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EQUATIONS AUX DERIVEES PARTIELLES

CONSTRUCTIBLE SHEAVES, WHITNEY FUNCTIONS AND SCHWARTZ'S DISTRIBUTIONS

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(joint work with Masaki Kashiwara)

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PIERRE SCHAPIRA (JOINT WORK WITH MASAKI KASHIWARA)

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1 Introduction

Let f be a proper morphism of real analytic manifolds. It is a natural problem to characterize the space of the integrals along f of all $\mathcal{C}^\infty$ or distribution densities. Such problems occur in particular when studying correspondences:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd}$$

as, for example, the Penrose correspondance or else the Radon transform (see [2]). The simple case of $X = \mathbb{C}(\simeq \mathbb{R}^2)$ and $f : X \longrightarrow X$ is the map $z \mapsto z^2$ shows that the constructible sheaf $f_*\mathbb{C}_X$, the direct image of the constant sheaf $\mathbb{C}_X$ on X by f, plays an essential role in this description.

In [5], Kashiwara has introduced the functor $TH_X(\cdot)$ of moderate cohomology. It is defined on the category of $\mathbb{R}$-constructible sheaves on the real analytic manifold X, and is characterized by the fact that it is exact and its value on $\mathbb{C}_Z$ for a closed subanalytic subset Z of X is the sheaf $\Gamma_Z(\mathcal{D}b_X)$ of distributions supported by Z. Here, we shall give another construction of $TH_X(\cdot)$ (that we prefere to denote by $\mathcal{T}hom(\cdot, \mathcal{D}b_X)$) and at the same time, we construct a new functor, dual to the preceding one, the Whitney functor $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$. This is again an exact functor characterized by the fact that its value on $\mathbb{C}_U$, for U an open subanalytic subset of X, is the subsheaf $\mathcal{I}^\infty_{X \setminus U, X}$ of $\mathcal{C}^\infty$-functions which vanish to infinite order on $X \setminus U$. If now X is a complex manifold, taking the Dolbeault complexes of the preceding functors, we get the functors $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ and $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$. The main result of this paper will be the adjunction formulas in section 6.

Section 1 to 6 of this paper are extracted from [7], and section 7 is extracted from [2].The redaction is due to PS, and does not involve the responsability of other authors.

2 Construction of functors on $\mathbb{R} - C(X)$

Let X be a real analytic manifold and denote by $Mod(\mathbb{C}_X)$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on X, by $\mathbb{R} - C(X)$ the abelian subcategory of $\mathbb{R}$-constructible sheaves and by $\mathbb{R} - C_c(X)$ the subcategory of $\mathbb{R} - C(X)$ of sheaves with compact support. Denote by $\mathcal{S}_X$ the category whose objects are the open subanalytic relatively compact subsets of X, the only morphisms being the inclusions $U \subset V$. Then $U \mapsto \mathbb{C}_U$ gives a faithful functor $\mathcal{S}_X \longrightarrow \mathbb{R} - C(X)$. Let $\mathcal{A}$ be an abelian category over $\mathbb{C}$. This means that $\mathrm{Hom}_{\mathcal{A}}(M, N)$ has a structure of $\mathbb{C}$-vector space for M, $N \in \mathcal{A}$, and the composition of morphisms is $\mathbb{C}$-bilinear. Let $\psi : \mathcal{S}_X \longrightarrow \mathcal{A}$ be a functor, and consider the conditions:

$$\psi(\emptyset) = 0. \tag{2.1}$$

$$\begin{cases} \text{for any } U,\ V \text{ in } \mathcal{S}_X, \text{ the sequence} \\ \psi(U \cap V) \longrightarrow \psi(U) \oplus \psi(V) \longrightarrow \psi(U \cup V) \longrightarrow 0 \\ \text{is exact.} \end{cases} \tag{2.2}$$

$$\text{for any open inclusion } U \subset V \text{ in } \mathcal{S}_X,\ \psi(U) \longrightarrow \psi(V) \text{ is a monomorphism} \tag{2.3}$$

Theorem 2.1 *(i) Assume (2.1) and (2.2). Then there is a right exact functor, unique up to isomorphism,*

$$\Psi : \mathbb{R} - C_c(X) \longrightarrow \mathcal{A}$$

such that $\Psi(\mathbb{C}_U) \simeq \psi(U)$ functorially in $U \in \mathbb{R} - C_c(X)$.

(ii) Assume (2.1), (2.2) and (2.3). Then Ψ is exact.

(iii) Let ψ_1 and ψ_2 be two functors from $\mathcal{S}_X$ to $\mathcal{A}$ both satisfying (2.1) and (2.2), and let Ψ_1 and Ψ_2 be the corresponding functors given in (i). Let $\theta : \psi_1 \longrightarrow \psi_2$ be a morphism of functors. Then θ extends uniquely to a morphism of functors

$$\Theta : \Psi_1 \longrightarrow \Psi_2.$$

(iv) In the situation of (i), assume that $\mathcal{A}$ is a subcategory of the category $Mod(\mathbb{C}_X)$ of sheaves of $\mathbb{C}$-vector spaces on X, and that $\mathcal{A}$ is local, that is: an object F of $\mathrm{Mod}(\mathbb{C}_X)$ belongs to $\mathcal{A}$ if for any relatively compact open U there exists F' in $\mathcal{A}$ such that $F|_U \simeq F'|_U$. Assume further that ψ is local, that is: $\mathrm{supp}(\psi(U)) \subset \bar{U}$ for any $U \in \mathcal{S}_X$.
Then ψ extends uniquely to $\mathbb{R} - C(X)$ as a right exact functor Ψ which is local, that is, $\Psi(F)|_U \simeq \Psi(F_U)|_U$ for any $F \in \mathbb{R} - C(X)$ and $U \in \mathcal{S}_X$. Moreover the assertion (ii) remains valid, as well as (iii), provided that both ψ_1 and ψ_2 are local.

3 The functors $\cdot\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X$ and $\mathcal{T}hom(\cdot,\mathcal{D}b_X)$

On a real analytic manifold X, we denote respectively by $\mathcal{A}_X, \mathcal{C}^\infty_X, \mathcal{D}b_X, \mathcal{B}_X$ the sheaves of real analytic functions, $\mathcal{C}^\infty$-functions, Schwartz's distributions and Sato's hyperfunctions. We denote by $\mathcal{A}^\vee_X$ the sheaf of real analytic densities and if $\mathcal{F}$ is an $\mathcal{A}_X$-module, we set $\mathcal{F}^\vee = \mathcal{F}\otimes_{\mathcal{A}_X}\mathcal{A}^\vee_X$. We denote by $\mathcal{D}_X$ the sheaf of finite order differential operators with coefficients in $\mathcal{A}_X$.

Theorem 3.1 *There exist exact functors:*

$$\begin{aligned}
&\cdot\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X : \mathbb{R}-C(X) \longrightarrow Mod(\mathcal{D}_X)\\
&\mathcal{T}hom(\cdot,\mathcal{D}b_X) : \mathbb{R}-C(X)^{op} \longrightarrow (Mod(\mathcal{D}_X)),
\end{aligned}$$

such that for any U (resp. Z) open (resp. closed) subanalytic subset of X, one has:

$$\begin{aligned}
&\mathbb{C}_U\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X \simeq \mathcal{I}^\infty_{X\setminus U,X}\,,\\
&\mathcal{T}hom(\mathbb{C}_Z,\mathcal{D}b_X) \simeq \Gamma_Z(\mathcal{D}b_X).
\end{aligned}$$

(Recall that for Z a closed subset of X, $\mathcal{I}^\infty_{Z,X}$ denotes the subsheaf of $\mathcal{C}^\infty_X$ of functions vanishing to infinite order on Z.)

Proof: This follows immediately from Theorem 2.1 and the Lojasiewicz's theorem (see [9]) which asserts that if Z_1 and Z_2 are closed subanalytic subsets of X, then the two sequences:

$$\begin{aligned}
0 \longrightarrow \mathcal{I}^\infty_{Z_1\cup Z_2,X} \longrightarrow \mathcal{I}^\infty_{Z_1,X}\oplus\mathcal{I}^\infty_{Z_2,X} \longrightarrow \mathcal{I}^\infty_{Z_1\cap Z_2,X} \longrightarrow 0,\\
0 \longrightarrow \Gamma_{Z_1\cap Z_2}\mathcal{D}b_X \longrightarrow \Gamma_{Z_1}\mathcal{D}b_X\oplus\Gamma_{Z_2}\mathcal{D}b_X \longrightarrow \Gamma_{Z_1\cup Z_2}\mathcal{D}b_X \longrightarrow 0
\end{aligned}$$

are exact. *q.e.d.*

Remark 3.2 (i) The functor $\mathcal{T}hom(\cdot,\mathcal{D}b_X)$ has been defined in [5], without using Theorem 2.1.

(ii) The sheaves $F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X$ and $\mathcal{T}hom(F,\mathcal{D}b_X)$ are sheaves of $\mathcal{C}^\infty_X$-modules, hence are soft.

(iii) The vector space $\Gamma(X;F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X)$ may naturally be endowed with a topology of type FN (Fréchet nuclear), the vector space $\Gamma_c(X;\mathcal{T}hom(F,\mathcal{D}b^\vee_X))$ with a topology of type DFN (dual of Fréchet nuclear) and this two spaces are dual to each other. This is proved by reducing to the case where $F=\mathbb{C}_Z$, for Z a closed subanalytic subset of X.

(iv) The functors $\cdot\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X$ and $\mathcal{T}hom(\cdot,\mathcal{D}b_X)$ being exact, they extend to the derived categories. Hence we obtain functors:

$$\begin{gathered}\cdot\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X : \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X) \longrightarrow \mathbf{D}^b(\mathcal{D}_X)\\ \mathcal{T}hom(\cdot,\mathcal{D}b_X) : \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)^{op} \longrightarrow \mathbf{D}^b(\mathcal{D}_X).\end{gathered}$$

(v) Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and denote by $D'F$ its dual, $D'F = R\mathcal{H}om(F,\mathbb{C}_X)$. then we have a commutative diagram:

$$\begin{tikzcd}
D'F\otimes\mathcal{A}_X \arrow[rr] \arrow[d] & & R\mathcal{H}om(F,\mathcal{A}_X) \arrow[d] \\
D'F\otimes\mathcal{C}^\infty_X \arrow[r] \arrow[d] & D'F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X \arrow[r] \arrow[d] & R\mathcal{H}om(F,\mathcal{C}^\infty_X) \arrow[d] \\
D'F\otimes\mathcal{D}b_X \arrow[r] \arrow[d] & \mathcal{T}hom(F,\mathcal{D}b_X) \arrow[r] & R\mathcal{H}om(F,\mathcal{D}b_X) \arrow[d] \\
D'F\otimes\mathcal{B}_X \arrow[rr] & & R\mathcal{H}om(F,\mathcal{B}_X)
\end{tikzcd}$$

4 Operations on $\cdot\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X$ and $\mathcal{T}hom(\cdot,\mathcal{D}b_X)$

Let $f : Y \longrightarrow X$ be a morphism of real analytic manifolds. As usual, one denotes by $\mathcal{D}_{Y\to X}$ (resp.$\mathcal{D}_{X\leftarrow Y}$) the sheaf $\mathcal{A}_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X$ (resp. $\mathcal{A}^\vee_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}(\mathcal{D}_X \otimes_{\mathcal{A}_X} \mathcal{A}^{\vee\otimes-1})$ endowed with its structure of a $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$-module (resp. $(f^{-1}\mathcal{D}_X,\mathcal{D}_Y)$-module). Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and let $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$, with f proper on supp G. There are natural (iso-)morphisms:

$$\mathcal{D}_{Y\to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}(F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X) \longrightarrow f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X \tag{4.1}$$

$$f^{-1}(F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X) \tag{4.2}$$

$$f_*G\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_X \simeq f_*R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, G\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_Y) \tag{4.3}$$

Morphism (4.2) is deduced from (4.1) by adjunction. Morphism (4.1) is constructed by using Theorem 2.1 which allows to reduce to the case where $F = \mathbb{C}_U$, U an open subanalytic subset of X. Then, setting $Z = X\setminus U$, one has the natural morphism:

$$\mathcal{A}_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{I}^\infty_{Z,X} \longrightarrow \mathcal{I}^\infty_{f^{-1}(Z),Y}.$$

When f is a closed embredding, (4.1) is an isomorphism and when f is smooth, (4.2) is an isomorphism. Morphism (4.3) is decuced from (4.2) by adjunction. To prove it is an isomorphism, one can treat separately the case of a closed embedding and that of a smooth map. In the first case, one is reduced to the case where Y is defined by an equation $\{t = 0\}$ and $G = \mathbb{C}_U$ for an open subanalytic subset U of Y. Then (4.3) follows from the isomorphism $\mathcal{I}^{\infty}_{X \setminus U, X} \simeq \mathcal{I}^{\infty}_{Y \setminus U, Y}[[t]]$. If f is smooth, one reduces to the case where $Y = X \times \mathbb{R}$, f is the projection and $G = \mathbb{C}_Z$ where Z is a closed subanalytic subset and the fibers of f on Z are closed intervals. Then one has to check that the sequence:

$$0 \longrightarrow \mathcal{I}^{\infty}_{f(Z),X} \longrightarrow f_* \mathcal{I}^{\infty}_{Z,Y} \overset{\partial/\partial t}{\to} f_* \mathcal{I}^{\infty}_{Z,Y} \longrightarrow 0$$

is exact (see [5]).

Similarly, one has a natural isomorphism, assuming f is proper on supp G:

$$f_!(\mathcal{T}hom(G, \mathcal{D}b^{\vee}_Y) \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y \to X}) \ \tilde{\to}\ \mathcal{T}hom(Rf_! G, \mathcal{D}b^{\vee}_X), \tag{4.4}$$

a natural isomorphism, assuming f is smooth:

$$R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \to X}, \mathcal{T}hom(f^{-1}F, \mathcal{D}b_Y)) \ \tilde{\to}\ f^{-1}\mathcal{T}hom(F, \mathcal{D}b_X), \tag{4.5}$$

and a natural isomorphism, assuming f is a closed embedding:

$$\mathcal{T}hom(f^{-1}F, \mathcal{D}b_Y) \quad \tilde{\to}\ R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Y}, \mathcal{T}hom(F, \mathcal{D}b_X)). \tag{4.6}$$

5 The functors $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $\mathcal{T}hom(\cdot, \mathcal{O}_X)$

Now assume X is a complex manifold, denote as usual by $\overline{X}$ the anti-holomorphic associated complex manifold, and identify X to the diagonal of $X \times \overline{X}$.

Definition 5.1 Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. One sets:

$$\begin{aligned} F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_X) \\ \mathcal{T}hom(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(F, \mathcal{D}b_X)) \end{aligned}$$

Example 5.2 (i) Let M be a real analytic manifold, $i : M \hookrightarrow X$ a complexification. Then:

$$i_* F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \simeq i_*(F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_M) \tag{5.1}$$

$$\mathcal{T}hom(i_* F, \Omega_X[d_X]) \simeq i_* \mathcal{T}hom(F, \mathcal{D}b^{\vee}_M) \tag{5.2}$$

Note that isomorphism (5.2) is a result of Andronikof [1], which extends a previous theorem of Martineau (who treated the case when $F = \mathbb{C}_M$).

(ii) Let Z be a closed complex analytic subset of X. Then:

$$\begin{aligned}\mathbb{C}_Z\overset{\mathrm{w}}{\otimes}\mathcal{O}_X &\simeq \mathcal{O}_X\widehat{|}_Z &(5.3)\\ \mathcal{T}hom(\mathbb{C}_Z,\mathcal{O}_Z) &\simeq R\Gamma_{[Z]}\mathcal{O}_X &(5.4)\end{aligned}$$

Here, $\mathcal{O}_X\widehat{|}_Z$ denotes as usal the formal completion of $\mathcal{O}_X$ along Z, and $R\Gamma_{[Z]}\mathcal{O}_X$ the algebraic cohomology of $\mathcal{O}_X$ supported by Z (see [4]).

In order to recall the main operations on these functors, we shall follow the notations of [6] for $\mathcal{D}$-modules. In particular if $f : Y \longrightarrow X$ is a morphism of complex manifolds and if $\mathcal{M}$ (resp. $\mathcal{N}$) is a $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$)-module, one sets:

$$\begin{aligned}\underline{f}^{-1}\mathcal{M} &= \mathcal{D}_{Y\to X}\otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M},\\ \underline{f}_!\mathcal{N} &= Rf_!(\mathcal{D}_{X\leftarrow Y}\otimes^L_{\mathcal{D}_Y}\mathcal{N}).\end{aligned}$$

Then, using isomorphisms (4.1)-(4.6), we get the following results.
Assuming that f is smooth, we have natural isomorphisms:

$$\begin{aligned}f^{-1}(F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X) &\simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X},f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{O}_Y), &(5.5)\\ R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X},\mathcal{T}hom(f^{-1}F,\mathcal{O}_Y)) &\simeq f^{-1}\mathcal{T}hom(F,\mathcal{O}_X). &(5.6)\end{aligned}$$

Assuming that f is a closed embedding, we have natutral isomorphisms:

$$\begin{aligned}\underline{f}^{-1}(F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X) &\simeq f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{O}_Y, &(5.7)\\ \mathcal{T}hom(f^{-1}F,\mathcal{O}_Y) &\simeq \underline{f}^{-1}\mathcal{T}hom(F,\mathcal{O}_X). &(5.8)\end{aligned}$$

Assuming that f is proper on $supp\,G$, we have natural isomorphisms:

$$\begin{aligned}f_!R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X},G\overset{\mathrm{w}}{\otimes}\mathcal{O}_Y) &\simeq f_!G\overset{\mathrm{w}}{\otimes}\mathcal{O}_X, &(5.9)\\ \underline{f}_!\mathcal{T}hom(G,\Omega_Y[d_Y]) &\simeq \mathcal{T}hom(Rf_!G,\Omega_X[d_X]). &(5.10)\end{aligned}$$

Remark 5.3 The functor $\mathcal{T}hom(\cdot,\mathcal{O}_X)$ has been microlocalized by Andronikof ([1]). The specialization of the functor $\cdot\overset{\mathrm{w}}{\otimes}\mathcal{O}_X$ is related to the notion of asymptotic expansions. This will be developed elsewhere.

6 Adjunction formulas

Consider a correspondence of complex manifolds:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

let $\mathcal{M} \in \mathbf{D}^b_{good}(\mathcal{D}_X)$, (the triangulated subcategory of $\mathcal{M} \in \mathbf{D}^b_{good}(\mathcal{D}_X)$ generated by the objects whose cohomology groups are coherent and may be endowed with a good filtration on each compact subset of X), and let $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. Assume:

$$f \text{ is non characteristic for } \mathcal{M}, \tag{6.1}$$
$$f \text{ is proper over } g^{-1}(\operatorname{supp} G), \tag{6.2}$$
$$g \text{ is proper over } f^{-1}(\operatorname{supp} \mathcal{M}). \tag{6.3}$$

Introduce the notations:

$$\phi_S(G) = Rf_! g^{-1} G\,[\mathrm{d}_S - \mathrm{d}_X]$$
$$\underline{\phi}_S(\mathcal{M}) = \underline{g}_* \underline{f}^{-1} \mathcal{M}.$$

Theorem 6.1 *Assume (6.1)-(6.3). Then there are natural isomorphisms:*

$$R\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \phi_S(G)\overset{\mathrm{w}}{\otimes}\mathcal{O}_X))\,[d_X] \simeq$$
$$R\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\phi}_S(\mathcal{M}), G\overset{\mathrm{w}}{\otimes}\mathcal{O}_Y))\,[d_Y]$$
$$R\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(\phi_S(G), \mathcal{O}_X)))\,[d_X] \simeq$$
$$R\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\phi}_S(\mathcal{M}), \mathcal{T}hom(G, \mathcal{O}_Y)))\,[d_Y].$$

Sketch of proof

It is enough to treat the case where $S = Y$ and the case where $S = X$. Using the isomorphisms (5.5)-(5.10), the remaining problem is to prove that if $f : Y \longrightarrow X$ is a morphism of complex manifolds, if $\mathcal{N} \in \mathbf{D}^b_{good}(\mathcal{D}_Y)$, f is proper over $\operatorname{supp}\mathcal{N}$, and $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$, then there are natural isomorphisms:

$$\begin{aligned} Rf_! R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \underline{f}^{-1}(F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X))[d_Y] &\simeq Rf_! R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{N}, F\overset{\mathrm{w}}{\otimes}\mathcal{O}_Y)[d_X], \\ Rf_! \mathcal{T}hom(f^{-1}F, \Omega_X) \otimes^L_{\mathcal{O}_Y} \mathcal{N} &\simeq \mathcal{T}hom(F, \Omega_X) \otimes^L_{\mathcal{O}_X} \underline{f}_!\mathcal{N}. \end{aligned}$$

The second isomorphism is deduced from the first one by a duality argument. The first isomorphism is equivalent to:

$$Rf_! R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \underline{f}^{-1}(F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X)) \simeq Rf_! R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, f^{-1}(F)\overset{\mathrm{w}}{\otimes}\mathcal{O}_Y).$$

Then the proof follows the main lines of [11]. One reduces to the case where $Y = Z \times X$ and f is the projection, $\mathcal{N} = \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{G}$, for $\mathcal{G}$ a coherent $\mathcal{O}_Y$-module, f being propre on $\operatorname{supp}\mathcal{G}$. Then one has to prove the isomorphism:

$$Rf_! R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_Y) \otimes^L_{\mathcal{O}_X} (F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X \simeq Rf_! R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, f^{-1}F\overset{\mathrm{w}}{\otimes}\mathcal{O}_Y).$$

Since one may represent $Rf_! R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_Y)$ by a complex of FN-free $\mathcal{O}_X$-modules and $F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X$ is an $\mathcal{O}_X$-module of type FN, one may apply Proposition 3.13 of [11], a variant of a theorem of Ramis-Ruget [10].

7 Application to the Radon transform

As an application of Theorem 6.1, we recall some results of [2].

Let $\mathbf{P}$ be a complex n-dimensional projective space, $\mathbf{P}^*$ the dual projective space, and $\mathbf{A}$ the hypersurface of $\mathbf{P}\times\mathbf{P}^*$ given by the incidence relation. If $[\xi]=[\xi_0,\ldots\xi_n]$ is a homogeneous coordinate system on $\mathbf{P}$, $[\eta]=[\eta_0,\ldots\eta_n]$ the dual system on $\mathbf{P}^*$, then:

$$\mathbf{A}=\{(\xi,\eta)\in\mathbf{P}\times\mathbf{P}^*;\ \langle\xi,\eta\rangle=0\}.$$

Let us consider the correspondence:

$$\begin{tikzcd} & \mathbf{A} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbf{P} & & \mathbf{P}^*, \end{tikzcd} \tag{7.1}$$

and denote by q_1 and q_2 the projections from $\mathbf{P}\times\mathbf{P}^*$ to $\mathbf{P}$ and $\mathbf{P}^*$, respectively.

For $k\in\mathbb{Z}$, we denote by $\mathcal{O}_{\mathbf{P}}(k)$ the $-k$-th power of the tautological line bundle $\mathcal{O}_{\mathbf{P}}(-1)$, and we set:

$$\mathcal{D}_{\mathbf{P}}(k)=\mathcal{D}_{\mathbf{P}}\otimes_{\mathcal{O}_{\mathbf{P}}}\mathcal{O}_{\mathbf{P}}(k).$$

For $(k,k')\in\mathbb{Z}\times\mathbb{Z}$, we set $\mathcal{O}_{\mathbf{P}\times\mathbf{P}^*}(k,k')=\mathcal{O}_{\mathbf{P}\times\mathbf{P}^*}\otimes_{\mathcal{O}_{\mathbf{P}}}\mathcal{O}_{\mathbf{P}}(k)\otimes_{\mathcal{O}_{\mathbf{P}^*}}\mathcal{O}_{\mathbf{P}^*}(k')$. To $k\in\mathbb{Z}$, we associate:

$$k^*=-n-1-k.$$

In [8], Leray introduced:

$$\begin{aligned}\omega^*(\xi) &= \sum_{i=0}^{n}(-1)^i\xi_i\,d\xi_0\wedge\ldots\wedge\widehat{d\xi_i}\wedge\ldots\wedge d\xi_n,\\ s_k &= s_k(\xi,\eta)=\frac{\omega^*(\xi)}{\langle\xi,\eta\rangle^{n+1+k}}.\end{aligned}$$

Then s_k is a well defined section of $\mathcal{O}^{(n,0)}_{\mathbf{P}\times\mathbf{P}^*}(-k,k^*)$ on $\mathbf{P}\times\mathbf{P}^*\setminus\mathbf{A}$. If $n+1+k>0$ (i.e. if $k^*<0$), s_k has meromorphic singularities on $\mathbf{A}$, and its image via the natural morphism

$$\Gamma(\mathbf{P}\times\mathbf{P}^*\setminus\mathbf{A};\mathcal{O}^{(n,0)}_{\mathbf{P}\times\mathbf{P}^*}(-k,k^*))\longrightarrow H^1_{\mathbf{A}}(\mathbf{P}\times\mathbf{P}^*;\mathcal{O}^{(n,0)}_{\mathbf{P}\times\mathbf{P}^*}(-k,k^*))$$

defines a section (that we denote by the same symbol):

$$s_k\in\Gamma(\mathbf{P}\times\mathbf{P}^*;\mathcal{B}^{(n,0)}_{\mathbf{A}}(-k,k^*)),$$

where $\mathcal{B}^{(n,0)}_{\mathbf{A}}(-k,k^*)$ is the sheaf $\mathcal{B}_{\mathbf{A}}=H^1_{[\mathbf{A}]}(\mathcal{O}_{\mathbf{P}\times\mathbf{P}^*})$ twisted by $\mathcal{O}^{(n,0)}_{\mathbf{P}\times\mathbf{P}^*}(-k,k^*)$. One sees easily that the Leray section s_k defines a $\mathcal{D}_{\mathbf{P}^*}$-linear morphism

$$\alpha(s_k):\mathcal{D}_{\mathbf{P}^*}(-k^*)\longrightarrow\underline{\phi}_{\mathbf{A}}(\mathcal{D}_{\mathbf{P}}(-k)) \tag{7.2}$$

The main result of [2] is that if $-n-1<k<0$, then $\alpha(s_k)$ is an isomorphism. Applying Theorem 6.1, we find for $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{P}})$, the isomorphisms:

$$R\Gamma(\mathbf{P}; F\overset{\mathrm{w}}{\otimes}\mathcal{O}_{\mathbf{P}}(k)) \simeq \qquad\qquad (7.3)$$
$$R\Gamma(\mathbf{P}^*; \phi_{\mathbf{A}}(F)\overset{\mathrm{w}}{\circ}\mathcal{O}_{\mathbf{P}^*}(k^*))$$
$$R\Gamma(\mathbf{P}; \mathcal{T}hom(F, \mathcal{O}_{\mathbf{P}}(k))) \simeq \qquad\qquad (7.4)$$
$$R\Gamma(\mathbf{P}^*; (\underline{\phi}_S(\mathcal{M}), \mathcal{T}hom(\phi_{\mathbf{A}}(F), \mathcal{O}_{\mathbf{P}^*}(k^*)))\,.$$

Denote by P and P^* a real projective space of dimension $n>1$ and its dual, and consider $\mathbf{P}$ and $\mathbf{P}^*$ as complexifications of P and P^*. Let $k \in \mathbb{Z}$, and let $\varepsilon, \overline{\varepsilon} \in \{0,1\}$ have different parity. We denote by $\mathcal{C}^\infty_P(k,\varepsilon)$ the locally constant sheaf of rank one over $\mathcal{C}^\infty_P$ whose global sections are represented by those functions f on $\mathbb{R}^{n+1}\setminus\{0\}$ satisfying the homogeneity condition:

$$f(\lambda x) = (\operatorname{sgn}\lambda)^\varepsilon \lambda^k f(x), \quad \text{for } \lambda \neq 0.$$

Using explicit integral formulas, Gelfand et al. [3] proved the isomorphisms for $-n-1<k<0$:

$$\Gamma(P; \mathcal{C}^\infty_P(k,\varepsilon)) \simeq \begin{cases} \Gamma(P^*; \mathcal{C}^\infty_{P^*}(k^*,\overline{\varepsilon})) & \text{for } n \text{ even,} \\ \Gamma(P^*; \mathcal{C}^\infty_{P^*}(k^*,\varepsilon)) & \text{for } n \text{ odd.} \end{cases}$$

We recover here these isomorphisms (and the similar ones with $\mathcal{C}^\infty$ replaced by $\mathcal{D}b$) by applying (7.3) to the case where either $F=\mathbb{C}_P$ or $F=K_P$, the canonical line bundle on P. In fact, $\mathcal{C}^\infty_P(k,0) \simeq \mathbb{C}_P \otimes \mathcal{O}_{\mathbf{P}}(k)$ and $\mathcal{C}^\infty_P(k,1) \simeq K_P\overset{\mathrm{w}}{\circ}\mathcal{O}_{\mathbf{P}}(k)$.

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