilon} \sum_y |a(x-y)|\psi_k(y) \\
&\leq \sup_{x\in\text{supp}\psi_j^\epsilon} \sum_{y\in x-\text{supp}\psi_k^\epsilon} |a(y)| \leq \|a\|_{\ell^1(\text{supp}\psi_j^\epsilon - \text{supp}\psi_k^\epsilon)} \\
&= \|a\|_{\ell^1(\text{supp}\psi_{j-k}^\epsilon - \text{supp}\psi_0^\epsilon)} \underset{\text{def}}{=} U^\epsilon(j-k),
\end{aligned}$$

where $\|U^\epsilon\|_{\ell^1} \to 0$, $\epsilon \to 0$. Similarly,

$$\sup_y \sum_k |\psi_j^\epsilon(x)a(x-y)\psi_k(y)| \leq \|a\|_{\ell^1(\text{supp}\psi_{j-k}^\epsilon - \text{supp}\psi_0^\epsilon)},$$

and combining this with (4.4), we get a new function U^ϵ tending to 0 in ℓ^1 when $\epsilon \to 0$, such that

$$\|[a*, \psi_j^\epsilon]\psi_k^\epsilon\| \leq U^\epsilon(j-k). \tag{4.7}$$

Replacing U^ϵ by a multiple, we then get from (4.3), that

$$\|\psi_j^\epsilon u\| \leq \|\psi_j^\epsilon(a*u)\| + \sum_k U^\epsilon(j-k)\|\psi_k^\epsilon u\|, \tag{4.8}$$

which we view as a convolution inequality for the function $j \mapsto \|\psi_j^\epsilon u\|$. Choose $\epsilon > 0$ small enough, so that $\|U^\epsilon\|_{\ell^1} \leq \frac{1}{2}$, and let $0 \leq V^\epsilon \in \ell^1$ be the function with

$$(\delta_0 + V^\epsilon) * (\delta_0 - U^\epsilon) = \delta_0, \text{ where } \delta_0 = 1_{\{0\}}. \tag{4.9}$$

Then from (4.8), we get,

$$\|\psi_j^\epsilon u\| \leq \|\psi_j^\epsilon(a*u)\| + \sum_k V^\epsilon(j-k)\|\psi_k^\epsilon(a*u)\|. \tag{4.10}$$

Chossing $u = b$ to be the ℓ^2 solution of $a * b = \delta_0$, we get in particular that,

$$\|b\|_{\ell^1} < \infty, \tag{4.11}$$

which is the conclusion of the Wiener lemma. Notice also that if $|a| \leq A$, $A \in \ell^1$ and $\|u\| \leq \|a * u\|$, for all $u \in \ell^2$, then $\exists B \in \ell^1$, such that $|b| \leq B$.

Remark. Since our proof does not use Fourier transform we have the following immediate generalization: Let $k(x,y)$ be a matrix with $|k(x,y)| \leq A(x-y)$, $x,y \in \mathbf{Z}^m$ for some $A \in \ell^1$. Assume further that $Ku(x) = \sum_y k(x,y)u(y)$ has a bounded inverse of norm ≤ 1 in $\mathcal{L}(\ell^2, \ell^2)$. Then there exists $B \in \ell^1$ depending only on A, such that the matrix $\ell(x,y)$ of the inverse, satisfies: $|\ell(x,y)| \leq B(x-y)$.

The proof above can be extended to a proof of the fact that if $a \in S_{\mathrm{w}}(\mathbf{R}^m)$ and $1/a$ is a bounded function, then $1/a \in S_{\mathrm{w}}$. This also seems to follow from abstract theory and we are still in the commutative setting. (See [Lo].)

We now attack the pseudor case. Let $m = 2n$ and let χ_j, $j \in \mathbf{Z}^{2n}$ be as before. Let χ_j^{w} be the Weyl quantization (and sometimes we shall drop the superscript when it is clear that we discuss pseudors.) Consider the operator $\chi : L^2 \ni u \mapsto (\chi_j^{\mathrm{w}} u)_{j \in \mathbf{Z}^{2n}} \in \oplus L^2$, and its adjoint: $\chi^* : (u_j) \mapsto \sum \chi_j^w u_j$. It is easy to see that χ is a bounded operator. (See for instance the beginning of [HeS].) Moreover χ^* has a bounded right inverse for the following reason: Let $\widetilde{\chi}_0 \in C_0^\infty$ be equal to 1 in a large region containing $\mathrm{supp}\chi_0$, and put $\widetilde{\chi}_j = \tau_j \widetilde{\chi}_0$. For a given $u \in L^2$, put $u_j = \widetilde{\chi}_j^{\mathrm{w}} u$. For a given $u \in L^2$, put $u_j = \widetilde{\chi}_j^{\mathrm{w}} u$. Then, $\chi^*(u_j) = \sum \chi_j^{\mathrm{w}} \widetilde{\chi}_j^{\mathrm{w}} u = u - \sum \chi_j^{\mathrm{w}}(1 - \widetilde{\chi}_j^{\mathrm{w}})u$. Here, we may arrange so that $\| \sum \chi_j^{\mathrm{w}}(1 - \widetilde{\chi}_j^{\mathrm{w}})\| \leq \frac{1}{2}$ and the claim follows. By duality, χ has a bounded left inverse and hence, $\|u\| \leq \mathcal{O}(1)\|\chi u\|$. It follows that $\chi^2 =_{\mathrm{def}} \sum (\chi_\nu^{\mathrm{w}})^2 = \chi^* \chi$ has a bounded inverse and we shall denote it by χ^{-2}. It is wellknown from the usual Beals lemma, that $\chi^{-2} \in \mathrm{Op}(S_{0,0}^0)$.

Put $\Psi_j^\epsilon = \sum \psi_j^\epsilon(\nu)\chi_\nu^{\mathrm{w}}$. Defining $(\Psi^\epsilon)^2$ as above, we conclude as there, that $(\Psi^\epsilon)^2$ has a bounded inverse $(\Psi^\epsilon)^{-2}$ belonging to a bounded set in $\mathrm{Op}(S_{0,0}^0)$, for $\epsilon > 0$ small enough. Also, if we define Ψ^ϵ in the same way as χ, we see as before that Ψ^ϵ has a bounded left inverse, and that $\|\Psi^\epsilon u\| \sim \|u\|$ uniformly with respect to $\epsilon > 0$.

If $B : \mathcal{S} \to \mathcal{S}'$, we consider $\chi_j B u = \sum_{k,\tilde{k}} \chi_j B \chi_k \chi_k \chi^{-4} \chi_{\tilde{k}}^2 u$. Assume that the matrix $(\|\chi_j B \chi_k\|)_{j,k}$ is in $\mathcal{L}(\ell^2, \ell^2)$. Then $\|\chi_j B u\| \leq \mathcal{O}(1) \sum_{k,\tilde{k}} \|\chi_j B \chi_k\| \langle k - \tilde{k}\rangle^{-N} \|\chi_{\tilde{k}} u\|$, so

$$\|(\|\chi_j B u\|)_j\|_{\ell^2} \leq \mathcal{O}(1) \|(\|\chi_j B \chi_k\|)_{j,k}\|_{\mathcal{L}(\ell^2,\ell^2)} \|(\|\chi_{\tilde{k}} u\|)\|_{\ell^2},$$

which gives,

$$\|\chi B u\| \leq \mathcal{O}(1)\|u\|,$$

and finally, $\|Bu\| \leq \mathcal{O}(1)\|u\|$, i.e. $B \in \mathcal{L}(L^2, L^2)$.

Now let $A \in \mathrm{Op}(S_{\mathrm{w}})$, so that according to Proposition 3.1, we have $\|\chi_j A \chi_k\| \leq U(j-k)$ for some $U \in \ell^1$. Consider,

$$[A, \sum \Psi_j^\epsilon] = \sum_\alpha (\psi_j^\epsilon(\alpha) - C)(A\chi_\alpha - \chi_\alpha A),$$

where we are free to choose the constant C. In order to estimate the norm of this commutator, we consider

$$\chi_\nu [A, \Psi_j^\epsilon] \chi_\mu = \sum_\alpha (\psi_j^\epsilon(\alpha) - \psi_j^\epsilon(\frac{\nu+\mu}{2}))\chi_\nu (A\chi_\alpha - \chi_\alpha A)\chi_\mu, \tag{4.12}$$

and write (with $N > m$):

$$\|\chi_\nu A\chi_\alpha\chi_\mu\| = \|\sum_\beta \chi_\nu A\chi_\beta\chi_\beta\chi^{-2}\chi_\alpha\chi_\mu\|$$
$$\leq \mathcal{O}(1)\sum_\beta U(\nu-\beta)\langle\beta-\alpha\rangle^{-N}\langle\alpha-\mu\rangle^{-N} = \mathcal{O}(1)(U*\langle\cdot\rangle^{-N})(\nu-\alpha)\langle\alpha-\mu\rangle^{-N}.$$

In the same way,

$$\|\chi_\nu\chi_\alpha A\chi_\mu\| \leq \mathcal{O}(1)\langle\nu-\alpha\rangle^{-N}(\langle\cdot\rangle^{-N}*U)(\alpha-\mu).$$

Now estimate,

$$\psi_j^\epsilon(\alpha)-\psi_j^\epsilon(\frac{\nu+\mu}{2}) = \mathcal{O}(1)\min(1,\epsilon|\alpha-\frac{\nu+\mu}{2}|). \tag{4.13}$$

Then, from (4.12) and the subsequent estimates with $U_N = \langle\cdot\rangle^{-N}*U$:

$$\|\chi_\nu[A,\Psi_j^\epsilon]\chi_\mu\| \leq$$
$$\mathcal{O}(1)\sum_\alpha(\min(1,\epsilon|\alpha-\nu|)\langle\nu-\alpha\rangle^{-N}U_N(\alpha-\mu)+\langle\nu-\alpha\rangle^{-N}\min(1,\epsilon|\alpha-\mu|)U_N(\alpha-\mu)+$$
$$\min(1,\epsilon|\alpha-\nu|)U_N(\nu-\alpha)\langle\alpha-\mu\rangle^{-N}+U_N(\nu-\alpha)\min(1,\epsilon|\alpha-\mu|)\langle\alpha-\mu\rangle^{-N}) =$$
$$\mathcal{O}(1)((\min(1,\epsilon|\cdot|)\langle\cdot\rangle^{-N})*U_N+\langle\cdot\rangle^{-N}*(\min(1,\epsilon|\cdot|)U_N))(\nu-\mu).$$

Since $\min(1,\epsilon|\cdot|)\langle\cdot\rangle^{-N}$ and $\min(1,\epsilon|\cdot|)U_N$ tend to zero in ℓ^1, we conclude that

$$\|\chi_\nu[A,\Psi_j^\epsilon]\chi_\mu\| \leq U_\epsilon(\nu-\mu),$$

with $U_\epsilon \to 0$ in ℓ^1 and hence,

$$\|[A,\Psi_j^\epsilon]\|_{\mathcal{L}(L^2,L^2)} \to 0,\ \epsilon\to 0,\ \text{(uniformly in } j). \tag{4.14}$$

In particular,

$$\|[A,\Psi_j^\epsilon]\Psi_k^\epsilon\| \to 0,\ \epsilon\to 0,\ \text{(uniformly in } j,k). \tag{4.15}$$

Let $|j-k|\geq C$, with C large enough, so that

$$\mathrm{dist}(\mathrm{supp}\psi_j^\epsilon,\mathrm{supp}\psi_k^\epsilon) \sim \frac{1}{\epsilon}|j-k|. \tag{4.16}$$

Consider,

$$\begin{aligned}\|\chi_\nu[A,\Psi_j^\epsilon]\Psi_k^\epsilon\chi_\mu\| &\leq \\ \sum_{\alpha,\beta}\|\chi_\nu A\psi_j^\epsilon(\alpha)\chi_\alpha\psi_k^\epsilon(\beta)\chi_\beta\chi_\mu\| &+ \sum_{\alpha,\beta}\|\chi_\nu\psi_j^\epsilon(\alpha)\chi_\alpha A\psi_k^\epsilon(\beta)\chi_\beta\chi_\mu\|.\end{aligned} \tag{4.17}$$

Write,

$$\chi_\nu A\chi_\alpha\chi_\beta\chi_\mu=\sum_{\nu_1}\chi_\nu A\chi_{\nu_1}\chi_{\nu_1}\chi^{-2}\chi_\alpha\chi_\beta\chi_\mu,$$

and conclude that

$$\begin{aligned}\|\chi_\nu A\chi_\alpha\chi_\beta\chi_\mu\|&\leq\mathcal{O}(1)\sum_{\nu_1}U(\nu-\nu_1)\langle\nu_1-\alpha\rangle^{-N}\langle\alpha-\beta\rangle^{-N}\langle\beta-\mu\rangle^{-N}\\&=\mathcal{O}(1)(U*\langle\cdot\rangle^{-N})(\nu-\alpha)\langle\alpha-\beta\rangle^{-N}\langle\beta-\mu\rangle^{-N}.\end{aligned}$$

Similarly, write

$$\chi_\nu\chi_\alpha A\chi_\beta\chi_\mu=\sum_{\nu_1,\nu_2}\chi_\nu\chi_\alpha\chi^{-2}\chi_{\nu_1}\chi_{\nu_1}A\chi_{\nu_2}\chi_{\nu_2}\chi^{-2}\chi_\beta\chi_\mu,$$

to conclude that,

$$\begin{aligned}\|\chi_\nu\chi_\alpha A\chi_\beta\chi_\mu\|&\leq\mathcal{O}(1)\sum_{\nu_1,\nu_2}\langle\nu-\alpha\rangle^{-N}\langle\alpha-\nu_1\rangle^{-N}U(\nu_1-\nu_2)\langle\nu_2-\beta\rangle^{-N}\langle\beta-\mu\rangle^{-N}\\&=\mathcal{O}(1)\langle\nu-\alpha\rangle^{-N}U_{N,N}(\alpha-\beta)\langle\beta-\mu\rangle^{-N},\end{aligned}$$

where $U_{N,N}=\langle\cdot\rangle^{-N}*\langle\cdot\rangle^{-N}*U$.

Using these estimates in (4.17), we get with $U_N=\langle\cdot\rangle^{-N}*U$:

$$\begin{aligned}\|\chi_\nu[A,\Psi_j^\epsilon]\Psi_k^\epsilon\chi_\mu\|&\leq\sum_{\alpha,\beta}U_N(\nu-\alpha)\psi_j^\epsilon(\alpha)\langle\alpha-\beta\rangle^{-N}\psi_k^\epsilon(\beta)\langle\beta-\mu\rangle^{-N}\\&+\sum_{\alpha,\beta}\langle\nu-\alpha\rangle^{-N}\psi_j^\epsilon(\alpha)U_{N,N}(\alpha-\beta)\psi_k^\epsilon(\beta)\langle\beta-\mu\rangle^{-N}.\end{aligned}\tag{4.18}$$

We are interested in the $\ell^2\to\ell^2$ norm of the corresponding matrix, $(\|\chi_\nu[A,\Psi_j^\epsilon]\Psi_k^\epsilon\chi_\mu\|)_{\nu,\mu}$. We use the Shur lemma and start by estimating $\sup_\nu\sum_\mu\|\chi_\nu[A,\Psi_j^\epsilon]\Psi_k^\epsilon\chi_\mu\|$:

We have:

$$\begin{aligned}&\sum_\mu\sum_{\alpha,\beta}U_N(\nu-\alpha)\psi_j^\epsilon(\alpha)\langle\alpha-\beta\rangle^{-N}\psi_k^\epsilon(\beta)\langle\beta-\mu\rangle^{-N}\\&=\sum_\alpha U_N(\nu-\alpha)\psi_j^\epsilon(\alpha)\psi_{k,N,N}^\epsilon(\alpha),\end{aligned}\tag{4.19}$$

where $\psi_{k,N,N}=\langle\cdot\rangle^{-N}*\langle\cdot\rangle^{-N}*\psi_k^\epsilon$. Using (4.16), we see that

$$\|\psi_j^\epsilon\psi_{k,N,N}^\epsilon\|_{\ell^\infty}\leq\mathcal{O}(1)\frac{\epsilon^N}{|j-k|^N},$$

so the expression (4.19) is $\leq\mathcal{O}(1)\epsilon^N|j-k|^{-N}\|U\|_{\ell^1}$. Also,

$$\begin{aligned}&\sum_{\mu,\alpha,\beta}\langle\nu-\alpha\rangle^{-N}\psi_j^\epsilon(\alpha)U_{N,N}(\alpha-\beta)\psi_k^\epsilon(\beta)\langle\beta-\mu\rangle^{-N}\\&\leq\mathcal{O}(1)\sup_{\alpha\in\operatorname{supp}\psi_j^\epsilon}\sum_\beta U_{N,N}(\alpha-\beta)\psi_k^\epsilon(\beta)\leq\mathcal{O}(1)\|U_{N,N}\|_{\ell^1(\operatorname{supp}\psi_j^\epsilon-\operatorname{supp}\psi_k^\epsilon)}\end{aligned}\tag{4.20}$$

as earlier. Summing up:

$$(4.21)\qquad \sup_\nu \sum_\mu \|\chi_\nu [A, \Psi_j^\epsilon]\Psi_k^\epsilon \chi_\mu\| \le \mathcal{O}(1)\Big(\frac{\epsilon^N \|U\|_{\ell^1}}{|j-k|^N} + \|U_{N,N}\|_{\ell^1(\mathrm{supp}\psi_j^\epsilon - \mathrm{supp}\psi_k^\epsilon)}\Big).$$

When estimating $\sup_\mu \sum_\nu$ of (4.18), the contribution from the last term in (4.18) can be treated as in (4.20) and we are left with estimating

$$\sum_{\nu,\alpha,\beta} U_N(\nu-\alpha)\psi_j^\epsilon(\alpha)\langle \alpha-\beta\rangle^{-N}\psi_k^\epsilon(\beta)\langle \beta-\mu\rangle^{-N}$$

$$\le \mathcal{O}(1)\|U_N\|_{\ell^1}\|\psi_{j,N}^\epsilon(\beta)\psi_k^\epsilon(\beta)\|_{\ell^\infty} \le \mathcal{O}(1)\|U_N\|_{\ell^1}\frac{\epsilon^N}{|j-k|^N},$$

so $\sup_\mu \sum_\nu (4.18)$ can be estimated by the RHS of (4.21), and we get for $|j-k|\ge C$:

$$(4.22)\qquad \begin{aligned} \|[A,\Psi_j^\epsilon]\psi_k^\epsilon\| &\le \mathcal{O}(1)\|(\|\chi_\nu[A,\Psi_j^\epsilon]\Psi_k^\epsilon\chi_\mu\|)_{\nu,\mu}\|_{\mathcal{L}(\ell^2,\ell^2)} \\ &\le \mathcal{O}(1)\Big(\frac{\epsilon^N}{|j-k|^N}\|U\|_{\ell^1} + \|U_{N,N}\|_{\ell^1(\mathrm{supp}\psi_j^\epsilon - \mathrm{supp}\psi_k^\epsilon)}\Big). \end{aligned}$$

Combining this with (4.15), we conclude that with a new U_ϵ:

$$(4.23)\qquad \|(\|[A,\Psi_j^\epsilon]\Psi_k^\epsilon\|)_{j,k}\|_{\mathcal{L}(\ell^2,\ell^2)} \le U_\epsilon(j-k),\ \ \|U_\epsilon\|_{\ell^1}\to 0,\ \epsilon\to 0.$$

Now assume that $\|u\|\le \|Au\|$ for all $u\in L^2$. Then

$$\|\Psi_j^\epsilon u\| \le \|A\Psi_j^\epsilon u\| \le \|\Psi_j^\epsilon A u\| + \|[A,\Psi_j^\epsilon]u\| =$$

$$\|\Psi_j^\epsilon Au\| + \|\sum_{k,\ell}[A,\Psi_j^\epsilon]\Psi_k(\Psi_k^\epsilon(\Psi^\epsilon)^{-2})(\Psi^\epsilon)^{-2}\Psi_\ell^\epsilon)\Psi_\ell^\epsilon u\|$$

$$\le \|\Psi_j^\epsilon Au\| + \mathcal{O}(1)\sum_{k,\ell} U_\epsilon(j-k)\langle k-\ell\rangle^{-N}\|\Psi_\ell^\epsilon\| = \|\Psi_j^\epsilon Au\| + \sum_\ell \widetilde{U}_\epsilon(j-\ell)\|\Psi_\ell^\epsilon u\|,$$

where $\widetilde{U}_\epsilon = \mathcal{O}(1)U_\epsilon * \langle\cdot\rangle^{-N}\to 0$ in ℓ^1 when $\epsilon\to 0$. For $\epsilon>0$ small enough, let $(\delta_0+V_\epsilon)*$ be the inverse of $(\delta_0-\widetilde{U}_\epsilon)*$. Then

$$(4.24)\qquad \|\Psi_j^\epsilon u\| \le \|\Psi_j^\epsilon Au\| + \sum_k V_\epsilon(j-k)\|\Psi_k^\epsilon Au\|.$$

Let B be a left inverse of A, so that:

$$\|\Psi_j^\epsilon Bu\| \le \|\Psi_j^\epsilon u\| + \sum_k V_\epsilon(j-\ell)\|\Psi_\ell^\epsilon u\|,$$

and in particular,

$$\|\Psi_j^\epsilon B\Psi_k^\epsilon u\| \le \|\Psi_j^\epsilon\Psi_k^\epsilon u\| + \sum_k V_\epsilon(j-\ell)\|\Psi_\ell^\epsilon\Psi_k^\epsilon u\|,$$

so that,

$$\|\Psi_j^\epsilon B \Psi_k^\epsilon\|_{\mathcal{L}(L^2,L^2)} \leq C_N(\langle j-k\rangle^{-N} + V_\epsilon * \langle\cdot\rangle^{-N}(j-k)),$$

and we conlude from Proposition 3.1, that $B \in \mathrm{Op}(S_{\mathrm{w}})$. Summing up, we have proved:

Theorem 4.1. *Let$A \in \mathrm{Op}(S_{\mathrm{w}})$ have an L^2 bounded left inverse B. Then $B \in \mathrm{Op}(S_{\mathrm{w}})$.*

5. The sharp Gårding inequality.

Proposition 5.1. *Let $a \in S_{\mathrm{w},2}$; $a = a_0(x,\xi) + ha_1(x,\xi)$ with $a_0(x,\xi) \geq 0$. Then there exists a constant $C > 0$ such that*

$$\Re(\mathrm{Op}_h(a)u|u) \geq -Ch\|u\|^2,\ u \in L^2(\mathbf{R}^n). \tag{5.1}$$

Our proof will follow the idea of the proof of the sharp Gårding inequality in [CF] and a very short proof can be obtained by adapting Exercise 4.9 of [GriS]. (See also [T].) We here give a slightly longer proof which illustrates the use of pseudors in the complex domain in the spirit of [S2,3]. See also [H2]. Our arguments below will be a little sketchy.

Let $\phi(x,y)$ be a holomorphic quadratic form on $\mathbf{C}^n \times \mathbf{C}^n$ with

$$\det\phi''_{xy} \neq 0, \tag{5.2}$$

$$\Im\phi''_{yy} > 0. \tag{5.3}$$

Then $\kappa : (y, -\phi'_y(x,y)) \mapsto (x, \phi'_x(x,y))$ is a linear canonical transformation. Put $\Phi(x) = \sup_{y\in\mathbf{R}^n} -\Im\phi(x,y)$. Then it is easy to check that Φ is strictly pluri-subharmonic, that

$$\Lambda_\Phi \underset{\mathrm{def}}{=} \{(x, \frac{2}{i}\frac{\partial\Phi}{\partial x}(x)); x \in \mathbf{C}^n\} \tag{5.4}$$

is I-Lagrangian and R-symplectic, in the sense that $\Im\sigma_{|\Lambda_\Phi} = 0$, $\Re\sigma_{|\Lambda_\Phi}$ is non-degenerate, where $\sigma = \sum d\xi_j \wedge dx_j$ is the complex symplectic form, and finally that $\Lambda_\Phi = \kappa(\mathbf{R}^{2n})$. Consider the "linearized" FBI (or generalized Bargmann) transform:

$$Tu(x;h) = C_\phi h^{-3n/4} \int e^{i\phi(x,y)/h} u(y)dy. \tag{5.5}$$

Let $H_\Phi = \mathcal{H}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n; e^{-2\Phi(x)/h}L(dx))$, where $\mathcal{H}(\mathbf{C}^n)$ denotes the space of entire functions and $L(dx) = d\Re x d\Im x$ denotes the Lebesgue measure. Choosing $C_\phi > 0$ suitably, we can arrange so that $T : L^2(\mathbf{R}^n) \to H_\Phi$ is isometric. Consider then the orthogonal projection $TT^* = H_\Phi \to T(L^2)$, where

$$T^*v(y;h) = C_\phi h^{-3n/4} \int e^{-i\overline{\phi(x,y)}/h} v(x) e^{-2\Phi(x)/h} L(dx) \tag{5.6}$$

is the complex adjoint of T.

Let $\psi(x,y)$ be the holomorphic quadratic form with $\psi(x,\overline{x}) = \Phi(x)$. It is easy to check that

$$TT^*v(x;h) = C_1 h^{-n} \int e^{2\psi(x,\overline{y})/h} v(y) e^{-2\Phi(y)/h} L(dy), \tag{5.7}$$

for some $C_1 > 0$ (and here $2\psi(x,\overline{y})$ is the critical value of $t \mapsto i(\phi(x,t) - \overline{\phi(y,\overline{t})})$. One can check directly the uniform boundedness of the RHS: $L^2(\mathbf{C}^n; e^{-2\Phi/h}L(dx)) \to L^2(\mathbf{C}^n; e^{-2\Phi/h}L(dx))$, by using the fact that $\Phi(x) + \Phi(y) - 2\Re\psi(x,\overline{y}) \sim |x-y|^2$.

Consider on the other hand the identity operator on H_Φ as a pseudor:

$$\Pi u(x;h) = \frac{1}{(2\pi h)^n} \int\int e^{\frac{i}{h}(x-y)\cdot\eta} v(y) dy d\eta \tag{5.8}$$

(equipped with a suitable integration contour). Write $2(\psi(x,\theta) - \psi(y,\theta)) = i\frac{2}{i}\frac{\partial\psi}{\partial x}(\frac{x+y}{2},\theta)\cdot(x-y)$. Then we relate η and θ by $\eta = \frac{2}{i}\frac{\partial\psi}{\partial x}(\frac{x+y}{2},\theta)$, so that $dyd\eta = Cdyd\theta$ with $C \neq 0$. Then with a new C:

$$\Pi u(x;h) = Ch^{-n} \int\int e^{\frac{2}{h}(\psi(x,\theta)-\psi(y,\theta))} u(y) dy d\theta. \tag{5.9}$$

A suitable integration contour, is given by $\theta = \overline{y}$ and since $L(dy) = \text{Const.}dyd\overline{y}$:

$$\Pi u(x;h) = \widetilde{C}_1 h^{-n} \int e^{2\psi(x,\overline{y})/h} v(y) e^{-2\Phi(y)/h} L(dy). \tag{5.10}$$

We know that $\Pi u = u$ for $u \in H_\Phi$ and that $TT^*u = u$ for some non-trivial $u \in H_\Phi$. Hence $\widetilde{C}_1 = C_1$, so $\Pi = TT^*$. In particular T is unitary.

If $a \in \mathcal{S}(\mathbf{R}^{2n})$ and $b \in \mathcal{S}(\Lambda_\Phi)$ are related by $b \circ \kappa = a$, then since we are in a metaplectic situation,

$$BT = TA, \tag{5.11}$$

where $A = \text{Op}_h(a)$ and

$$Bu(x) = \text{Op}_h(b)u(x) = \frac{1}{(2\pi h)^n} \int\int e^{i(x-y)\cdot\eta/h} b(\frac{x+y}{2},\eta) u(y) dy d\eta. \tag{5.12}$$

Here the (only possible) integration contour (in general) is given by $\eta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2})$. It is well adapted to Φ, since we then have $-\Phi(x) + \Re(i(x-y)\cdot\eta) + \Phi(y) = 0$.

For $0 \leq t \leq \frac{1}{2}$, consider operators of the form

$$A_t u(x;h) = Ch^{-n} \int\int_{\Gamma_t} e^{\frac{2}{h}(\psi(x,\theta)-\psi(y,\theta))} a_t(tx + (1-t)y, \theta) u(y) dy d\theta, \tag{5.13}$$

where $\Gamma_t(x)$ is given by $\theta = \overline{tx + (1-t)y}$, and $a_t \in \mathcal{S}(\{(x,\theta) \in \mathbf{C}^{2n}; \theta = \overline{x}\})$. For $t = 1/2$, we can change variables $\theta \to \eta$ and we then get the same integration contour as in (5.12). For $t=0$, we get $\theta=\overline{y}$ and the operator becomes very similar to (5.10). For intermediate values of t, we also have a well adapted contour, since $\psi(x,tx+(1-t)y)-\psi(y,tx+(1-t)y)$ is affine linear in t.

For $t=0$, we also notice that $A_0u=\Pi a_0u$, where $a_0=a_0(y,\overline{y})$, so A_0 is a Toeplitz operator and in particular we have $A_0\geq 0$ in the sense of self-adjoint operators on H_Φ if $a_0\geq 0$.

We now require a_t to depend smoothly on t, and ask when A_t is independent of t (when acting on H_Φ). The t-derivative of the RHS of (5.13) is

$$Ch^{-n}\int\!\!\int_{\Gamma_t} e^{\frac{2}{h}(\psi(x,\theta)-\psi(y,\theta))}\Big(\frac{\partial a_t}{\partial t}+(x-y)\cdot\frac{\partial a_t}{\partial x}\Big)(tx+(1-t)y,\theta)u(y)dyd\theta,$$

and the contribution from $(x-y)\cdot\frac{\partial a_t}{\partial x}$ can be transformed by means of integrations by parts in θ (or rather by means of Stokes' formula). We see that A_t is independent of t if

$$\frac{\partial a_t}{\partial t}=\frac{h}{2}(\psi''^{-1}_{\theta x}\frac{\partial}{\partial\theta})\cdot\frac{\partial}{\partial x}a_t,\quad a_t=a_t(x,\theta). \tag{5.14}$$

Recalling that we work on $\theta=\overline{x}$, we can rewrite this as

$$\frac{\partial a_t}{\partial t}=\frac{h}{2}(\Phi''^{-1}_{\overline{x}x}\frac{\partial}{\partial\overline{x}})\cdot\frac{\partial}{\partial x}a_t. \tag{5.15}$$

Identifying $\mathbf{C}_\xi$ with $\mathbf{R}^{2n}$ in the standard way we see that the symbol of $(\Phi''^{-1}_{\overline{x}x}\frac{\partial}{\partial\overline{x}})\cdot\frac{\partial}{\partial x}$ is $-\frac{1}{4}\Phi''^{-1}_{\overline{x}x}\xi\cdot\overline{\xi}$ which is <0 when $0\neq\xi\in\mathbf{C}^n$. Hence (5.15) is a heat equation with the solution

$$a_t=\exp\Big(\frac{th}{2}\Phi''^{-1}_{\overline{x}x}\frac{\partial}{\partial\overline{x}}\cdot\frac{\partial}{\partial x}\Big)a_0 \tag{5.16}$$

for any given $a_0\in\mathcal{S}$. The results of section 1 about $e^{-ih\Phi^{-1}(D_x)}$ carry over to $e^{-h\Phi^{-1}(D_x)}$, when Φ^{-1} is positive definite, so if a_0 is independent of h and all its derivatives of order ≤ 2 belong to S_{w}, then $a_{\frac{1}{2}}=a_0+hr(x,\theta;h)$, with $r(\cdot;h)$ bounded in S_{w}. We can partly reverse this and observe that if $a_{\frac{1}{2}}=a_0(x,\theta)+ha_1(x,\theta;h)\in S_{\mathrm{w},2}$ and if $a_0\geq 0$, then $A_{\frac{1}{2}}=\Pi a_0+\mathcal{O}(h)$ in $\mathcal{L}(H_\Phi,H_\Phi)$ and in particular, $\Re(A_{\frac{1}{2}}u|u)\geq -Ch\|u\|^2_{H_\Phi}$. However from the change of variables $\theta\leftrightarrow\eta$, we see that $A_{\frac{1}{2}}$ is a h-pseudor in the Weyl quantization and with the symbol a symbol in $S_{\mathrm{w},2}$ with leading part ≥ 0. The relation (5.11) allows us to go from the $L^2(\mathbf{R}^n)$ setting to that of H_Φ and the theorem follows. $\sharp$

It would be interesting to know if there is a corresponding low regularity version of the Melin inequality.

It seems possible also to prove the L^2 boundedness directly in the FBI presentation in two different ways: either by contour deformation (and suitable almost analytic extension) or by estimating the kernel of $\Pi B\Pi$ (illustrating the general point of view of [S3]) and then in both cases by applying Shur's lemma. We leave this open until needed.

References.

[Be] R.Beals, *Characterization of P.D.O. and applications,* Duke math. J. 44(1977), 45-57.

[B] J.M.Bony, *Opérateurs intégraux de Fourier et calcul de Weyl-Hörmander. (Cas d'une métrique symplectique),* Proceedings of the "Journées d'équations aux dérivées partielles" St Jean de Monts, 30 mai- 3 juin 1994. Ecole Polytechnique, Université de Nantes, Université de Rennes 1

[CF] A.Cordoba, C.Fefferman, *Wave packets and Fourier integral operators,* Comm. P.D.E. 3(11)(1978), 979-1006.

[GriS] A.Grigis, J.Sjöstrand, *Microlocal analysis for differential operators,* London Math. Soc. Lect. Notes Series 196, Cambridge Univ. Press (1994).

[GraSch] B.Gramsch, E.Schrohe, *Submultiplicativity of Boutet de Monvel's algebra for boundary value problems,* Preprint Max-Planck-Institut für Mathematik Bonn (1994).

[GraUeW] B.Gramsch, J.Ueberberg, K.Wagner, *Spectral invariance and submultiplicativity for Fréchet algebras with applications to pseudodifferential operators and Ψ^*-quantization,* pages 71-98 in "Operator calculus and spectral theory", Symposium on operator calculus and spectral theory, Lambrecht (Germany), December 1991, Ed. by M.Demuth..., Birkhäuser 1992, Operator theory, Vol. 57.

[HeS] B.Helffer, J.Sjöstrand, *Analyse semi-classique pour l'équation de Harper.* Bull. de la SMF, 116(4)(1988), mémoire no 34.

[H1] L.Hörmander, *The analysis of linear partial differential operators, I-IV,* Grundlehren, Springer, 256, 257, 274, 275 (1983-85).

[H2] L.Hörmander, *Quadratic hyperbolic operators,* Springer L.N.M., 1495 (1991), 118-160.

[KN] J.J.Kohn, L.Nirenberg, *On the algebra of pseudo-differential operators,* Comm. Pure Appl. Math. 18(1965), 269-305.

[LKa] P.G.Lemarié-Rieusset, J.P.Kahane, Book to appear, chapter 10.

[Lo] L.Loomis, *An introduction to abstract harmonic analysis,* Van Nostrand, Toronto, New York, London, (1953).

[Sh] M.S.Shubin, *Almost periodic functions and partial differential equations,* Uspechi Math. Nauk 33(2)(1978), 243-275, Russian Math. Surveys 33(2)(1978), 1-52.

[S1] J.Sjöstrand, *An algebra of pseudodifferential operators,* Math. Res. Lett. 1(1994),185-192.

[S2] J.Sjöstrand, Unpublished lecture notes from Lund (1985-86).

[S3] J.Sjöstrand, *Singularités analytiques microlocales,* Astérisque, 95(1982).

[T] M.Taylor, *Pseudodifferential operators,* Princeton Univ. Press (1981).

[U] A.Unterberger, *Les opérateurs métadifférentiels,* Lecture Notes in Physics, 126(1980), 205-241.

[Z] A.Zygmund, *Trigonometric series,* Cambridge Univ. Press, 2nd edition (1959). See vol. 1, page 246 and note on page 380.