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EQUATIONS AUX DERIVEES PARTIELLES

THE INVERSE N-BODY PROBLEM A GEOMETRICAL APPROACH

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The Inverse N-Body Problem. A Geometrical Approach.

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1 Introduction

In this talk I present recent results on the inverse N-body scattering problem in quantum mechanics that were obtained with a simple geometrical time-dependent method. We study the high velocity limit of the scattering operator and we obtain formulas with error term for the reconstruction of the potential. In particular we prove that any one of the Dollard scattering operators determines uniquely the potential. We also consider the inverse N-body scattering problem in the case where the particles are in the presence of a constant external electric field. This case is particularly interesting because potentials that are of long range in the absence of a constant external electric field become of short range when the electric field is added. For example in the two-body case the potentials that decay at infinity as $V(\mathbf{x}) \approx |\mathbf{x}|^{-\gamma}, 1/2 < \gamma \leq 1$, are of long range when there is no electric field and they require the introduction of a modified free time evolution in order to define the wave and scattering operators. It is a remarkable fact that when a constant external electric field is added the same potentials are of short range in the sense that the ordinary wave and scattering operators exist, i. e. it is not necessary to introduce a modified free time evolution. In the N-body case when the electric field is present we take the standard free time evolution for the relative motion of the pairs of particles whose reduced charge is different from zero and a Dollard modified free time evolution for the relative motion of the pairs of particles with reduced charge zero. We study the high velocity limit of the modified scattering operator and we obtain formulas with error term for the reconstruction of the potential. In particular we prove that any one of the modified scattering operators in a constant external electric field determines uniquely the potential. In the particular case when the relative charge of all pairs of particles is different from zero we uniquely reconstruct the potential from the canonical scattering operator defined with the standard free time evolution in the presence of a constant external electric field. In all of the above mentioned results in uniqueness and reconstruction it is only necessary to know the high velocity limit of the corresponding scattering operator. The results presented in this talk are contained in the papers by Enss and Weder [1], [2], [3] and by Weder [4].

2 The Results

Let m_j and $\tilde{\mathbf{x}}_j \in \mathbb{R}^n, j = 1, \ldots, N, n \geq 2$, be respectively, the masses and the positions of the particles. The free Hamiltonian of the system is

$$\tilde{H}_0 = \sum_{j=1}^{N} (2m_j)^{-1} \tilde{\mathbf{p}}_j^2, \tilde{\mathbf{p}}_j = -i\nabla_{\tilde{\mathbf{x}}_j}.$$

As usual we work on the center of mass frame. The kynetic energy of the center of mass is given by

$$H_{CM} = \left(2\sum_{j=1}^{N} m_j\right)^{-1} \left(\sum_{j=1}^{N} \tilde{\mathbf{p}}_j\right)^2 .$$

We separate off the motion of the center of mass and we obtain the free Hamiltonian

$$H_0 := \tilde{H}_0 - H_{CM}. \tag{2.1}$$

The abstract Hilbert space of states, $\mathcal{H}$, in the center of mass frame is represented in configuration space by wave functions ϕ in the space

$$L^2(\mathbf{X}), \mathbf{X} = \{(\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_N) : \sum_{j=1}^{N} m_j \tilde{\mathbf{x}}_j = 0\} \cong \mathbb{R}^{n(N-1)}$$

with the measure induced on $\mathbf{X}$ by the norm on $\mathbb{R}^{nN}$ $|||\mathbf{x}||| = \left[\sum_{j=1}^{N} m_j \tilde{\mathbf{x}}_j^2\right]^{1/2}$. Similarly the set of momentum space wave functions $\hat{\phi}$ is the space

$$L^2(\hat{\mathbf{X}}), \hat{\mathbf{X}} = \{(\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_N) : \sum_{j=1}^{N} \tilde{\mathbf{p}}_j = 0\} \cong \mathbb{R}^{n(N-1)},$$

where we give to $\hat{\mathbf{X}}$ the dual metric induced by $\left[\sum_{j=1}^{N} (m_j)^{-1} \tilde{\mathbf{p}}_j^2\right]^{1/2}$. The configuration and momentum space wave functions are related to each other by Fourier transform. The free Hamiltonian H_0 is self-adjoint on the domain $D(H_0) = W^{2,2}(\mathbf{X})$. We consider potentials that are a sum of pair potentials that are operators of multiplication by real valued functions.

$$V = \sum_{j<k} V_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j).$$

We split each pair potential into a short-range part and a long-range part. The short-range part is allowed to have singularities but it decays integrably to zero at infinity. The long-range part has continuous derivatives, but it decays slowly at infinity

$$V_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) = V_{jk}^s(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V_{jk}^l(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j). \tag{2.2}$$

The class of short-range real valued potentials is defined as

$$\begin{aligned} \mathcal{V}_{SR} = \{V^s = \sum_{j<k} V_{jk}^s(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) : V_{jk}^s \text{ is Kato-small,} \\ \text{and } \int_0^\infty (1+R)^\rho \|V_{jk}^s(\mathbf{y})(-\Delta + I)^{-1} F(|\mathbf{y}| > R)\| dR < \infty, \\ \text{for some } 0 \le \rho \le 1\}. \end{aligned} \tag{2.3}$$

For any set $O \in \mathbb{R}^n$ we denote by $F(\mathbf{x} \in O)$ the operator of multiplication by the characteristic function of O. The class of long-range real valued potentials is given by

$$\mathcal{V}_{LR} = \{V^l = \sum_{j<k} V^l_{jk}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) : V^l_{jk} \in C^1_\infty(\mathbb{R}^n), \\ |(\nabla V^l_{jk})(\mathbf{y})| \leq C(1 + |\mathbf{y}|)^{-\gamma}, \gamma > 3/2\}, \tag{2.4}$$

where $C^1_\infty(\mathbb{R}^n)$ is the space of continuously differentiable functions that tend to zero towards infinity. To simplify the notation we will use $\gamma < 2$. Clearly the separation into short-and long-range parts is not unique, and it is well known (see [5]) that without lossing generality it can be made in such a way that furthermore

$$V^l_{jk} \in C^4(\mathbb{R}^n), |D^\alpha V^l_{jk}(\mathbf{y})| \leq C(1 + |\mathbf{y}|)^{-1-|\alpha|(\varepsilon+1/2)} \tag{2.5}$$

$1 \leq |\alpha| \leq 4$, where D^α denotes the derivatives with the usual multi-index notation and $\varepsilon > 0$. In what follows we assume that a spliting according to (2.4), (2.5) has been made and is kept fixed. The interacting Hamiltonian is $H = H_0 + V, D(H) = D(H_0)$, and it is a self-adjoint operator by the Kato-Rellich theorem.

The Dollard modified free time evolution is generated by the following time-dependent Hamiltonian

$$H_D(t) := H_0 + \sum_{j<k} V^l_{jk}(t\mathbf{p}_{jk}/\mu_{jk}) \tag{2.6}$$

where $\mathbf{p}_{jk}/\mu_{jk} = (\tilde{\mathbf{p}}_k/m_k - \tilde{\mathbf{p}}_j/m_j)$ is the relative velocity of the particles j and k, and $\mu_{jk} = m_j m_k/(m_j + m_k)$ is their reduced mass. The Dollard propagator is given by

$$U^D(t) := \mathrm{e}^{-itH_0}\exp\left[-i\sum_{j<k}\int_0^t ds V^l_{jk}(s\mathbf{p}_{jk}/\mu_{jk})\right]. \tag{2.7}$$

The modified Dollard wave and scattering operators are defined as follows

$$\Omega^D_\pm = s - \lim_{t\to\pm\infty} e^{itH}U^D(t), \tag{2.8}$$

$$S^D = (\Omega^D_+)^*\Omega^D_- = S^D(V^l; V^s). \tag{2.9}$$

The existence of the strong limits in (2.8) is well known.

In order to define the high velocity states we introduce appropriate coordinates. We will reconstruct the pair potentials V_{jk} one by one. Let us assume that the pair of interest consists of particles 1 and 2. As usual we take as one coordinate their relative distance and the corresponding momentum

$$\mathbf{x} = \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1, \qquad \mathbf{p} = -i\nabla_{\tilde{\mathbf{x}}}.$$

As the remaining $N-2$ coordinates we take the distance of particle $j = 3, \ldots, N$ to the center of mass of particles 1 and 2, and the corresponding momentum

$$\mathbf{x}_j := \tilde{\mathbf{x}}_j - (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)/(m_1 + m_2), \\ \mathbf{p}_j = \mu_j(\tilde{\mathbf{p}}_j/m_j - (\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2)/(m_1 + m_2)),$$

with

$$\mu_j = m_j(m_1+m_2)/(m_j+m_1+m_2),$$

$j = 3, \ldots N$. Let Φ_0 be a state in $\mathcal{H}$ whose momentum space wave function (given by Fourier transform) is a product function

$$\Phi_0 \sim \hat{\phi}_{12}(\mathbf{p}) \qquad \hat{\phi}_3(\mathbf{p}_3, \ldots, \mathbf{p}_N),$$

where $\hat{\phi}_{12} \in C_0^\infty(\mathbb{R}^n)$, $\qquad \hat{\phi}_3 \in C_0^\infty(\mathbb{R}^{n(N-2)})$, and $\|\Phi_3\| = 1$. The high velocity state is defined as follows

$$\Phi_{\mathbf{v}} \sim \hat{\phi}_{12}(\mathbf{p}-m\mathbf{v}) \quad \hat{\phi}_3(\mathbf{p}_3 - m\mathbf{v}_3, \ldots, \mathbf{p}_N - m\mathbf{v}_N), \tag{2.10}$$

or equivalently

$$\Phi_{\mathbf{v}} = e^{im\mathbf{v}\cdot\mathbf{x}} \Pi_{j=3}^N e^{i\mu_j \mathbf{v}_j \cdot \mathbf{x}_j} \Phi_0, \tag{2.11}$$

where $\mathbf{v} = v\hat{\mathbf{v}}, |\hat{\mathbf{v}}| = 1$, and for $j = 3, \ldots, N, \mathbf{v}_j = v^2 \mathbf{e}_j \neq 0, \mathbf{e}_j \neq \mathbf{e}_k, j \neq k$. In the high velocity state $\Phi_{\mathbf{v}}$ the average relative velocity of the pair (1 2) is v and all other particles have minimal velocity proportional to v^2 relative to all other particles.

The intuitive idea behind our reconstruction method is that the time that the particles spend in the interaction region is proportional to $1/v$. So in the high velocity limit the interaction goes to zero and the scattering operator goes to the identity. Then substracting the identity from the scattering operator and after rescaling we are able to reconstruct the potential from the leading term as $v \to \infty$. Here it is relevant that in the high velocity limit the spreading of the wave packet under the free time evolution can be neglected and it only remains a translation. The reconstruction formulas are given in the following Theorem.

Theorem 2.2 (Enss and Weder [3]) suppose that $V^s \in \mathcal{V}_{SR}$ and $V^l \in \mathcal{V}_{LR}$. Then for all $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ as in (2.11)

$$\begin{aligned} &v\big[(i(S^D - I)\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) - \int_{-\infty}^{+\infty} dt((V_{12}^l(\mathbf{x}+\tau\hat{\mathbf{v}}) - V_{12}^l(t\mathbf{p}/m)) \\ &U^D(t)\Phi_{\mathbf{v}}, U^D(t)\Psi_{\mathbf{v}})\big] = (\int_{-\infty}^{+\infty} d\tau V_{12}^s(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_{12}, \Psi_{12}) \\ &+\begin{cases} o(v^{-\rho}), & 0 \le \rho < 1, \\ O(v^{-1}), & \rho = 1. \end{cases} \end{aligned} \tag{2.12}$$

Moreover suppose that in (2.3) $0 \le \rho < \gamma - 1$. Then for any $1 \le j_0 \le n$,

$$\begin{aligned} &v(i[S^D, p_{j_0}]\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) = \int_{-\infty}^{+\infty} d\tau \big[(V_{12}^s(\mathbf{x}+\tau\hat{\mathbf{v}})p_{j_0}\Phi_{12}, \Psi_{12}) \\ &-(V_{12}^s(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_{12}, p_{j_0}\Psi_{12})\big] + i\int_{-\infty}^{+\infty} d\tau((\partial_{j_0}V_{12}^l)(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_{12}, \Psi_{12}) \\ &+o(v^{-\rho}). \end{aligned} \tag{2.13}$$

If the long-range part of the potential, V^l, is a priori known, we obtain from (2.11) the Radon transform of V_{12}^s. Inverting this Radon transform we reconstruct V_{12}^s. This reconstruction method is of particular interest in the short-range case, $V^l \equiv 0$. Using formula (2.13) we obtain from the high velocity limit of the commutator of S^D with a component of $\mathbf{p}$ the Radon transform of the corresponding derivative of the potential V_{12}. Inverting this Radon tranform and integrating we reconstruct V_{12}. Since we actually reconstruct the scalar product of the potential between states we only need to consider $\hat{\mathbf{v}}$ in a two dimensional subspace in both reconstruction methods. See [3] for details. Moreover we obtain the following result in the uniqueness of the inverse problem.

Corollary 2.2

Any one of the modified Dollard scattering operators, S^D, determines uniquely the potential V. In particular the scattering map $S^D(V^l, \cdot) : \mathcal{V}_{S,R} \to \mathcal{L}(\mathcal{H})$ is injective.

Let us now consider the case of scattering in the presence of a constant external electric field. We will only discuss here the two-body case. For the N-body case see Weder [4]. The Hamiltonian of a particle, or of a two-body system in the center of mass frame, in a constant external electric field is the following operator in $L^2(\mathbb{R}^n)$

$$H_0^{\mathbf{E}} = (2m)^{-1}\mathbf{p}^2 + q\mathbf{E}\cdot\mathbf{x},$$

$q \neq 0$ is the electric charge of the particle, or the reduced charge of the two-body system. We will take an electric field directed along the $-x_1$ direction, i.e. $\mathbf{E} = (-E, 0, \ldots, 0), E = |\mathbf{E}| > 0. H_0^{\mathbf{E}}$ is essentially self-adjoint on the space of Schwartz. We also denote by $H_0^{\mathbf{E}}$ its unique self-adjoint realization. In what follows we always assume that $n \geq 3$. We introduce now an appropriate class of real valued potentials.

Definition 2.3

We denote by $\mathcal{V}_{\mathbf{E}}$ the class of potentials, $V^{\mathbf{E}}(\mathbf{x})$, such that

$$V^{\mathbf{E}}(\mathbf{x}) = V^{\mathbf{E},s}(\mathbf{x}) + V^{\mathbf{E},l}(\mathbf{x}),$$

with $V^{\mathbf{E},s} = V_1^{\mathbf{E},s}(\mathbf{x}) + V_2^{\mathbf{E},s}(\mathbf{x})$, $(1+|x_1|)V_1^{\mathbf{E},s}(\mathbf{x})$ Kato-small, $V_2^{\mathbf{E},s}(\mathbf{x})$ bounded,

$$|V^{\mathbf{E},l}(\mathbf{x})| \leq C(1+|\mathbf{x}|)^{-\gamma}, 3/4 < \gamma \leq 1, \tag{2.14}$$

and for some $0 \leq \rho \leq 4\gamma - 3$

$$\int_0^\infty (1+R)^\rho \|V^{\mathbf{E},s}(\mathbf{x})(-\Delta + I)^{-1}F(|\mathbf{x}| > R)\| dR < \infty, \tag{2.15}$$

$$(1+|\mathbf{x}|)^\rho |(\partial_{j_0} V^{\mathbf{E},l})(\mathbf{x})| \leq C(1+|\mathbf{x}|)^{-1-\varepsilon}, \tag{2.16}$$

for some $2 \leq j_0 \leq n, \varepsilon > 0$.

The interacting Hamiltonian is now defined as

$$H^{\mathbf{E}} = H_0^{\mathbf{E}} + V^{\mathbf{E}}(\mathbf{x}), D(H) = D(H_0). \tag{2.17}$$

The wave operators are defined as

$$\Omega_\pm^{\mathbf{E}} = s - \lim_{t \to \pm\infty} e^{itH^{\mathbf{E}}} e^{-itH_0^{\mathbf{E}}}, \tag{2.18}$$

and the scattering operator is given by

$$S^{\mathbf{E}} = (\Omega_+^{\mathbf{E}})^* \Omega_-^{\mathbf{E}}. \tag{2.19}$$

The existence of the strong limits in (2.18) is well known (see [4] for references to the original contributions).

The high velocity states are now defined as

$$\Phi_{\mathbf{v}} = e^{im\mathbf{v}\cdot\mathbf{x}}\Phi_0 \Leftrightarrow \hat{\phi}_{\mathbf{v}}(\mathbf{p}) = \hat{\phi}(\mathbf{p} - m\mathbf{v}), \tag{2.20}$$

with $\hat{\phi} \in C_0^\infty(\mathbb{R}^n)$.

The reconstruction formula when the constant external electric field is present is given in the following theorem. Recall that $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$.

Theorem 2.4 (Weder [4])
Suppose that $V^{\mathbf{E}} \in \mathcal{V}_{\mathbf{E}}$.
Then for all $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ as in (2.20) with $\mathbf{v}\cdot\mathbf{E} = 0$,

$$\begin{aligned} v\left(i[S^{\mathbf{E}}, p_{j_0}]\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right) &= \int_{-\infty}^{+\infty} d\tau \big[(V^{\mathbf{E},s}(\mathbf{x}+\tau\hat{\mathbf{v}})p_{j_0}\Phi_0, \Psi_0) \\ &-(V^{\mathbf{E},s}(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0, p_{j_0}\Psi_0)\big] + i\int_{-\infty}^{+\infty} d\tau((\partial_{j_0}V^{\mathbf{E},l})(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_0, \Psi_0) \\ &+\begin{cases} o(v^{-\rho}), 0 \le \rho < 4\gamma - 3, \\ 0(v^{-\rho}), \rho = 4\gamma - 3 < 1, \\ 0((\ln v)^2 v^{-1}), \rho = 4\gamma - 3 = 1. \end{cases} \end{aligned} \tag{2.21}$$

Using (2.21) we reconstruct from the high velocity limit of the commutator of $S^{\mathbf{E}}$ with p_{j_0} the Radon transform of the corresponding derivative of the potential, and integrating we reconstruct $V^{\mathbf{E}}$ (see [4]). In particular we prove the uniqueness of the inverse problem.

Corollary 2.5
The scattering map $S^{\mathbf{E}}(\cdot) : \mathcal{V}_{\mathbf{E}} \to \mathcal{L}(\mathcal{H})$ is injective.

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