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EQUATIONS AUX DERIVEES PARTIELLES

SEMILINEAR DIFFRACTION OF CONORMAL WAVES

(joint work with Melrose and Sa Barreto)

M. ZWORSKI

Exposé n° II

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Maciej Zworski

1. Introduction. The purpose of this exposé is to describe the results of [27] on the conormal regularity for a class of mixed problems for the semi-linear hyperbolic equations and to indicate the general approach which is used in that paper.

The study of C^∞ regularity of solutions to non-linear wave equations has had two main directions: finding estimates on the strength of the anomalous singularities, i.e. those not present in the linear interaction, and obtaining geometric restrictions on the location of singularities. Our work is of the latter type. The strength of singularities for non-linear mixed problems has already been investigated with considerable success in [38, 9, 16, 40]. The estimates on the location of singularities are much finer, so stronger assumptions are needed on the incoming waves or the initial data. The most striking example of this was provided by [2] where it is shown that wave-front set restrictions alone still allow the self-spreading of of singularities, making the singular support propagate essentially in the same way as the support of the solution. Thus, in full generality, the location of singularities cannot be related to the original geometry except in a trivial way. A technically more challenging construction of a similar example for gliding mixed problems was then given in [39].

The appropriate class of distributions to consider for the incoming waves or the initial data are the *conormal distributions*, as was first noted in [6]. The conormal distributions appear naturally in the linear theory and are a subclass of the *Lagrangian distributions* motivated by geometrical optics. The interaction of conormal waves for interior problems has been investigated in [33, 24, 7, 8, 3, 35, 26] and the formation of non-linear caustics in [13, 14, 10, 20, 36, 37]. For mixed problems, with only transversal reflections allowed, it was shown in [4, 5] that no anomalous singularities appear. One should also mention that examples of 'new' non-linear singularities were provided at an early stage in [32]: namely, the interaction of three plane waves carrying conormal singularities produces a conic surface of new singularities propagating from the triple interaction point. However, in more complicated settings such as the propagation of the swallowtail or diffraction, where the 'new' cones are expected, no examples have yet been constructed. For interior problems [15] provides a systematic approach to such constructions.

We consider a mixed hyperbolic problem with a diffractive boundary (see Sect.2 for a review of definitions). Our object of study is the semi-linear equation:

$$Pu = f(x,u) \text{ in } X,\ u|_{\partial X} = 0,\ u|_{X_-} = u_0 \tag{0.1}$$

where P is a strictly hyperbolic operator, X is a C^∞ manifold with the boundary ∂X, $X_- = \{x \in X : \phi(x) < -T\}$ with $\phi \in C^\infty(X)$ a time function for P and the time T fixed. The nonlinearity is quite general, $f \in C^\infty(X, \mathbf{C})$.

The initial data is assumed to be conormal to the *incident front* F. We assume that

$$(F \cap \bar{X}) \cap \partial X = \emptyset.$$

The reflection rule of geometrical optics produces the *reflected front* R. With the motivation coming again from the geometric optics we define the *shadow boundary* on ∂X as

$$\Gamma = \partial X \cap \operatorname{cl}[R \cap F \setminus \partial X]$$

The front obtained from the nonlinear interaction is the *forward half-cone*, S_+, of $P-$bicharacteristics starting on Γ. Let us also denote by D_+ and B_+ the two components of the set of glancing characteristics on S_+. A more detailed discussion of the fronts is presented in Sect.2. Fig.1 shows three different time slices and Fig.2 is a space-time picture. Note that R and F are hypersufaces with singular boundaries.

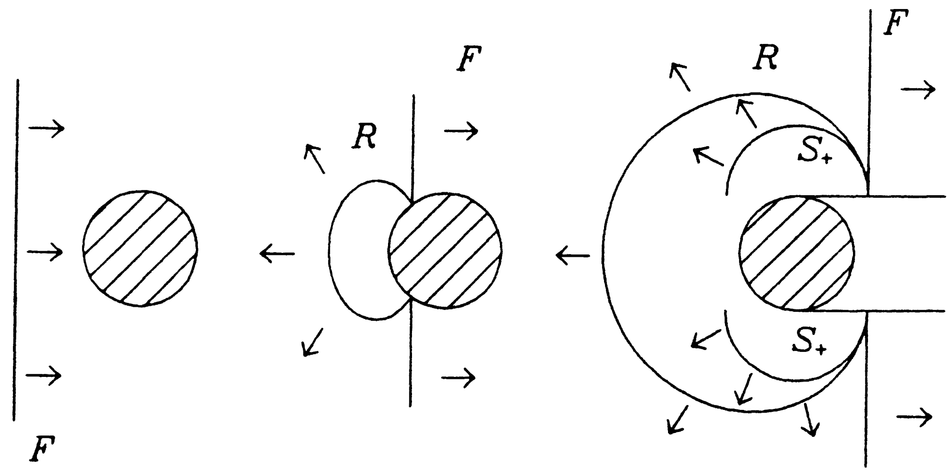

Figure 1: The fronts projected to the space variables at fixed times

The crudest form of our result is

Theorem 1 *Let $u \in L^\infty(X)$ be a bounded solution of of (0.1) with*

$$u_0 \in I_\infty L^2(X_-; F)$$

Then

$$WF_b(u) \subset {}^bN^*R \cup {}^bN^*F \cup {}^bN^*S_+ \cup {}^bN^*B_+ \cup {}^bN^*D_+ \cup {}^bT^*_\Gamma X \setminus 0$$

We refer the reader to [18] and [11], Sect.18.3 for the definition of the b-wave front set, WF_b, which reduces to the ordinary WF away from the boundary ∂X. We use the natural map $\jmath : T^*X \setminus 0 \to {}^bT^*X \setminus 0$ (see the references given above) to define ${}^bN^*\Sigma = \jmath(N^*\Sigma)$.

Theorem 1 immediately gives the singular support statement:

Corollary 1 *Under the assumptions of Theorem 1*

$$\text{sing supp } u \subset F \cup R \cup S_+$$

Since the data u_0 is conormal, one would like to describe precisely the conormal regularity of the solution u. In fact the proof is based on the construction of an appropriate space with good multiplicative and propagative properties – see Sect.5. Since the precise definition of this 'strong', but not quite conormal, space is rather involved we shall content ourselves with a weaker statement here, referring the reader to Definition 2 and Theorem 5 in Sect.5 for the full result.

Theorem 2 *Let $u \in L^\infty(X)$ be a bounded solution of (0.1) with*

$$u_0 \in I_k L^2(X_-; F)$$

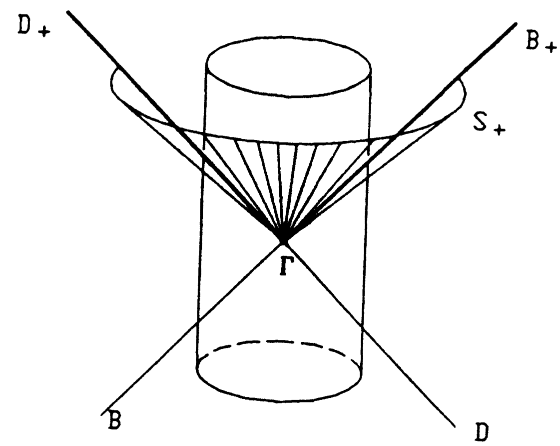

Figure 2: The forward half-cone and the glancing boundaries B and D.

If Ω is an open subset of X such that

$$\Omega \cap (D_+ \cup B_+) = \emptyset$$

then

$$u|_\Omega \in I_kL^2(\Omega, F) + I_kL^2(\Omega, R) + I_kL^2(\Omega, S_+)$$

Already in the transversal case this is slightly stronger than the result in [4] as conormal singularities with respect to the boundary are excluded.

Our conclusions are concerned purely with the L^2-based regularity. The present existence theory [38] requires higher Sobolev regularity for u_0 to guarantee local existence of bounded solutions, so one needs to assume $u_0 \in I_kL^2(X_-; F) \cap H_{(s)}(X_-)$ for $s > n/2$. However, the conormal results described above should lead to to an improvement in the style of [34]. It should be noted that our present method does not treat the fully semi-linear equation $Pu = f(x, u, \nabla u)$, essentially because the iteration procedure in k proceeds in steps of $1/2$ – see Theorem 4 below.

2. Diffractive geometry. First we describe the interaction of a characteristic hypersurface for a second-order hyperbolic operator with a bicharacteristically concave (diffractive) boundary. In particular we point out in Proposition 1 that the reflected front has cusp singularity when continued across the boundary.

Let X be a manifold with boundary equipped with a pseudo-Riemannian metric of hyperbolic signature, $+, -, -, - \ldots$. The metric symbol $p \in S^2(T^*X)$ is therefore a polynomial of degree two on each fibre and it can be reduced, in linear coordinates in each fibre, to

$$\tau^2 - \xi_1^2 - \ldots - \xi_n^2, \quad \dim X = n + 1.$$

The boundary of X is said to be time-like if p is negative-definite on $N^*\partial X$; this is always assumed below. It will be convenient to assume that X is time oriented; this amounts to the continuous selection of one of the solid cones, $p > 0$, in the fibres. A function $t \in C^\infty(X)$ is a time-function if $p(dt) > 0$.

The assumption that ∂X is time-like means that it carries an induced pseudo-Riemannian metric of hyperbolic signature. If g is the dual quadratic form to p, on TX, then $g_\partial = g|_{T\partial X}$

fixes the induced structure. Let p_∂ denote the metric symbol on $T^*\partial X$. In $T^*\partial X\backslash 0$, set

$$\begin{aligned}\mathcal{H} &= \{p_\partial > 0\} = \mathcal{H}_+ \cup \mathcal{H}_- \\ \mathcal{G} &= \{p_\partial = 0\} \\ \mathcal{E} &= \{p_\partial < 0\},\end{aligned}$$

respectively the hyperbolic, glancing and elliptic regions of $T^*\partial X\backslash 0$. The time-orientation of X induces a time-orientation of ∂X, giving the decomposition of the hyperbolic region.

The points of $\mathcal{G}$ are further distinguished by the behaviour of the second Poisson bracket:

$$\begin{aligned}\tilde{\mathcal{G}}_d &= \{m \in \tilde{\mathcal{G}}; H_p^2 q(m) > 0\} \\ \tilde{\mathcal{G}}_h &= \{m \in \tilde{\mathcal{G}}; H_p^2 q(m) = 0\} \\ \tilde{\mathcal{G}}_g &= \{m \in \tilde{\mathcal{G}}; H_p^2 q(m) < 0\}\end{aligned}$$

These, and similarly their images in $\mathcal{G}$ under ι^*, are respectively the sets of diffractive, higher-order and gliding points. The boundary of X is said to be diffractive (or bicharacteristically concave) if $\mathcal{G} = \mathcal{G}_d$; *this is the assumptions made in our work.*

We will be concerned with the local geometry near a base point $x_0 \in \partial X$, so we are free to shrink X as necessary. In this sense the assumption that the boundary is diffractive is really that $\mathcal{G} \cap T^*_{x_0}\partial X \subset \mathcal{G}_d$. In case $X = \mathbf{R} \times Y$ carries a product metric, $g = dt^2 - h$, with h a Riemann metric on Y, the boundary is diffractive if and only if ∂Y is strictly geodesically concave. In case $Y = \mathbf{R}^n \backslash K$ where K is an open, smoothly bounded region and h is the Euclidean metric this is equivalent to the strict convexity of K (cf. [19]).

It is convenient to consider an extension, $\tilde{X}$, of X to a manifold without boundary. A corresponding extension of this pseudo-Riemannian structure will be denoted $\tilde{p}$. The defining function $x \in C^\infty(X)$ extends to $\tilde{x} \in C^\infty(\tilde{X})$ and if $\tilde{X}$ is chosen small enough, $\partial X = \{\tilde{x} = 0\}$ is an embedded hypersurface. The freedom to shrink X will be used to choose $\tilde{X}$ to be bicharacteristically convex.

In $\tilde{X}$ we consider a closed characteristic hypersurface for $\tilde{p}$, passing through this point x_0. Thus $\tilde{F} \subset \tilde{X}$ satisfies

$$\tilde{F} = \{\tilde{f} = 0\},\ \tilde{f} \in C^\infty(\tilde{X}),\ d\tilde{f} \neq 0 \text{ on } \tilde{F},\ \tilde{p}|_{N^*\tilde{F}} = 0.$$

The characteristic hypersurface $\tilde{F}$ is to be thought of as the extension through the boundary of X of the incident front. It is important to separate which parts of $\tilde{F}$ are intrinsic and which depend on the choice of extension—the latter being necessarily irrelevant to the final form of the results.

By assumption $N^*\tilde{F}$ is closed, so it is the union of the maximally extended bicharacteristic interval, i.e. integral curve of H_p, through each of its points. Set

$$F = \{z \in \tilde{F} \cap X;\ \text{the bicharacteristics through } N_z^*\tilde{F} \text{ stay in } T^*X \text{ for } t \leq t(z)\}.$$

Here, t is a time function. The submanifold $\Gamma \subset F$ is the singular locus in F near which it is not even a manifold with corners. Indeed the boundary of F consists of two smooth manifolds with boundary (each of codimension two in X)

$$\partial F = F_\partial \cup B,\ F_\partial \cap B = \partial F_\partial = \partial B = B \cap \partial X = \Gamma. \tag{0.2}$$

Here F_∂ is half of $\widetilde{F}_\partial$ and B, the shadow boundary, is the projection into X of the forward half-bicharacteristic starting at points of $N^*_\Gamma \widetilde{F}$.

The main objective of this section is to consider the reflected front generated by $\widetilde{F}$ and ∂X. To do so we need to recall the notion of a hypersurface with cusp singularity. By definition a cusp hypersurface is one which is diffeomorphic to $C = \{x_2^3 = x_1^2\}$ in $\mathbf{R}^n$, $n \geq 2$.

A simple characterization can be obtained in terms of the closure of the conormal bundle to the regular part of the hypersurface. As is easily checked

$$\Lambda_C = \mathrm{cl} N^*\{x_2^3 = x_1^2; x_2 > 0\} \subset T^*\mathbf{R}^n \backslash 0$$

is a smooth, homogeneous Lagrangian. Now a point of the singular locus, $L = \{x_1 = x_2 = 0\}$,

$$\begin{aligned} &\pi\ \Lambda_C \longrightarrow \mathbf{R}^n \text{ has differential with} \\ &\text{two} - \text{dimensional null space at } \Lambda_C \cap T_l^*\mathbf{R}^n, l \in L. \end{aligned}$$

Moreover, any vector field V on $T^*\mathbf{R}^n$ which is tangent to $T_l^*\mathbf{R}^n$ and takes the value $v \in T_m\Lambda_C \cap T_m(T_l^*\mathbf{R}^n)$ at m is only simply tangent to Λ_C at m. Conversely (see Arnol'd [1]) if these two conditions hold for Λ_C near $m \in T_l^*\mathbf{R}^n \cap \Lambda_C$ then the projection of a neighbourhood of $m \in \Lambda_C$ is a cusp. We use this abstract characterization, with $\mathbf{R}^n$ replaced by $\widetilde{X}$ (as can obviously be done) to analyze the reflected front.

Set $\widetilde{\Lambda}_R^0 = I_\partial(N^*_{\partial X}\widetilde{F})$ and let $\widetilde{\Lambda}_R$ be the H_p-flow-out in $T^*\widetilde{X}\backslash 0$ of $\widetilde{\Lambda}_R^0$. Thus $\widetilde{\Lambda}_R$ is just the union of the maximally extended H_p integral curves passing through points of $\widetilde{\Lambda}_R^0$.

Proposition 1 *If $\widetilde{F} \subset \widetilde{X}$ is a smooth characteristic hypersurface for which $x_0 \in \partial X$ is a diffractive point then, for $\widetilde{X}$ shrunk to a sufficiently small bicharacteristically convex neighbourhood of x_0, $\widetilde{\Lambda}_R \subset T^*\widetilde{X}\backslash 0$ is a smooth closed conic Lagrangian submanifold which is the closure of the conormal bundle to a hypersurface with cusp singularity, $\widetilde{R}$, through x_0.*

Clearly the cusp locus $L \subset \widetilde{R}$ passes through Γ. It is important to check that

$$L\backslash\Gamma \subset \widetilde{X}\backslash X \text{ and } \widetilde{L}_R \text{ is simply tangent to } \partial X \text{ at } \Gamma.$$

Since the tangent space to L is just the image of the tangent space to $\widetilde{\Lambda}_R$ under the projection, L is certainly tangent to ∂X at Γ. In the case of the wave equation in the exterior of a convex obstacle Proposition 1 was given in [41]. In that case the cusp locus L projected to the space variables is the envelope of the reflected rays, see Fig.3.

We also remark that although the extension $\tilde{p}$ was used in the definition of Λ_R, the part of $\widetilde{R}$ corresponding to the true reflection is determined by p and F alone. It will be denoted by R and is defined as follows. We can easily prove that $\widetilde{R}\backslash\widetilde{F}$ has four components, two of which are disjoint from L. We now take as R the closure of the one for which $R \cap \partial X = F \cap \partial X$. A more natural but longer definition can be given in terms of tracing of the bicharacteristics in Λ_R.

The bicharacteristic cone over the shadow boundary in ∂X, Γ is now defined in the standard way, as the union of the maximally extended bicharacteristic intervals over $N^*\Gamma \cap \Sigma$. We denote it by Λ_S and its projection by $\widetilde{S}$. We note however that although $\widetilde{S}|_X$ depends on the extension $\tilde{p}$, the half cone S defined by the glancing boundaries B and D, $B \subset F$ does not – see Fig.2. We can separate the forward and retarded components, $S_\pm$, respectively, and similarly denote by $\widetilde{S}_+$ the full forward cone over Γ. We also denote by B_+ and D_+ the intersections of B and D with cl S_+ respectively.

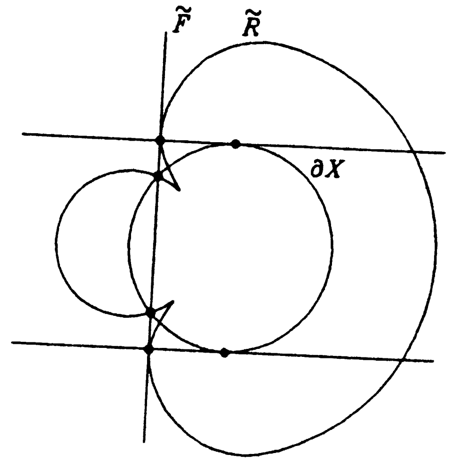

Figure 3: The extended reflected front projected to the space variables at a fixed time.

In the non-linear interaction more geometry is present. In addition to the cone over Γ we will also have to include, in a very residual way, smooth characteristic surfaces tangent to $\widetilde{S}$ at D – see Sect.5. Thus we define

$$\mathcal{R} = \{H \subset \widetilde{X} \text{ smooth hypersurface } : p|_{N^*H} = 0,\ N^*_{D\setminus\Gamma}H = N^*_{D\setminus\Gamma}\widetilde{S}\}, \tag{0.3}$$

with the first easy observation that $\bigcap_{H\in\mathcal{R}} H = D$.

3. The general strategy. The purpose of this section is to outline the general method for finding the geometric location of singularities of solutions to semilinear hyperbolic equations. The particular approach used in the study of semi-linear diffraction [27] originates from [24]. For the purpose of the general discussion we will consider the interior problem and thus assume that the bicharacteristically convex region X is open (i.e. has no boundary). It will also be convenient to take $\bar{X}$ compact and contained in a larger bicharacteristically convex region. For such X we want to study

$$Pu = f(x,u) \text{ in } X,\quad u|_{X_-} = u_0 \tag{0.4}$$

where X is the domain of influence of X_- and the location of singularities of u_0 is given in some appropriate sense. To simplify the presentation and to avoid the L^2_{loc} and $H^{\text{loc}}_{(k)}$ based spaces, we shall always tacitly assume that the prescribed regularity of u_0 extends to a larger open set. We assume that the solution exists in X (taken sufficiently small) and that $u \in L^\infty(X)$. The simplest case of propagation of regularity involves no geometry:

Example. If $u \in L^\infty(X)$ satisfies (0.4) with $u_0 \in H_{(k)}(X_-)$, $k \in \mathbb{N}_0$, then $u \in H_{(k)}(X)$.

Proof: Consider $\chi \in C^\infty(\bar{X})$ such that $\chi \equiv 0$ in $X \setminus X_-$ and $\chi \equiv 1$ in $X'_- \subset X_-$, where X_- is the domain of influence of X'_-. We can then solve the following linear equation:

$$Pu_1 = \chi(x)f(x,u_0) \text{ in } X,\quad u_1|_{X'_-} = u_0|_{X'_-}. \tag{0.5}$$

Since $u_0 \in H_{(k)}(X_-) \cap L^\infty(X_-)$, the Leibnitz rule shows that $\chi(x)f(x,u_0) \in H_{(k)}(X_-)$. Thus the energy estimates give $u_1 \in H_{(k)}(X)$. We then consider a nonlinear equation

$$Pu_2 = (1-\chi)(x)f(x,u_1+u_2) \text{ in } X,\quad u_2|_{X'_-} = 0$$

with the aim of showing that $u_2 \in H_{(k)}(X)$ as we observe that $u = u_1 + u_2$. This is easily proved by induction:
Step 1. Since $u_2 \in L^2(X)$ (as $u_1 + u_2 \in L^\infty(X), u_1 \in H_{(k)}(X)$) the energy estimate implies that $u_2 \in H_{(1)}(X)$.
Step $l \leq k$. We know that $u_1 + u_2 \in H_{(l)}(X) \cap L^\infty(X)$ so the Leibnitz rule shows that $f(x, u_1 + u_2) \in H_{(l)}(X)$. The energy estimate now gives $u_2 \in H_{(l+1)}$ and consequently $u_1 + u_2 \in H_{(\min(l+1,k))}(X)$.

We observe that only very special properties of $H_{(k)}(X)$ were used in the proof of the elementary example and if, in this case, we define $J_k L^2(X) = H_{(k)}(X)$ they were

C^∞-algebra property

$$u \in J_k L^2(X) \cap L^\infty(X) \Longrightarrow f(u) \in J_k L^2(X), \text{ for all } f \in C^\infty(\mathbf{C})$$

C^∞-module property

$$u \in J_k L^2(X) \Longrightarrow gu \in J_k L^2(X) \text{ for all } g \in C^\infty(\bar{X})$$

Propagation property

$$Pu = v \in J_k L^2(X),\ u|_{X_-} = 0 \Longrightarrow u \in J_{k+1} L^2(X)$$

$$Pu = 0,\ u|_{X_-} \in J_k L^2(X)|_{X_-} \Longrightarrow u \in J_k L^2(X).$$

We will refer to the first two properties as **(A)** and to the third one as **(P)**. We will always assume that $J_0 L^2(X) = L^2(X)$. The simple example now generalizes to

Proposition 2 *If a vector subspace $J_k L^2(X) \subset L^2(X)$ satisfies the conditions* **(A)** *and* **(P)** *above and $u \in L^\infty(X)$ is a solution of (0.4) with $u_0 \in J_k L^2(X)|_{X_-}$ then $u \in J_k L^2(X)$.*

We can replace the property **(P)** by a weaker property

$$Pu = v \in J_s L^2(X),\ u|_{X_-} = 0 \Longrightarrow u \in J_{s+\epsilon} L^2(X), \tag{0.6}$$

for some fixed $\epsilon > 0$ as long as we define $J_s L^2(X)$ satisfying **(A)** (in the example it is obviously $H_{(s)}(X)$). The second condition in **(P)** can also be modified if we are interested in a more restricted set of initial data.

We now want to pass to the case of non-trivial geometry. The simplest is provided by the conormal distributions associated to a smooth hypersurface. If $F \subset X$ is a C^∞ hypersurface in a C^∞ manifold X, let $\mathcal{V} = \mathcal{V}(X, F)$ be the Lie algebra of C^∞ vector fields in X tangent to F. The space of distributions of finite L^2-based conormal regularity with respect to F is then defined by the stability of regularity under the applications of the elements of $\mathcal{V}$:

$$I_k L^2(X, F) = \{u \in L^2(X) \ :\ V_1 \cdots V_l u \in L^2(X) \text{ for } l \leq k \text{ and } V_i \in \mathcal{V}\}$$

This modifies the definition of the Sobolev space $H_{(k)}$ by placing some geometric restrictions on the differentiations. Nevertheless, as observed in [24], bounded conormal functions have very good multiplicative properties in view of Gagliardo-Nirenberg type inequalities.

In fact any Lie algebra of vector fields could be taken in place of $\mathcal{V}(X, F)$. Thus for any variety $\mathcal{S}$ consisting of characteristic surfaces, their singular loci and intersections, we can define a space $I_kL^2(X,\mathcal{S})$ by taking as $\mathcal{V}$ all vector fields tangent to all components of $\mathcal{S}$. As varieties we could take in $X = \mathbf{R}^3$:

$$\mathcal{S} = \{F_1 \setminus F_2, F_2 \setminus F_1, F_1 \cap F_2\},\ F_i = \{x_i = x_3\},\ P = D_{x_3}^2 - D_{x_1}^2 - D_{x_2}^2, \tag{0.7}$$
$$\mathcal{S} = \{C \setminus L, L\},\ C = \{x_3^2 = x_2^3\},\ L = \{x_2 = x_3 = 0\},\ P = 4D_{x_2}^2 - 9x_2D_{x_3}^2 - D_{x_1}D_{x_3}. \tag{0.8}$$

For the spaces $I_kL^2(X,\mathcal{S})$ the algebraic property **(A)** comes for free and the main effort goes into establishing **(P)**. In the two examples (0.7) and (0.8) above that is quite easy by commuting P through $\mathcal{V}$ (i.e. using the P-completeness of the Lie algebras of vector fields – see [24]) but consider instead

$$\mathcal{S}_0 = \{C \setminus (L \cup H), H \setminus C, L \setminus C, C \cap H \setminus L, C \cap L \cap H\} \tag{0.9}$$

with C, L and P as in (0.8) and $H = \{x_1 = 0\}$ – see Fig.4. This is the problem studied in [26] and as we shall see in Sect.5 it is highly relevant in diffraction. The space $I_kL^2(\mathbf{R}^3, \mathcal{S}_0)$ cannot have property **(P)** as it includes any function conormal to the origin. Thus u in **(P)** could be singular on the cone Q obtained by projecting to $\mathbf{R}^3$ the null bicharacteristics passing through $T^*_{(0,0,0)}\mathbf{R}^3 \setminus 0$. In other words the union of the conormal bundles of the components of $\mathcal{S}_0$, $N^*\mathcal{S}_0$ is not closed under the Hamilton flow of the symbol of P. Thus we need to enlarge our variety to

$$\mathcal{S} = \mathcal{S}_0 \cup \{Q \setminus (C \cup H), Q \cap C \setminus L, Q \cap H \setminus L\} \tag{0.10}$$

for which $N^*\mathcal{S}$ is closed under the Hamilton flow. The space $I_kL^2(\mathbf{R}^3, \mathcal{S})$ has the property **(A)** but it is *not* known to satisfy **(P)** (and it most likely does not) as the P-completeness property for the Lie algebra of tangent vector fields does not hold. The same problem is encountered in the case of triple interaction, swallowtail and diffraction.

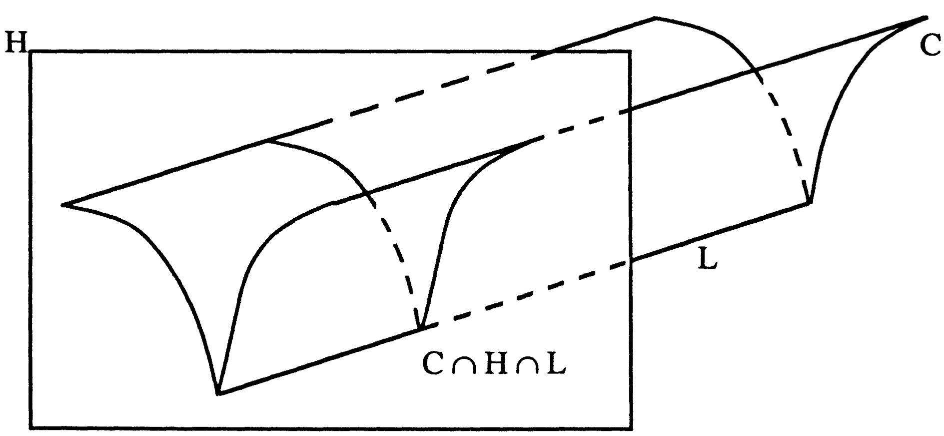

Figure 4: The cusp and a transversal plane.

To define a conormal space with reasonable propagation of regularity for P, one follows the method originating from [24] and subsequently applied in [20, 26, 35, 36, 37]. Its essence is the resolution of singularities and the use of the vector fields tangent to the lifted geometry in the resolved space. The insistence on conormality is motivated by the good multiplicative properties of bounded conormal functions, as already indicated above and the conviction that conormal regularity excludes any hidden singularities that could produce self-spreading.

Roughly speaking the method can be described as follows. By a successive application of real blow-ups (see Sect.4 for an example) we obtain a resolution

$$X_\beta \xrightarrow{\beta} X$$

of X with the blow-down map β. The space X_β is a manifold with corners with a natural measure ν_β pushing forward to the measure on X. The blow-ups are supposed to resolve the geometry in the sense that $\beta^*\mathcal{S} \sqcup \partial X_\beta$ consists of cleanly intersecting submanifolds (see [21, 22]). We then define $I_k L^2_{\nu_\beta}(X_\beta, \beta^*\mathcal{S} \sqcup \partial X_\beta)$ same as before and take as the new space

$$J_k L^2(X) = \beta_* I_k L^2_{\nu_\beta}(X_\beta, \beta^*\mathcal{S} \sqcup \partial X_\beta).$$

Again, the space $J_k L^2(X)$ satisfies the algebraic property **(A)** immediately and the main difficulty lies in proving **(P)**. In fact we cannot in general hope for the propagation and the variety $\beta^*\mathcal{S} \sqcup \partial X_\beta$ needs to be extended. However the clean intersection property achieved in X_β brings us closer to the P-completeness, referred to after (0.7) and (refeq:24) – now in X_β. Although that can be exploited only partially it explains a better chance for the property **(P)** to hold.

Let us now adopt an opposite point of view and concentrate on defining spaces for which the propagation property **(P)** holds automatically but **(A)** is hard to verify (or simply is not true!). We start again with the simplest example of a C^∞ hypersurface F and define the Lagrangian distributions associated to the Lagrangian $\Lambda_F = N^*F \setminus 0$:

$$\begin{aligned} I_k L^2(X, \Lambda_F) &= \{u \in L^2(X) : A_1 \cdots A_l u \in L^2(X) \text{ for any } l \leq k \\ &\qquad \text{and } A_i \in \Psi^1_{\text{phg}}(X) \text{ with } \sigma_1(A_i)|_{\Lambda_F} = 0\}. \end{aligned} \tag{0.11}$$

Of course, this is just the conormal space associated to F and defined above. This definition generalizes however to any conic Lagrangian Λ and if Λ lies in the characteristic variety of P, then the space $I_k L^2(X, \Lambda)$ has the property **(P)**. That is quite easy and can for instance be seen by conjugating P and Λ to a suitable model.

In the analogy with the varieties in the base space X we now consider Lagrangian varieties consisiting of conormal bundles of the components of the varieties in the base and all their succesive intersections. For example, in the cases (0.7) and (0.8) we now have

$$\begin{aligned} \mathcal{L} = \{&N^*F_1 \setminus N^*(F_1 \cap F_2), N^*F_2 \setminus N^*(F_1 \cap F_2), N^*(F_1 \cap F_2) \setminus (N^*F_1 \cup N^*F_2), \\ &N^*(F_1 \cap F_2) \cap N^*F_1, N^*(F_1 \cap F_2) \cap N^*F_2\}, \end{aligned} \tag{0.12}$$

$$\mathcal{L} = \{\mathrm{cl}(N^*(C \setminus L)) \setminus N^*L, N^*L \setminus \mathrm{cl}(N^*(C \setminus L)), N^*L \cap \mathrm{cl}(N^*(C \setminus L))\}. \tag{0.13}$$

We could again apply the direct analogy with the case of one Lagrangian and define a space as in (0.11) by demanding that the symbols of the defining operators vanish on all components of $\mathcal{L}$. When the intersections are clean as in (0.12) that in fact is quite sufficient, but when they are not as in (0.13) the space could be too big for propagation (though in this particular example it is not). To define better spaces we break the Lagrangian varieties into nested families $\mathcal{K}$:

$$\Lambda \supset K_1 \supset K_2 \supset \cdots \supset K_0$$

where Λ is a conic Lagrangian and K_i's are conic embedded submanifolds. In example (0.12) there were three such families and in (0.13) two:

$$\mathrm{cl}(N^*(C\setminus L)) \supset \mathrm{cl}(N^*(C\setminus L))\cap N^*L, \quad N^*L \supset \mathrm{cl}(N^*(C\setminus L))\cap N^*L.$$

For a nested family $\mathcal{K}$, Melrose [23] defined a natural class of *marked Lagrangian distributions*:

$$\begin{aligned} I_kL^2(X;\mathcal{K}) \;=\; & \{u\in L^2(X): A_1\cdots A_l u \in L^2(X) \text{ for any } l\le k \text{ and } A_i\in\Psi^1_{\mathrm{phg}}(X) \\ & \text{with } \sigma_1(A_i)|_{\Lambda_F}=0,\ H_{\sigma_1(A_i)} \text{ tangent to all elements of } \mathcal{K}\}. \end{aligned}$$

These spaces have nice propagation properties as is illustrated in the following

Proposition 3 *Let $p=\sigma_2(P)$ and let $\Lambda\subset T^*X\setminus 0$ be a C^∞ homogeneous Lagrangian. Suppose that $K=\Lambda\cap\{p=0\}$ is a C^∞ hypersurface in Λ along which H_p is tangent to Λ exactly to some fixed order and transversal to K. Then, if Λ' is the H_p flow-out of $\Lambda\cap\{p=0\}$, the space*

$$J_kL^2(X) = I_kL^2(X;\Lambda,K)+I_kL^2(X;\Lambda',K)$$

has the property **(P)**.

This proposition can be stated more generally, allowing in particular multiple markings, but it already indicates that in 'reasonable' situations (such as those given by (0.12) and (0.13)) we can obtain microlocally defined spaces associated to a given geometry and for which **(P)** holds. For instance for the cusp we take [20]:

$$J_kL^2(X) = I_kL^2(X;\Lambda_C,\Lambda_C\cap\Lambda_L)+I_kL^2(X;\Lambda_L,\Lambda_C\cap\Lambda_L),\ \Lambda_C=\mathrm{cl}(N^*(C\setminus L)),\ \Lambda_L=N^*L. \tag{0.14}$$

Ideally, the space obtained by summing up the marked Lagrangian contributions coming from the full geometry could be equal to the one obtained by pushing forward the conormal space in X_β. It would then satisfy **(A)** and **(P)** automatically and in view of Proposition 2 could be applied to the study of regularity for semi-linear problems. That is however rarely the case. In fact the microlocally defined spaces are usually larger and may not even be algebras. To produce the actual space we need to 'play' on both sides and in addition use the properties of the lifted equation in X_β, such as the second microlocal ellipticity [36] or the propagation of the support [27], Sect.5.

In this rough outline I was not able to indicate the essential new difficulties encountered in the study of the mixed problem (1). Nevertheless, the general strategy applies and we aim at obtaining a space $J_kL^2(X)$, with X a manifold with boundary, such that **(A)** holds. We can only obtain a modification of **(P)**, (0.6), with $\epsilon=1/2$. That in particular bars at the moment the study of the fully semi-linear mixed problem.

4. An example. To illustrate the general discussion in Sect.3 we shall now present an example [20, 36] of relating microlocal and conormal spaces. This will also give us a chance to introduce the sub-marked Lagrangian distributions [42] which together with the super-marked ones ([27], Sect.4) are crucial in the diffractive estimates ([27], Sect.7).

Thus we recall (0.8)

$$X=\mathbf{R}^3,\ C=\{x_2^3=x_3^2\}\subset X,\ L=\{x_2=x_3=0\}\subset C,$$

with the Lagrangians

$$\Lambda_C=\mathrm{cl}N^*(C\setminus L)\setminus 0,\ \Lambda_L=N^*L\setminus 0,$$

and the marking

$$K = \Lambda_C \cap \Lambda_L.$$

The singularity of C at L can be resolved in one non-homogeneous blow-up of L:

$$\beta : X_\beta \longrightarrow X,\ X_\beta = \mathbf{R} \times \mathbf{R}_+ \times \mathbf{S}^1_{2-3},\ \ \mathbf{S}^1_{2-3} = \{(\omega_2, \omega_3) : \omega_2^6 + \omega_3^4 = 1\},$$

$$\beta : (z, r, \omega) \longmapsto (z, r^2\omega_2, r^3\omega_3).$$

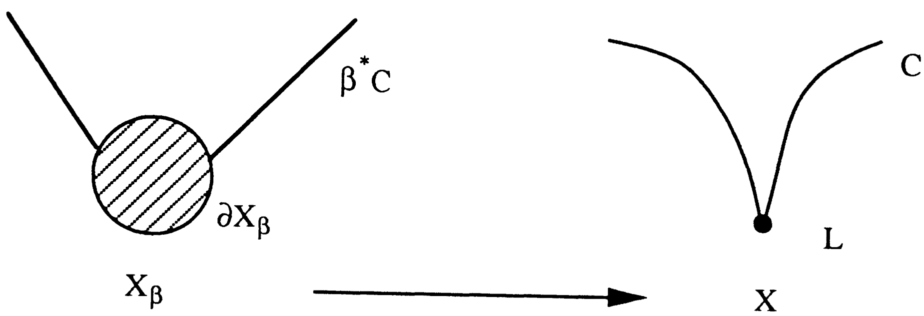

Figure 5: The lifted variety $\beta^*\mathcal{S}$.

The lift of the cusp, $\beta^*C = \mathrm{cl}\beta^{-1}(C \setminus L)$, is now very nice (see Fig.5) and we consider the lifted variety $\beta^*\mathcal{S} = \beta^*C \sqcup \partial X_\beta$. To describe it, we introduce projective coordinates in four neighbourhoods near $\beta^*\{x_i = 0, \pm x_j > 0\}$. For instance, for $i = 2, j = 3$ and $+$ we take (z, r, X) such that

$$\beta : (z, r, X) \mapsto (z, r^2, r^3X),\ \partial X_\beta = \{r = 0\},\ \beta^*C = \{X = \pm 1\},\ \nu_\beta = r^6 dz dX \frac{dr}{r}. \quad (0.15)$$

We easily define the conormal space associated to $\beta^*\mathcal{S}$ and, for instance in the coordinates (0.15), that means requiring stability under $\partial_z, r\partial_r$ and $(1 - X^2)\partial_X$. The space

$$J_k^\natural L^2(X) = \beta_* I_k L^2_{\nu_\beta}(X_\beta, \beta^*\mathcal{S}) \quad (0.16)$$

automatically satisfies $(\mathbf{A})$ and we will come to back to its propagative properties at the end of this section.

We now turn to the microlocal picture and start with $I_kL^2(X, \Lambda_L)$. The symbols of operators defining this space (see (0.11)) are generated over S^0_{phg} by

$$\xi_1,\ x_2\xi_2,\ x_3\xi_2,\ x_2\xi_3,\ x_3\xi_3, \quad (0.17)$$

and thus

$$I_kL^2(X, \Lambda_L) = \{u \in L^2(X) : D_{x_1}^{k_1}(x_2D_{x_2})^{k_2}(x_3D_{x_2})^{k_3}(x_2D_{x_3})^{k_4}(x_3D_{x_3})^{k_5} \in L^2(X), \sum k_i \leq k\}.$$

To define $I_kL^2(X;\Lambda,K)$ we need to modify (0.17) by demanding that the Hamilton vector fields of the generators are tangent to K:

$$\xi_1,\ x_2\xi_2,\ x_3\xi_2,\ x_2^2\xi_3,\ x_3\xi_3, \tag{0.18}$$

so that

$$I_kL^2(X;\Lambda_L,K)=\{u\in L^2(X):D_{x_1}^{k_1}(x_2D_{x_2})^{k_2}(x_3D_{x_2})^{k_3}(x_2^2D_{x_3})^{k_4}(x_3D_{x_3})^{k_5}\in L^2(X),\sum k_i\le k\}.$$

Comparing (0.17) and (0.18) we see that the only difference is in the generator $x_2^{1+\alpha}\xi_3$ with $\alpha=0$ and $\alpha=1$ respectively. Thus one expects that changing α in some suitable way should vary the rate of marking. For $0<\alpha<1$ that introduces sub-marking and for $1<\alpha<\infty$ super-marking. The precise definition is more involved especially in the latter case where to have invariance more geometric information needs to be introduced (see [27], Sect.4). Here we will discuss the relevant case of $\alpha=\frac{1}{2}$ which amounts to changing (0.18) to

$$\xi_1,\ x_2\xi_2,\ x_3\xi_2,\ x_2^3\xi_3^2,\ x_3\xi_3, \tag{0.19}$$

where the operator corresponding to the fourth symbol has now *weight* two in the following sense:

$$\begin{aligned}&I_kL^2(X;\Lambda_L,K,1/2)=\{u\in L^2(X): \qquad\qquad (0.20)\\ &D_{x_1}^{k_1}(x_2D_{x_2})^{k_2}(x_3D_{x_2})^{k_3}(x_2^3D_{x_3}^2)^{k_4}(x_3D_{x_3})^{k_5}\in L^2(X),\sum_{i\neq 4}k_i+2k_4\le 2l=k\},\end{aligned}$$

for even $k=2l$. For odd $k=2l+1$ we define the space by complex interpolation between the even-indexed neighbours.

Since the spaces are essentially defined by vector fields the lifting and push-forward under β are quite easy:

$$I_kL^2(X;\Lambda_L,K)\xleftrightarrow{\beta_*}I_kL^2_{\nu_\beta}(X_\beta;\partial X_\beta\sqcup\mathcal{P}_1) \tag{0.21}$$

$$I_kL^2(X;\Lambda_L,K,1/2)\xleftrightarrow{\beta_*}I_kL^2_{\nu_\beta}(X_\beta;\partial X_\beta\sqcup\mathcal{P}_{\frac{1}{2}}), \tag{0.22}$$

where

$$\mathcal{P}_1=(\beta^*\{x_2=0\}\cup\beta^*\{x_3=0\})\cap\partial X_\beta,\quad \mathcal{P}_{\frac{1}{2}}=\beta^*\{x_2=0\}\cap\partial X_\beta.$$

Thus, we are getting more in the lift than one might naively expect, with an improvement however when a finer microlocal space is used.

To discuss the cusp we observe that symplectically

$$\begin{array}{ccc}\Lambda_L & \longleftrightarrow & \Lambda_C\\ (x_1,x_2,x_3;\xi) & \longleftrightarrow & (x_1,x_2-(2\xi_2/3\xi_3)^2,x_3+(2\xi_2/3\xi_3)^3;\xi)\end{array}$$

and thus after some computations

$$\begin{aligned}&I_kL^2(X;\Lambda_C,K)=\{u\in L^2(X):D_{x_1}^{k_1}(2x_2D_{x_2}+3x_3D_{x_3})^{k_2}\cdot\\ &(27x_3D_{x_3}^3+8D_{x+2}^3)^{k_3}(3x_2^2D_{x_3}+2x_3D_{x_2})^{k_4}u\in H_{(-2k_3)}(X),\sum k_i\le k\},\end{aligned}$$

and for even $k = 2l$,

$$I_k L^2(X;\Lambda_C, K, 1/2) = \{u \in L^2(X) : D_{x_1}^{k_1}(2x_2D_{x_2} + 3x_3D_{x_3})^{k_2} \cdot$$
$$(27x_3D_{x_3}^3 + 8D_{x_2}^3)^{k_3}(4D_{x_2}^2 - 9x_2D_{x_3}^2)^{k_4}u \in H_{(-2k_3-4k_4)}(X), \sum_{i\le 3} k_i + 2k_4 \le 2l = k\},$$

and then for $k = 2l+1$ by complex interpolation. The lifting of these spaces is relatively easy but already additional care is needed to deal with the higher order operators:

$$I_kL^2(X;\Lambda_C,K) \xrightarrow{\beta^*} I_kL^2_{\nu_\beta}(X_\beta;\partial X_\beta \sqcup \beta^*C \sqcup \mathcal{P}_1), \tag{0.23}$$
$$I_kL^2(X;\Lambda_C,K,1/2) \xrightarrow{\beta^*} I_kL^2_{\nu_\beta}(X_\beta;\partial X_\beta \sqcup \beta^*C). \tag{0.24}$$

The push-forward is considerably harder again in view of the presence of higher order operators. In particular, we cannot characterize the lift of the spaces associated to Λ_C alone:

$$I_kL^2_{\nu_\beta}(X_\beta;\partial X_\beta \sqcup \beta^*C \sqcup \mathcal{P}_1) \xrightarrow{\beta_*} I_kL^2(X;\Lambda_C,K) + I_kL^2(X;\Lambda_L,K) \tag{0.25}$$
$$I_kL^2_{\nu_\beta}(X_\beta;\partial X_\beta \sqcup \beta^*C \sqcup \mathcal{P}_{\frac{1}{2}}) \xrightarrow{\beta_*} I_kL^2(X;\Lambda_C,K,1/2) + I_kL^2(X;\Lambda_L,K,1/2) \tag{0.26}$$

Combining (0.22),(0.23),(0.25) and (0.22),(0.24),(0.26) we see that the microlocally defined spaces

$$J_k^\flat L^2(X) = I_kL^2(X;\Lambda_C,K) + I_kL^2(X;\Lambda_L,K),$$
$$J_k^\sharp L^2(X) = I_kL^2(X;\Lambda_C,K,1/2) + I_kL^2(X;\Lambda_L,K,1/2),$$

have the property **(A)**. By Proposition 3 the first space has property **(P)** and since that proposition holds also for the sub-marked spaces, so does the second. Thus in view of Proposition 2 both spaces propagate for the semi-linear equation.

As observed by Sá Barreto [36] the natural conormal space $J_k^\natural L^2(X)$ obtained by the push-forward (0.16) also propagates! An outline of his argument gives us an opportunity to see an example of the analysis 'upstairs' in the blown-up space X_β. Thus let us consider the lift of the operator $P = 4D_{x_2}^2 - 9x_2D_{x_3}^2 - D_{x_1}D_{x_3}$:

$$P_\beta = r^4\beta^*P\beta_* \in \mathrm{Diff}_b^2(X_\beta).$$

Since from (0.22),(0.23) and (0.25) we know that $\beta_*I_kL^2_{\nu_\beta}(X_\beta;\partial X_\beta \sqcup \beta^*C \sqcup \mathcal{P}_1)$ propagates, we only need to eliminate the singularities at

$$(\beta^*\{x_3 = 0\} \cup \beta^*\{x_2 = 0\}) \cap \partial X_\beta.$$

Roughly speaking that is done by using the ellipticity (in the totally characteristic sense of Melrose [18]) of P_β in the relevant region. In fact, in the coordinates (0.15) $\beta^*\{x_3 = 0\} \cap \partial X_\beta = \{X = r = 0\}$ and modulo lower order terms

$$P\beta \equiv 9(X^2-1)D_X^2 - 6XrD_rD_X + (rD_r)^2 - rD_XD_z.$$

Since we already have the stability under rD_r and D_z we can restrict our attention (microlocally in ${}^bT^*X_\beta \setminus 0$) to the region where ${}^b\sigma_1(D_X) \gg {}^b\sigma_1(rD_r), {}^b\sigma_1(D_z)$. Once $X \sim 0$, P_β is indeed elliptic there. More care is of course needed to exploit this properly and we should also remark that the argument can be made independent of propagation for the microlocal space $J_k^b L^2(X)$. One only needs the very easy propagation of $I_k L^2(X, \mathcal{S})$ with $\mathcal{S}$ as in (0.8).

5. The pseudo-conormal space for the diffractive problem As described in Sect.2 the interaction geometry is quite complicated as it involves cusp and conic singularities. Following the strategy outlined in Sect.3 we want to resolve the singularities and the method of resolution is similar to that used in [26, 37]. In particular it involves a *non-homogeneous blow-up.* To describe it let us consider

$$\mathbf{R}^n = \mathbf{R}^3 \times \mathbf{R}^{n-3}, \quad z = (x, y), \; z \in \mathbf{R}^n, x \in \mathbf{R}^3, y \in \mathbf{R}^{n-3}$$

on which we define an $\mathbf{R}_+$-action T_δ^{1-2-3}:

$$T_\delta^{1-2-3}(x, y) = (\delta x_1, \delta^2 x_2, \delta^3 x_3, y), \quad \delta \in \mathbf{R}_+ \tag{0.27}$$

We start with a definition of spaces of functions with given non-homogeneous orders of vanishing:

$$M_r^{1-2-3}(\widetilde{X}) \subset C^\infty(\widetilde{X}), \quad u \in M_r^{1-2-3}(\widetilde{X}) \Longleftrightarrow T_\delta^* u = \mathcal{O}(\delta^r), \; \delta \to 0 \tag{0.28}$$

This allows us to the define a filtration of the differential operators in terms of homogeneity. Thus

$$Q \in \mathrm{Diff}^k_{p,(1-2-3)}(\widetilde{X}) \Longleftrightarrow Q : M_r^{1-2-3}(\widetilde{X}) \longrightarrow M_{r-p}^{1-2-3}(\widetilde{X}) \quad \text{for} \quad r \geq p.$$

The homogeneous differential operator important in our discussion is Friedlander's operator in $\mathbf{R}^3$:

$$P_0 = 4D_{x_2}^2 - 9x_2 D_{x_3}^2 - 6D_{x_3} D_{x_1}.$$

A suitable coordinates are now given by

Proposition 4 *There exist coordinates $(x, y) \in \mathbf{R}^n$, $x \in \mathbf{R}^3, y \in \mathbf{R}^{n-3}$ in $\widetilde{X}$ such that*

$$P = P_0 + Q, \quad Q \in \mathrm{Diff}^2_{3,(1-2-3)}(\widetilde{X}), \tag{0.29}$$

$$\partial X = \{(x, y) : x_2 - \frac{1}{4}x_1^2 = 0\}, \tag{0.30}$$

and with the notation of Sect.2 and any $H \in \mathcal{R}$ given by (0.3)

a) $\Gamma = \{(0, y) : y \in \mathbf{R}^{n-3}\} \cap \widetilde{X}$

b) $\widetilde{R} = \{(x, y) : x_3^2 - x_2^3 = 0, \; x \in \mathbf{R}^3, y \in \mathbf{R}^{n-3}\}$

c) $\widetilde{F} = \{(x, y) : 2x_3 - 3x_2 x_1 + x_1^3 + f = 0, \; x \in \mathbf{R}^3, y \in \mathbf{R}^{n-3}\}$ *with* $f \in M_4^{1-2-3}(\widetilde{X})$,

d) $\widetilde{S} = \{(x, y) : x_1^4 + 8x_1 x_3 - 6x_1^2 x_2 - 3x_2^2 + s = 0, \; x \in \mathbf{R}^3, y \in \mathbf{R}^{n-3}\}$ *with* $s \in M_5^{1-2-3}(\widetilde{X})$,

e) $H = \{(x, y) : x_1 + h = 0\}$ *with* $h \in M_2^{1-2-3}(\widetilde{X})$.

We will consider the surfaces on the right-hand side as the *model geometry.* The sense in which they are models can be explained as follows. The model surface for $\widetilde{F}$ in *c)* is characteristic for Friedlander's operator P_0 and the cusp $\widetilde{R}$ is obtained from that model surface by reflection (according to the rules of geometric optics given in Proposition 2.1) through the boundary $x_2 - \frac{1}{4}x_1^2 = 0$. Note that this surface, although microlocally diffractive near $N^*\widetilde{R}$, is not globally diffractive for P_0 : it contains the characteristic $\{x_1 = x_2 = 0\}$. Thus we see that $Q \neq 0$ and essentially it has to contain a term of the form $-cx_2 D_{x_1}^2$ which destroys the degeneracy of the characteristic $\{x_1 = x_2 = 0\}$. The surface defined by the right hand side of d) is the cone over $0 \in \mathbf{R}^3$ with respect to the characteristic flow-out by P_0.

In view of Proposition 4 it is natural to resolve the geometry using the 1-2-3 blow-up given by the $\mathbf{R}_+$-action (0.27). Thus we define the space

$$\widetilde{X}_1 = (\widetilde{X} \setminus \Gamma) \sqcup (\mathbf{S}^2_{1-2-3} \times \mathbf{R}^{n-3}) \simeq \mathbf{R}_+ \times \mathbf{S}^2_{1-2-3} \times \mathbf{R}^{n-3} \tag{0.31}$$

where $\mathbf{S}^2_{1-2-3}$ is a non-round sphere $\{\omega \in \mathbf{R}^3 : \sum_{1 \le i \le 3} \omega_i^{12/i} = 1\}$ and where the C^∞ structure on $\widetilde{X}_1$ is given by the second identification (see [21]). We now have the blow-down map

$$\widetilde{X}_1 \xrightarrow{\beta_1} \widetilde{X}, \quad (r, \omega, y) \longmapsto (r\omega_1, r^2\omega_2, r^3\omega_3, y)$$

which is a diffeomorphism on $\widetilde{X}_1 \setminus \partial\widetilde{X}_1$. Thus following [21] we define the *pull-back* of Y to be

$$\beta_1^* Y = \mathrm{cl}[\beta_1^{-1}(Y \setminus \Gamma)], \quad Y \subset \widetilde{X}.$$

The lifts $\beta_1^*\widetilde{F}, \beta_1^*\widetilde{S}$ and $\beta_1^*\partial X$ are smooth hypersurfaces in $\widetilde{X}_1$ intersecting the boundary $\partial\widetilde{X}_1$ cleanly, and $\beta_1^*\widetilde{R}$ has a cusp singularity transversal to $\partial\widetilde{X}_1$. Also,

$$\begin{aligned}
&\beta_1^*\widetilde{R} \cap \partial\widetilde{X}_1 = \beta_1^*\{2x_3 - 3x_2x_1 + x_1^3 = 0\} \cap \partial\widetilde{X}_1, \\
&\beta_1^*\widetilde{S} \cap \partial\widetilde{X}_1 = \beta_1^*\{x_1^4 + 8x_1x_3 - 6x_1^2x_2 - 3x_2^2 = 0\} \cap \partial\widetilde{X}_1, \\
&\beta_1^* H \cap \partial\widetilde{X}_1 = \beta_1^*\{x_1 = 0\} \cap \partial\widetilde{X}_1.
\end{aligned}$$

The boundary of the resolved space $\partial\widetilde{X}_1$ and the above intersections are shown in Fig.6.

The cone on the right hand side of d) in Proposition 4 is essentially symmetric with respect to the interchange of x_1 and x_3 (a 1-2-3 homogeneous change of variables transforms Q to $4x_3x_1 - x_2^2$, see [26]).Roughly speaking, an additional blow-up near $\beta_1^*(Q \cap H) \cap \partial\widetilde{X}_1$ is needed to undo the asymmetry of the 1-2-3 blow-up.

To introduce it we first change coordinates in a 1-2-3 homogeneous way, preserving $\widetilde{R}$ and taking Q to $4x_1x_3 - x_2^2$. We then apple an almost-homogeneous change of variables to preserve Q but take H to $\{x_1 = 0\}$. Using the lift of these coordinates, we blow-up with the 2-1-1 homogeneity the codimension three submanifold $\partial\widetilde{X}_1 \cap \beta_1^*\{x_1 = x_2 = 0, x_3 > 0\} = \partial\widetilde{X}_1 \cap \beta_1^* D_+$:

$$\widetilde{X}_2 \xrightarrow{\beta_{12}} \widetilde{X}_1 \xrightarrow{\beta_1} \widetilde{X}, \quad \beta_2 = \beta_1 \circ \beta_{12}$$

$$\widetilde{X}_2 = \widetilde{X}_1 \setminus (\partial\widetilde{X}_1 \cap \beta_1^* D_+) \sqcup (\mathbf{S}^2_{2-1-1_+} \times \mathbf{R}^{n-3}),$$

where $\mathbf{S}^2_{2-1-1_+}$ is a half non-round sphere $\{\nu \in \mathbf{R}^3 : \nu_1^2 + \nu_2^4 + \nu_3^4 = 1, \nu_3 \ge 0\}$ and

$$\beta_{12}(\rho, \nu, y) = (\rho^2\nu_1, \rho\nu_2, \rho\nu_3, y),$$

with the coordinates in $\widetilde{X}_1$ near $\partial\widetilde{X}_1 \cap \beta_1^* D_+$ chosen so that $\beta_1(X_1, X_2, r, y) = (rX_1, r^2X_2, r^3, y) \in \widetilde{X}$. The manifold $\widetilde{X}_2$ has a codimension two corner and $\partial\widetilde{X}_2$ is shown in Fig.7.

Since $\widetilde{S}$ and H are simply tangent at D another blow-up is still needed:

$$\widetilde{X}_3 \xrightarrow{\beta_{23}} \widetilde{X}_2 \xrightarrow{\beta_{12}} \widetilde{X}_1 \xrightarrow{\beta_1} \widetilde{X}, \quad \beta_3 = \beta_1 \circ \beta_{12} \circ \beta_{23}$$

Here, the line $\beta_2^* D_+$ is blown-up with the 2-1-0 homogeneity in the coordinates where $\beta_2^* H \cap N = \{X_1 = 0\} \cap N$ and $\beta_2^*\widetilde{S} \cap N = \{4X_1 - X_2^2 = 0\} \cap N$ where N is a neighbourhood of $\beta_2^* D_+$ – see [36].

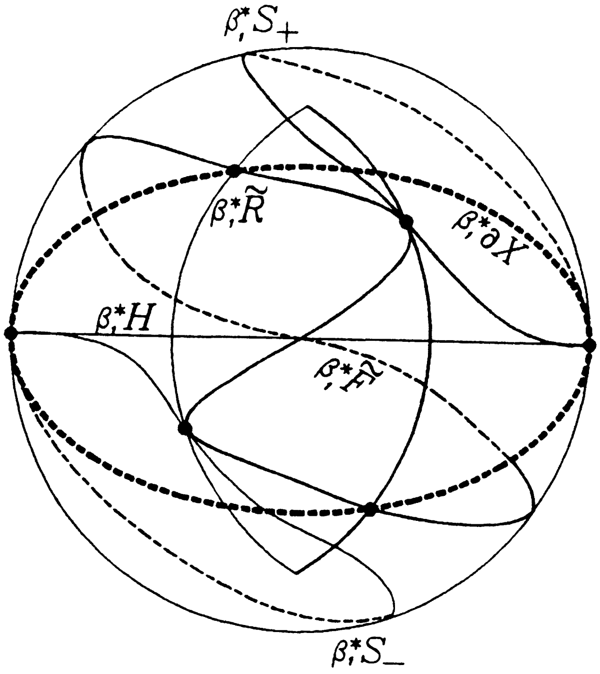

Figure 6: The geometry on $\partial\widetilde{X}_1$ as seen from the positive x_2 direction.

There are additional tangencies and singularities that have not yet been resolved: the tangencies described in Proposition 2.5 persist in $\widetilde{X}_1$ at $\beta_1^* B$ as does the cusp singularity of $\beta_1^*\widetilde{R}$ at $\beta_1^* L$. The former is resolved using a successsion of normal blow-ups [25] (see Fig.8) and the latter using the 3-2 blow up [36], only at $\beta_1^* B_+$ and $\beta_1^* L_+$ respectively ($\beta_1^* L_+ = \beta_1^*(L \cap \{x_1 \geq 0\}$. This leads to the space $\widetilde{X}_4$:

$$\widetilde{X}_4 \xrightarrow{\beta_{34}} \widetilde{X}_3 \xrightarrow{\beta_3} \widetilde{X}, \quad \beta = \beta_4 = \beta_3 \circ \beta_{34}$$

see Fig.7.

For future reference we also define $\widetilde{X}_5$, analogously to $\widetilde{X}_4$ but obtained by applying the same blow-ups at the lifts of $D_\pm, L_\pm, B_\pm$ rather than of D_+, L_+, B_+ only:

$$\widetilde{X}_5 \xrightarrow{\beta_{35}} \widetilde{X}_3 \xrightarrow{\beta_3} \widetilde{X}, \quad \beta_5 = \beta_3 \circ \beta_{35}.$$

We shall now define the C^∞-algebra $J_k L_c^2(\widetilde{X}, H)$ associated to the geometry in the open manifold $\widetilde{X}$. In the notation we stress the dependence on the 'artificial' characteristic hypersurface $H \in \mathcal{R}$.

Let us first consider the surfaces in $\widetilde{X}_4$ obtained from the geometry in $\widetilde{X}$:

$$\beta^*\widetilde{R},\ \beta^*\widetilde{F},\ \beta^*\widetilde{S}_+,\ \beta^*(\widetilde{F} \cap \widetilde{R} \setminus B),\ \beta^*(\widetilde{S}_+ \cap \widetilde{R} \setminus B),\ \beta^* H$$

where we note that the lifts of B_+, D_+ and L_+ are included in the boundary of $\widetilde{X}_4$. Let $\mathcal{S}$ be the variety obtained by taking a disjoint union of the five submanifolds above with $\partial\widetilde{X}_4$:

$$\mathcal{S} = \beta^*\widetilde{F} \sqcup \beta^*\widetilde{R} \sqcup \beta^*\widetilde{S}_+ \sqcup \beta^*(\widetilde{F} \cap \widetilde{R} \setminus B) \sqcup \beta^*(\widetilde{S}_+ \cap \widetilde{R} \setminus B) \sqcup \beta^* H \sqcup \partial\widetilde{X}_4 \qquad (0.32)$$

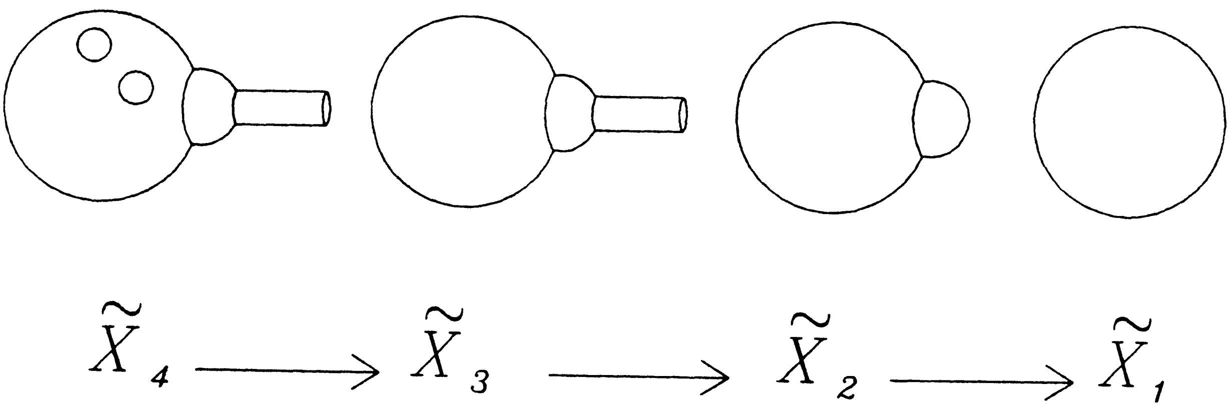

Figure 7: The hierarchy of blow-ups

Ideally, we would want to define $J_k L^2_c(\widetilde{X}, H)$ as the β-pushforward of the conormal spaces associated to $\mathcal{S}$ which is in fact done for the interior problem. Here, however, this would be disastrous.

We also need to define $K_1 = K_1(\epsilon) \subset \widetilde{X}_1$ which in some sense constitues a 'non-homogeneous' past. Thus we start by defining $q_0 \in C^\infty(\widetilde{X}_1)$ independent of r and y as

$$q_0(\omega) = r^{-4}(\beta_1^* q)(r, \omega), \quad q(x) = x_1^4 + 8x_1x_2 - 6x_1^2x_2 - 3x_2^2.$$

Thus $\beta_1^* Q = \{(r, \omega, y) : q_0(\omega) = 0\}$ is the model cone (see Proposition 4). We then consider $\widetilde{X}_1 \setminus \{(r, \omega, y) : q_0(\omega) = -\epsilon\}$, which for small $\epsilon > 0$ has three components. We take as K_1 the component which contains $\beta_1^* Q_-$ where Q_- is the model retarded cone over Γ. We can take ϵ small enough so that

$$\beta_1^*(\widetilde{F} \cap \widetilde{R} \setminus B) \subset \widetilde{X}_1 \setminus K_1(\epsilon) \tag{0.33}$$

Since the all the higher generation blow-ups occur away from K_1 we can think of it as a subset of $\widetilde{X}_4$ (or $\beta_{14}^* K_1 = K_1$).

Definition 1. For $k \in \mathbb{N}_0$, we define

$$J_k L^2_c(\widetilde{X}, H) = \beta_* \{U \in I_k L^2_\nu(\widetilde{X}_4; \mathcal{V}(\widetilde{X}_4, \mathcal{S})) \ : \ U|_{K_1} \in I_k L^2_\nu(\widetilde{X}_4, \beta^* \widetilde{F} \sqcup \partial \widetilde{X}_4)|_{K_1}\},$$

where the variety $\mathcal{S}$ is given by (0.32) and $K_1 = K_1(\epsilon)$ is given above with ϵ such that (0.33) is satisfied. The norm is defined using the norm of the lift:

$$\|u\|_{J_k L^2(\widetilde{X}, H)} = \|\beta^* u\|_{I_k L^2_\nu(\widetilde{X}_4, \mathcal{V}(\widetilde{X}_4, \mathcal{S}))} + \|\beta^* u|_{K_1}\|_{I_k L^2_\nu(\widetilde{X}_4, \beta^* \widetilde{R} \sqcup \partial \widetilde{X}_4)|_{K_1}}.$$

We also define $J^1_k L^2(\widetilde{X}, H)$ by demanding that $u, Du \in J_k L^2_c(\widetilde{X}, H)$, with the obvious norm.

For non-integral values of the order of regularity we use complex interpolation and define:

$$J_{k+s} L^2(\widetilde{X}, H) = [J_k L^2(\widetilde{X}, H), J_{k+1} L^2(\widetilde{X}, H)]_s, \quad 0 < s < 1, \tag{0.34}$$

and similarly for $J^1_{k+s} L^2(\widetilde{X}, H)$.

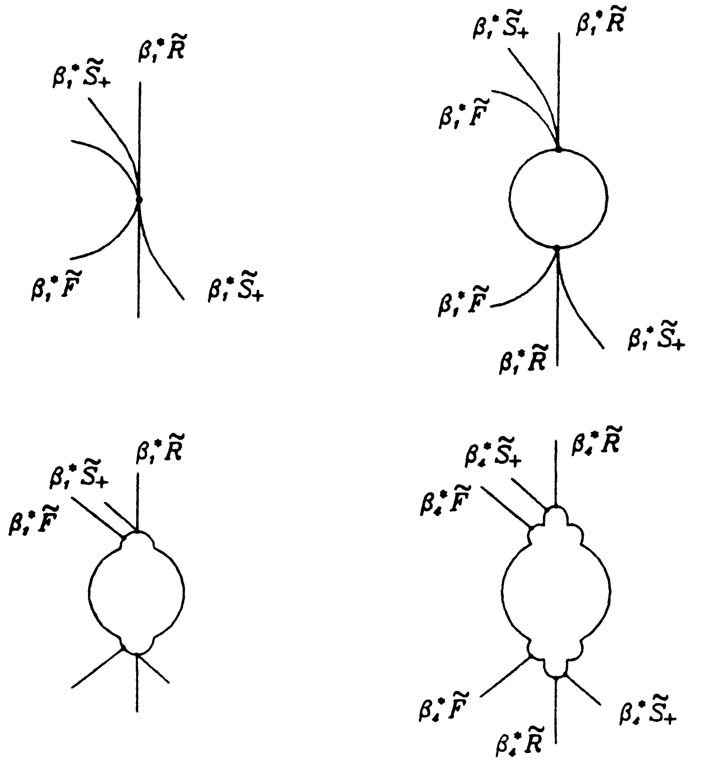

Figure 8: The three normal blow-ups of $\beta_1^* B$

This is a *pseudo-conormal* space as it involves an additional condition in K_1. The corresponding pseudo-conormal space for the manifold with boundary X is essentially obtained by restriction with an additional singular support condition:

Definition 2. For $s \geq 0$, $s \in \mathbf{R}$ we define

$$\begin{aligned} J_s L^2(X) = \quad & \{u \in L^2(X) : \text{ for every } H \in \mathcal{R} \text{ there exists } \tilde{u} \in J_s L^2(\widetilde{X}, H) \\ & \text{with } \tilde{u}|_X = u \text{ and } \operatorname{sing\,supp}^{(s)} u \cap (\widetilde{F} \setminus F \cup \widetilde{R} \setminus R \cup \widetilde{S}_+ \setminus S_+) = \emptyset \ \}. \end{aligned}$$

We recall that using the regularity function $s_u(x)$ (cf. [11], Sect.18.1), we define $\operatorname{sing\,supp}^{(s)} u = \{x : s_u(x) \leq s\}$ which by lower semi-continuity of s_u is closed. The space $J_s L^2(X)$ is *not* a normed space and although it can be made into a Fréchet space we shall not need this fact here.

Remark. Although the definition of the blow-up involves the choice of H, it can in fact be made independent of it. It is also true that away from Γ the spaces $J_s L^2(X)$ is the same as the space defined without including the lift of H in the defining variety. That statement is non-trivial only near D.

The complications of the definitions are now compensated by the simplicity of the proof of the following

Theorem 3 *The spaces $J_s L^2(\widetilde{X}, H) \cap L^\infty_{\mathrm{loc}}(\widetilde{X})$ and $J_s L^2(X) \cap L^\infty_{\mathrm{loc}}(X)$ given by Definitions 1 and 2 respectively are C^∞-modules and C^∞-algebras.*

As in the earlier work on conormal regularity the difficult part is the propagation theorem. For the interior problem it follows from the results of [26] and refined estimates in the 'non-homogeneous' past K_1.

The main result of [27] is the propagation theorem for the Dirichlet problem and the space $J_k L^2(X)$:

Theorem 4 *If* $f|_{X_-} = 0$ *and* $u \in L^2(X)$ *satisfies*

$$Pu = f \text{ in } X,\ u|_{\partial X} = 0,\ u|_{X_-} = 0,$$

then

$$f \in J_s L^2(X) \Longrightarrow u \in J_{s+\frac{1}{2}} L^2_{\mathrm{loc}}(X).$$

Theorems 3 and 4 provide almost all of modified properties **(A)** and **(P)** needed for the semi-linear propagation. We still need to check that

$$Pw = 0,\ w|_{\partial X} = 0,\ w|_{X_-} \in I_k L^2(X_-, F) \Longrightarrow w \in J_k L^2_{\mathrm{loc}}(X)$$

which is however comparatively easy with the right hand side much larger than necessary. Thus we can finally give our main result:

Theorem 5 *Let* $u \in L^\infty_{\mathrm{loc}}(X)$ *be the solution of the semi-linear mixed problem:*

$$Pu = f(x, u) \text{ in } X,\ u|_{\partial X} = 0,\ u|_{X_-} = u_0$$

where $f \in C^\infty(X, \mathbf{C})$ *and* $u_0 \in I_k L^2_{\mathrm{loc}}(X_-, F)$. *Then* $u \in J_k L^2_{\mathrm{loc}}(X)$.

The results presented in Sect.1 are easy consequences of Theorem 5 and the definitions.

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