S. Miyatake

Neumann operator for wave equation in a half space and microlocal orders of singularities along the boundary

Séminaire Équations aux dérivées partielles (Polytechnique) (1992-1993), exp. nº 16, p. 1-6

<http://www.numdam.org/item?id=SEDP_1992-1993____A16_0>

CENTRE
DE
MATHEMATIQUES

Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE

F-91128 PALAISEAU Cedex (FRANCE)
Tél. (1) 69 33 40 91
Fax (1) 69 33 30 19 ; Télex 601.596 F

Séminaire 1992-1993

EQUATIONS AUX DERIVEES PARTIELLES

NEUMANN OPERATOR FOR WAVE EQUATION IN A HALF SPACE AND MICROLOCAL ORDERS OF SINGULARITIES ALONG THE BOUNDARY

S. MIYATAKE

Exposé n° XVI

18 Mai 1993

1 Introduction

As for fixed initial boundary value problems for hyperbolic equation of second order, Dirichlet and Neumann problems are celebrated. However strict best estimates are not so easy to obtain, and between both problems there is a certain difference of order by 1/2 as for norms of boundary data. Now we try to study these foundations further and clarify that this difference is inevitable. For simplicity we consider wave equation $Pu = 0$ in a half space $R^+ \times R^{n-1} \times R \ni (x, y, t), P$ being $D_x^2 + \sum_j D_{y_j}^2 - D_t^2$. Namely we are concerned at first with Neumann problem :

$$(N.P) \qquad Pu(x,y,t) = 0 \text{ in } R^+ \times R^{n-1} \times R, \quad D_x u|_{x=0} = g(y,t).$$

We consider also Dirichlet problem

$$(D.P) \qquad Pu(x,y,t) = 0 \text{ in } R^+ \times R^{n-1} \times R, \quad u|_{x=0} = h(y,t),$$

since the solution u of (N.P) also satisfies (D.P) if we define $h(y,t)$ by $u_{|x=0}$. The existence and uniqueness theorems for both problems give the Neumann operator in our case, wich we denote by Q.

$$Qg = h. \tag{1.1}$$

Namely Neumann operator is defined using the solution of (N.P). More precisely we use the forward progressing solution satisfying the following condition :

$$\inf \underset{t}{\mathrm{Proj}} \ \ \mathrm{supp} \ \ u = \inf \underset{t}{\mathrm{Proj}} \ \ \mathrm{supp} \ \ g = t_0, \text{if} \ \ t_0 > -\infty \tag{1.2}$$

Then u is also the forward solution of (D.P) in the sense that $h(y,t)$ satisfies (1.2) replaced g by h. We can refer [6] and [3] concerning the existence and uniqueness theorems for these problems. These theorems can be considered in the spaces with norms of negative order in parallel and weak solution can be constructed by Laplace-Fourier inversion formula. We can then obtain a concrete expression of the solution to (D.P) and g is found by taking the normal derivative at the boundary. Since Laplace-Fourier inversion formula gives one to one onto correspondance we have the expression of Neumann operator as follows :

$$h(y,t) = (\frac{1}{2\pi})^n \int\int\int\int e^{i\phi}(\tau^2 - |\eta|^2)^{-1/2} g(y', t') dy' dt' d\eta d\tau, \tag{1.3}$$

where $\tau = \sigma - i\gamma, \gamma > 0$ and $\phi = (y - y')\eta + (t - t')\tau . h(y,t)$ is shown to be independent of $\gamma > 0$. Now denote $Q(Qg) = v(y,t)$. Then v satisfies

$$\Box v = (D_t^2 - \sum_{j=1}^{n-1} D_{y_j}^2) v = g(y,t),$$

and has the forward progressing property (1.2) replaced u by v. Therefore we can write the Neumann operator.

$$Qg = \Box^{-1/2} g. \tag{1.4}$$

Here $(\tau^2 - |\eta|^2)^{1/2}$ stand for a square root of $\tau^2 - |\eta|^2$, which is single valued and analytic in $C - [-|\eta|, |\eta|]$ and satisfies

$$(\tau^2 - |\eta|^2)^{1/2} < 0 \quad \text{for real} \quad \tau > |\eta|. \tag{1.5}$$

In this way our main purpose is to reseach the properties of $-1/2$ power of d'Alembertian.

At first applying a contour integral to the expression (1.3) we can reduce the integral in τ to two times integral in $\sigma = Re\tau$ from $-|\ \eta|$ to $|\ \eta\ |$. Then we use a special partition of unity in $(\eta, \sigma)-$ plane in order to get the following decomposition ;

$$Q = Q_0 + Q_+ + Q_-, \tag{1.6}$$

where Q_0 is a pseudodifferential operator with symbol

$$\sigma(Q_0) = p_0(y, t, \eta, \sigma) \in S^{-2/3}_{1/3,0}\ . \tag{1.7}$$

Therefore, if we define respectively

$$h_* = Q_* g$$

h_0 is a C^∞ function in $\{(y,t); t > t_1\}$. Here we have assumed

$$g \in \mathcal{E}' \quad \text{and} \quad \operatorname*{Proj}_t \ \text{supp} \ \ g = [t_0, t_1]\ , \tag{1.8}$$

Hence the main part of $h = Qg$ is $h_+ = Q_+ g$ and $h_- = Q_- g$, which we can write down as follows.

$$h_\pm(y,t) = Q_\pm g = \int_{t_0}^{t_1} P_{\phi\pm}(t - t') g(\cdot, t') dt' \tag{1.9}$$

where $P_{\phi\pm}(t - t')$ is a family of Fourier integral operators smoothly depending on a parameter $t - t'$, and t_1 in (1.9) is replaced by t if $t_1 > t$. Here the phase functions are of the forms

$$\phi_\pm = y\eta \pm (t - t')|\eta|, \tag{1.10}$$

and their amplitude functions satisfy

$$P_\pm = P_\pm(y, \eta; t - t') \in S^{-1/2}_{2/3,0} \tag{1.11}$$

if t is larger than t'. Moreover we can obtain concrete expression of $p_\pm$ and properties of $h_\pm$, which we see later. Here we remark only that we use Frenel integral and second mean value theorem concerning the integral of product functions. Finally notice that theorem C in [5] and (1.9) - (1.11) give the informations on the propagations of microlocal orders of singularities of the solution along the boundary, since h means the trace of u to the boundary.

Specially in the case where g is a shock pulse like

$$g(y',t') = \delta(t'-t')g_1(y') \ . \tag{1.12}$$

Then (1.9) becomes

$$h_\pm(y,t) = P_{\phi\pm}(t-t_1')\ g_1 \ . \tag{1.9}$$

Now applying Theorem 3.1 in [4] or Theorem C in [5] we can verify

$$OS(h_\pm(\cdot,t);y,\eta) = OS(g_1;y'_\pm,\eta) - 1/2, \text{ where } y-y' = \mp(t-t_1')\frac{\eta}{|\eta|} \ , \tag{1.13}$$

for so small $t-t_1' > 0$ that condition I in [5] is fulfilled. Even if $t-t_1'$ is larger, we can show (1.13) deviding $P_\phi(t-t_1')$ as follows, $\pm$ being omitted,

$$P_\phi(t-t_1') = (P_\phi(t-t_1')P_\phi^{-1}(t_k'-t_1'))\cdots(P_\phi(t_3'-t_2')P_\phi^{-1}(t_2'-t_1'))P_\phi(t_2'-t_1'). \tag{1.14}$$

Then each factor of (1.4) has an elliptic amplitude function and becomes a Fourier integral operator with phase function satisfying condition I. Thus we have (1.13) also in this case.

2 Neumann operator and -1/2 power of d'Alembertian

At first we give the expressions for solutions to (D.P) and (N.P) as follows respectively

$$u(x,y,t) = (\frac{1}{2\pi})^n \int\int\int\int e^{ix}(\tau^2-|\eta|^2)^{1/2}e^{i\phi}h(y',t')dy'dt'd\eta d\tau \ , \tag{2.1}$$

$$u(x,y,t) = (\frac{1}{2\pi})^n \int\int\int\int e^{ix(\tau^2-|\eta|^2)^{1/2}}e^{i\phi}\frac{g(y't')}{(\tau^2-|\eta|^2)^{1/2}}dy'dt'd\eta d\tau \ , \tag{2.2}$$

where the notation (1.5) is used. Therefore we can verify (1.3) and suitable estimates for u in (2.1) and (2.2) and for h in (1.3). Moreover if we take care that u and h are independent of $\gamma > 0$, we can show estimates which are very proper to forward progressing properties.

In order to caliculate, concerning h in (1.3) we can use Fubini theorem if we suppose $g \in \mathcal{D}$. For $g \in \varepsilon'$, we define Qg by $(Qg, \phi) = (g, Q^*\phi), Q^*\phi$ having backward properties. Even in this case our arguments are rather similar, we concentrate ourselves to the caliculus of $h = Qg$ in (1.3).

Let start to consider

$$(2.3) \qquad F = \int_{-\infty}^{\infty} \frac{e^{i(t-t')\tau}}{(\tau^2 - |\eta|^2)^{1/2}} d\sigma, \quad \text{for} \gamma \ > 0, |\eta| \neq 0 \ .$$

In order to make γ tend to zero, we prepare the following lemma :

Lemma 1.— *For real $p \neq 0$,*

$$F_\gamma(p) = \int_{=-\mathrm{Im} z>0} \frac{e^{izp}}{(z^2-1)^{1/2}} dz$$

is independent of $\gamma > 0$. $F_\gamma(p) = F(p)$ satisfies $F_\gamma(p) = 0$ if $p < 0$ and $F(p) = -2i \int_{-1}^{1} e^{ipx}(1-x^2)^{-1/2} dx$, if $p > 0$.

Using this lemma we can verify that (1.3) is equal to

$$(2.4) \qquad h(y,t) = (\frac{1}{2\pi})^n \frac{2}{i} \int_{R^n} \int_{-\infty}^{t} \int_{R^n} \int_{-|\eta|}^{|\eta|} \frac{e^{i\phi} g(y,t)}{(|\eta|^2 - \tau^2)^{1/2}} dy' dt' d\eta d\sigma.$$

Now decompose above h together with h in (1.3), Take a C^∞ function β satisfying,

$$(2.5) \qquad \beta(t) = 1 \ \text{ if } \ |t| \leqq 1/4 \ \text{ and } \ \beta(t) = 0 \ \text{ if } \ |t| > 1/2.$$

Then put

$$(2.6) \qquad 1 = \beta(|\eta|) + (1 - \beta(|\eta|)) = \beta_0(\eta) + \beta_1(\eta),$$

$$(2.7) \qquad \beta_1 = \alpha_+(\eta,\sigma) + \alpha_-(\eta,\sigma) + \alpha_0(\eta,\sigma), \text{where}$$

$$\alpha_+(\eta,\sigma) = \beta_1 \beta(|\eta|^{-1/3}(|\eta| - \sigma)), \alpha_-(\eta,\sigma) = \alpha_+(\eta,-\sigma),$$

and

$$\alpha_0(\eta,\sigma) = \beta_1(\eta) - \alpha_+(\eta,\sigma) - \alpha_-(\eta,\sigma).$$

Substituting (2.6) in the integrand of (1.3) we have

$$(2.6)' \qquad h(y,t) = h_0(y,t) + h_1(y,t) \ ,$$

where h_0 is considered for fixed $\gamma > 0$. As for h_1 we use Lemma 1 to obtain the same representation as (2.4). More precisely $h_1(y,t)$ has the form $h(y,t)$ in (2.4) with integrand multiplyed by $\beta_1(\eta,\sigma)$. Then corresponding to (2.7) it holds

$$(2.7)' \qquad h_1(y,t) = h_+(y,t) + h_-(y,t) + h_{1o}(y,t) \ .$$

h_0 and h_{10} are the images of g by pseudodifferential operators with amplitude functions satisfying respectively

$$(2.8) \qquad P_0 \in S_{10}^{-1} \text{ and } P_{10} \in S_{1/3,0}^{-2/3} \ .$$

As for $h_\pm$ we show that (1.9) holds satisfying (1.10), (1.11) and the following asymptotic form of amptitude function $P_\pm$.

$$(2.9) \qquad P_\pm = \mp i(\frac{\pi}{2})^{1/2} e^{\pm\pi i/4}((t-t')|\eta|)^{-1/2}\{1 + 0((t-t')|\eta|)^{-1/2}\} \ .$$

Since h_- is similar to h_+ we concentrate ourselves to h_+. Main part of culculus concerns with

$$(2.10) \qquad J = \int_{-|\eta|}^{|\eta|} \frac{e^{i(t-t')\sigma}}{(|\eta|^2 - \sigma^2)^{1/2}} \beta_1 \alpha_+(\eta,\sigma) d\sigma, \text{ for } \quad t - t' > 0 \ .$$

Now we change variable σ to s :

$$(2.11) \qquad s = ((t-t')|\eta|)^{1/2}\zeta = ((t-t')|\eta|)^{1/2}(1 - \frac{\sigma}{|\eta|})^{1/2} \ .$$

Then we have

$$(2.10)' \qquad J = 2e^{i\mu}\mu^{-1/2} \int_0^\nu e^{-is^2} \frac{1}{(1-\mu s^2)^{1/2}} \beta(\frac{s^2}{4\nu^2}) ds,$$

where $(\cdot - \cdot)^{1/2} \geqq 0$ in (2.10) and we have used notations

$$(2.11)' \qquad \mu = (t-t')|\eta| \ \text{ and } \ \nu = \frac{1}{2}(t-t')^{1/2}|\eta|^{1/6}.$$

Here we arrive at the following integrals

$$(2.12) \qquad I_1 = \int_0^\nu e^{is^2} \beta((\frac{s}{2\nu})^2) ds$$

which we can estimate using the second mean value theorem. We consider also

$$(2.13) \qquad I_2 = \int_0^\nu e^{is^2} \{2 - \frac{1}{(1-\mu s^2)^{1/2}}\} \beta((\frac{s}{2\nu})^2) ds.$$

Then it suffices to prove

(2.14) I_1 and I_2 belongs to $S^0_{2/3,0}$,

since the part of integral in (2.9)' is equal to $I_2 - I_1$. Moreover in the process of the proof of (2.12) we can obtain (2.9) if we use the following facts essentially depend on second mean value theore and integration by parts.

(2.15) $$F(\infty) = (\frac{\pi}{4})^{1/2}e^{\pi i/4}, F(\infty) - F(\nu) \in S^{-1/6}_{1,0} \quad \text{for} \quad F(r) = \int_0^r e^{is^2} ds$$

(2.16) $$\int_0^\nu e^{is^2} s^k ds \leq M\nu^k,$$

(2.17) $$\frac{\partial}{\partial \eta_j}\beta((\frac{s}{2\nu})^2) = \frac{\partial}{\partial s}\beta((\frac{s}{2\nu})^2)\cdot(-s)\frac{\eta_j}{|\eta|^2} \; j = 1, 2, \cdots, n.$$

REFERENCES

[1] L. Hörmander - The analysis of linear partial differencial operators III, Springer-Verlag, (1985).

[2] H. Kùmano-go - Pseudodifferential and Fourier integral operators with applications fundamental solutions to hyperbolic equations, Lecture note (1983).

[3] S. Miyatake - Mixed problems for hyperbolic equations of second order with first order complex boundary operators, J. Math. Kyoto Univ. (1975).

[4] S. Miyatake - Microlocal orders of singularities for distributions and an application to Fourier integral operators,J. Math. Kyoto Univ. 29, (1989).

[5] S. Miyatake - Microlocal orders of singularities for distributions and trace formulas of hyperbolic type, Lecture note in Trieste (1993).

[6] R. Sakamoto - Mixed problem for hyperbolic equation I and II, J. Math. Kyoto Univ. 8, (1970).

Nara Women's University

Kita-Uoya Nishimachi, Nara 630, Japa