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# Some globally stable approximations for the Navier-Stokes equations and for some other equations of viscous incompressible fluids

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EQUATIONS AUX DERIVEES PARTIELLES

# SOME GLOBALLY STABLE APPROXIMATIONS FOR THE NAVIER-STOKES EQUATIONS AND FOR SOME OTHER EQUATIONS OF VISCOUS INCOMPRESSIBLE FLUIDS

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We describe some approximations for the two-dimensional Navier-Stokes equations which are globaly stable and have the minimal global $B$-attractors ($= MIGBA_s$) in $\varepsilon$-vicinities of the $MIGBA$ for the investigated problem. In the end of the lecture we point out some other equations for which we have analogous results.

In all our theorems $\Omega$ is any bounded domain (b.d.) in $\mathbf{R}^2$.

## 1 The Galerkin-Faedo (G.-F.) approximations.

Consider the problem

$$(1_1) \qquad \partial_t v(t) - \nu \Delta v(t) + v(t).\nabla v(t) = -\nabla p(t) + f \ ,$$

$$(1_2) \qquad \text{div } v(t) = 0 \ , \ v(t)_{|\partial\Omega} = 0 \ , \ v(0) = \varphi \ ,$$

in a b.d. $\Omega \subset \mathbf{R}^2$ and $t \in \mathbf{R}^+ = [0, \infty)$. Here $v(t) : \Omega \to \mathbf{R}^2$ and $p(t) : \Omega \to \mathbf{R}^1$ are unknown functions, $v(t).\nabla v(t) = \sum_{k=1}^{2} v_k(t) \partial_{x_k} v(t)$ , $v_k(t)(k = 1, 2)$ are components of $v(t)$ , $\nu$ is a positive constant, $\varphi$ and $f$ are known functions independent on $t$, $\partial_t v$ and $\partial_{x_k} v$ are partial derivatives of $v$ in $t$ and $x_k$.

Let $H_0$ be the closure in the norm $\| \cdot \|$ of $L^2 \equiv L^2(\Omega, \mathbf{R}^2)$ of the set

$$\overset{\bullet}{J}{}^{\infty}(\Omega) = \{u | u \in C^{\infty}(\Omega, \mathbf{R}^2), \text{div } u = o \ , \text{supp } u \quad \text{is a compact in} \quad \Omega\} \ ,$$

$H_1$ be the closure of $\overset{\bullet}{J}{}^{\infty}(\Omega)$ in the norm $\| \cdot \|_1$ of Dirichlet integral, i.e. in the norm

$$\|u\|_1 = \left[ \int_\Omega \sum_{i,k=1}^{2} (\partial_{x_k} u_i(x))^2 dx \right]^{1/2} = \|\partial_x u\| \ ,$$

and $H_{-1}$ be the dual space to $H_1$ relative to $H_0$ with the standard norm $\| \cdot \|_{-1}$ . We denote by $(u, v)$ the inner product $u$ and $v$ in $H_0$.

It is known ([1]-[3]) that for any $f \in H_{-1}$ the solution operators $V_t : \varphi \to v(t, \varphi)$ for the problem $(1_k)$ exist in the whole $H_0$ and form the continuous bounded semi-group $\{V_t, t \in \mathbf{R}^+, H_0\}$. For it, the ball

$$B_R = \{u | u \in H_0, \|u\| \leq R\}, R > R_0^- = (\nu \sqrt{\lambda_1})^{-1} \|f\|_{-1},$$

is $B$-absorbing set and the intersection

$$\bigcap_{t \geq 0} V_t(B_R) \equiv \mathfrak{M}$$

is MIGBA. Here $-\lambda_1$ is the first eigenvalue of Stokes operator $\tilde{\Delta}$ with $\mathcal{D}(\tilde{\Delta}) \subset H_1([4], ch.2)$. $\mathfrak{M}$ is a bounded subset of $H_1$ and there is a majorant $\phi_1$ for

$$(2) \qquad \sup_{\varphi\in\mathfrak{M}} \sup_{t\in\mathbf{R}^1} \{\|\varphi\|_1 , \|\partial_t v(t,\varphi)\|, \int_t^{t+1} \|\partial^2_{x\tau} v(\tau,\varphi)\|^2 d\tau\} \leq \phi_1(\|f\|_{-1}, \nu^{-1}) .$$

These facts and some other properties of $\mathfrak{M}$ (for $f \in H_0$) were proved by us in [1] (see also [2],[3]). But the method of prooving the basic estimates given in [1] required a smoothness of $\partial\Omega$ and could be applied only to the Rothe approximations and to the G.-F. approximations with the eigenfunctions $\{\varphi_k\}_{k=1}^\infty$ of $\tilde{\Delta}$ as the coordinate functions in $H_1$. In [5],[6] we have given an other method of estimating the solutions to the Navier-Stokes equations which can be applied directly to the G.-F. approximations with the arbitrary coordinate $\{\psi_k\}_{k=1}^\infty$ in $H_1$.

Denote by $v^m(t,\varphi) = \sum_{k=1}^m c_k^m(t,\varphi)\psi_k$ the G.-F. approximations and by $V_t^m : \varphi \to v^m(t,\varphi)$ the solution operators for the G.-F. equations. The family $\{V_t^m, t \in \mathbf{R}^+, H_0^m\}$ for each $m = 1, 2, \cdots$, is a continuous semi-group. Here $H_0^m =$ span $\{\psi_1, \cdots, \psi_m\}$ is considered as a subspace of $H_0$. The following facts are true :

**Theorem 1.**— *Let $\Omega$ be a b.d. in $\mathbf{R}^2$ and $f \in H_{-1}$. The Galerkin-Faedo approximations with arbitrary coordinate functions in $H_1$ have MIGBAs $\mathfrak{M}^m (m = 1, 2, \cdots)$ lying in $H_0^m$ and having the properties :*

$$(3) \qquad \sup_{\varphi\in\mathfrak{M}^m} \sup_{t\in\mathbf{R}^1} \{\|\varphi\|_1, \|\partial_t v^m(t,\varphi)\|, \int_t^{t+1} \|\partial^2_{x\tau} v^m(\tau,\varphi)\|^2 d\tau\} \leq \phi_1(\|f\|_{-1}, \nu^{-1})$$

*with the same $\phi_1$ as in (2). For any $\varepsilon > 0$ exists a number $m(\varepsilon) \in \mathbf{N}^+$ such that*

$$(4) \qquad \mathfrak{M}^m \subset 0_\varepsilon(\mathfrak{M}) \quad \textit{for} \quad m \geq m(\varepsilon) .$$

*Here $0_\varepsilon(\mathfrak{M})$ is the $\varepsilon$-vicinity of $\mathfrak{M}$ in $H_0$* ∎

It is useful to bear in mind the following known fact :

**Lemma 1.**— *If $\{\varphi_k\}_{k=1}^\infty$ is a coordinate system in $\overset{\circ}{W}{}_2^2(\Omega)$ then $\{\psi_k = \nabla^\perp\varphi_k \equiv (-\partial_{x_2}\varphi_k, \partial_{x_1}\varphi_k)\}_{k=1}^\infty$ is the coordinate system in $H_1$. The inverse statement is also true : if $\{\psi_k\}_{k=1}^\infty$ is a coordinate system in $H_1$, then each $\psi_k$ determines a function $\varphi_k \in \overset{\circ}{W}{}_2^2(\Omega)$ and $\{\varphi_k\}_{k=1}^\infty$ is a coordinate system in $\overset{\circ}{W}{}_2^2(\Omega)$* ∎

The proof of Theorem 1 is based on some a priori estimates for $v^m$. They are the same as for solutions $v(t)$ of problem $(1_k)$ proved in [5] (see also [6]). These estimates are derived only from the inequalities :

$$(5_1) \qquad \frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \nu\|\partial_x v(t)\|^2 = (f, v(t)) \leq \|f\|_{-1}\|\partial_x v(t)\| ,$$

$$(5_2) \qquad \|\partial_t v(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\partial_x v(t)\|^2 = -(v(t).\nabla v(t), \partial_t v(t)) +$$

$$+(f, \partial_t v(t)) \leq \|v(t)\|_{4,\Omega}\|\partial_x v(t)\| \ \|\partial_t v(t)\|_{4,\Omega} + \|f\|_{-1}\|\partial^2_{xt} v(t)\| \ ,$$

$$(5_3) \qquad \frac{1}{2}\frac{d}{dt}\|\partial_t v(t)\|^2 + \nu\|\partial^2_{xt} v(t)\| = -(\partial_t v(t).\nabla v(t) + v(t).\nabla \partial_t v(t),$$

$$\partial_t v(t)) = -(\partial_t v(t).\nabla v(t), \partial_t v(t)) \leq \|\partial_t v(t)\|^2_{4,\Omega}\|\partial_x v(t)\| \ .$$

Here $\|\cdot\|_{4,\Omega}$ is the standard norm in $L^4(\Omega; \mathbf{R}^2)$.

The assertion (4) can be proved by *reductio ad absurdum.* Suppose that (4) is not true. Then there is an $\varepsilon > 0$ and a sequence $a_{m_j} \in \mathfrak{M}^{m_j}, m_j \to \infty$, such that

$$(4') \qquad \text{dist}\ \{a_{m_j}, \mathfrak{M}\} > \varepsilon \ .$$

Due to (3) the set $\bigcup_{m=1}^{\infty} \mathfrak{M}^m \cup \mathfrak{M}$ lies in the ball $B_{\phi_1}(H_1)$ of $H_1$ (with radius $\phi_1 = \phi_1(\|f\|_{-1}, \nu^{-1})$), is a precompact in $H_0$ and lies in a ball $B_{R_1}(H_0)$ of the space $H_0$. In particular, $a_{m_j} \in B_{R_1}(H_0) \cap B_{\phi_1}(H_1)$. Choose $T \in \mathbf{R}^+$ such that

$$(4'') \qquad V_T(B_{R_1}(H_0)) \subset 0_{\varepsilon/2}(\mathfrak{M}).$$

Each $a_{m_j}$ determines a $\varphi_{m_j} \in \mathfrak{M}^{m_j}$ for which $V_T^{m_j}(\varphi_{m_j}) = a_{m_j}$. The set $\{\varphi_{m_j}\}$ also belongs to $B_{R_1}(H_0) \cap B_{\phi_1}(H_1)$. Therefore we can choose a subsequence $\{\varphi_{m'_j}\}$ converging to a $\varphi$ in the space $H_0$ and $\varphi \in B_{R_1}(H_0) \cap B_{\phi_1}(H_1)$. We have for $\{V_t^{m'_j}(\varphi_{m'_j})\}$ the estimates (3). They permit to extract a subsequence $m''_j$ for which $V_t^{m''_j}(\varphi_{m''_j})$ converge to a $v(t)$ uniformly in $t \in [0, T]$ in the norm of $H_0$. In particular,

$$(4''') \qquad \|V_T^{m''_j}(\varphi_{m''_j}) - v(T)\| < \frac{\varepsilon}{2} \quad \text{for} \quad m''_j \geq m_0 \ .$$

Following standard arguments we prove that $v(t)$ is the solution $V_t(\varphi)$ of the problem $(1_k)$ with $\varphi \in B_{R_1}(H_0)$. Due to (4″) and (4‴) dist $\{a_{m''_j}, \mathfrak{M}\} < \varepsilon$, but this contradicts to the hypothesis (4').

**Remark :** The MIGBAs which we have in this lecture are invariant compact connected sets in the phase spaces choosed by us. They have all properties of $\mathfrak{M}$ for the problem $(1_k)$ proved in [1]-[3]. We can give for them common majorants for the number of determining modes and for their fractal dimensions.

## 2 A discretization of $t$.

For the study and computations of attractors some discretizations of $t$ can be useful. One of them for the F.-G. approximations has the form

$$(6)\qquad (v_{\bar t}^m(t),\psi_k)+\nu(\nabla v^m(t),\nabla\psi_k)+(v^m(t-\tau).\nabla v^m(t),\psi_k)=$$

$$=(f,\psi_k)\ ,\ k=1,2,\cdots m\ ,\ v^m(0)=\varphi^m,$$

for $t=\ell\varepsilon,\ell\in\mathrm{N}^+,\tau=\mathrm{const}>0\ ;\ v_{\bar t}^m(t)=\tau^{-1}[v^m(t)-v^m(t-\tau)]$. The systems (6) determine successively in $t=\tau,2\tau,\cdots$, the velocity fields $v^m(t,\varphi^m,\tau)$. The solution operators $V_t^{m.\tau}:\varphi\to v^m(t,\varphi,\tau)$ form the discrete semi-group $\{V_{\ell\tau}^{m,\tau},\ell\in\mathrm{N}^+,H_0^m\}$. It has MIGBA $\mathfrak{M}^{m,\tau}$. Let $\tau=\tau_k\to 0$ when $k\to\infty$.

**Theorem 2.—** *Let the conditions of Theorem 1 be fulfiled and $\mathfrak{M}^m$ be attractors from Theorem 1. For any $\delta>0$ exists a number $n(\delta,m)\in\mathrm{N}^+$ such that*

$$\mathfrak{M}^{m,\tau_k}\subset 0_\delta(\mathfrak{M}^m)\quad\text{for}\quad k\geq n(\delta,m)\ .$$

*Here $0_\delta(\mathfrak{M}^m)$ is $\delta$-vicinity of $\mathfrak{M}^m$ in $H_0^m$ and $\tau_k\to 0$. There is a common majorant for all $\mathfrak{M}^{m,\tau_k}$ in $H_1$, i.e.*

$$\sup_{\varphi\in\mathfrak{M}^{m,\tau_k}}\|\varphi\|_1\leq\phi_2(\|f\|_{-1},\nu^{-1},\tau_0)\ \blacksquare$$

The proof of Theorem 2 is based on a priori estimates for $v^m(t),t=\ell\tau,\ell=1,2,\cdots$, which we derived from the following relations :

$$(7_1)\qquad \|v^m(t)\|^2-\|v^m(t-\tau)\|^2+\|v^m(t)-v^m(t-\tau)\|^2+2\tau\nu\|\partial_x v^m(t)\|^2=$$

$$=2\tau(f,v^m(t))\leq 2\tau\|f\|_{-1}\|\partial_x v^m(t)\|\ ,$$

$$2\tau\|v_{\bar t}^m(t)\|^2+\nu\|\partial_x v^m(t)\|^2-\nu\|\partial_x v^m(t-\tau)\|^2+\nu\|\partial_x v^m(t)-$$

$$(7_2)\qquad -\partial_x v^m(t-\tau)\|^2=-2\tau(v^m(t-\tau).\nabla v^m(t),v_{\bar t}^m(t))+2\tau(f,v_{\bar t}^m(t))\leq$$

$$\leq 2\tau\|v^m(t-\tau)\|_{4,\Omega}\|\partial_x v^m(t)\|\ \|v_{\bar t}^m(t)\|_{4,\Omega}+2\tau\|f\|_{-1}\|\partial_x v_{\bar t}^m(t)\|,$$

and

$$\|v_{\bar t}^m(t)\|^2-\|v_{\bar t}^m(t-\tau)\|^2+\|v_{\bar t}^m(t)-v_{\bar t}^m(t-\tau)\|^2+2\tau\nu\|\partial_x v_{\bar t}^m(t)\|^2=$$

$$(7_3)\qquad =-2\tau(v^m(t-2\tau).\nabla v_{\bar t}^m(t)+v_{\bar t}^m(t-\tau).\nabla v^m(t),v_{\bar t}^m(t))=$$

$$=-2\tau(v_{\bar t}^m(t-\tau).\nabla v^m(t),v_{\bar t}^m(t))\leq 2\tau\|v_{\bar t}^m(t-\tau)\|_{4,\Omega}\ .$$

$$\|\partial_x v^m(t)\|\ \|v_{\bar t}^m(t)\|_{4,\Omega}\ ,\ t=\ell\tau\ .$$

They are corollaries of (6) and they are difference analogues of the relations $(5_k)$.

## 3 An $\varepsilon$-approximation

Consider the $\varepsilon$-approximation

$$(8_1) \qquad \partial_t v - \nu\Delta v - \varepsilon^{-1}\nabla \text{div } v + v.\nabla v + \frac{1}{2} v \text{div } v = f \ ,$$

$$(8_2) \qquad v_{|\partial\Omega} = 0 \ , \ v_{|t=0} = \varphi \ , \ \varepsilon \in (0,1] \ ,$$

of the problem $(1_k)$. The following statement holds for $(8_k)$ :

**Theorem 3.—** *Let $\Omega$ be a b.d. in $\mathbf{R}^2$ and $f \in \overset{\circ}{W}{}_2^{-1}(\Omega)$. The solution operators $V_t^\varepsilon : \varphi \to v^\varepsilon(t,\varphi)$ form the continuous semi-group $\{V_t^\varepsilon, t \in \mathbf{R}^+, L^2(\Omega, \mathbf{R}^2)\}$. It belongs to the class 1 and has a compact MIGBA $\mathfrak{M}^\varepsilon$. There is a common majorant $\phi_z$ for*

$$\sup_{\varepsilon\in(0,1]} \quad \sup_{\varphi\in\mathfrak{M}^\varepsilon} \sup_{t\in\mathbf{R}^1} \{\|\varphi\|_1, \varepsilon^{-1/2}\|\text{div } \varphi\|, \|\partial_t v^\varepsilon(t,\varphi)\|,$$

$$\int_t^{t+1} (\varepsilon^{-1}\|\text{div}\partial_\tau v^\varepsilon(\tau,\varphi)\|^2 + \|\partial_{x\tau}^2 v^\varepsilon(\tau,\varphi)\|^2) d\tau\} \le \phi_3(\|f\|_{-1}, \nu^{-1}) \ .$$

*For any $\delta > 0$ there is a $\varepsilon(\delta) > 0$ such that*

$$\mathfrak{M}^\varepsilon \subset 0_\delta(\mathfrak{M}) \quad \text{for} \quad \varepsilon \in (0, \varepsilon(\delta)] \ .$$

*Here $0_\delta(\mathfrak{M})$ is $\delta$-vicinity of $\mathfrak{M}$ in the space $L^2(\Omega,\mathbf{R}^2)$. For the solving of $(8_k), k = 1,2,$ can be used the approximations of n.n.1 and 2 with arbitrary coordinates $\{\psi_k\}_{k=1}^\infty$ in the space $\overset{\circ}{W}{}_2^1|\Omega)$.* ■

Write down the relations from which we derive a priori estimates for the solutions of $(8_k)$ :

$$(9_1) \qquad \frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \nu\|\partial_x v(t)\|^2 + \varepsilon^{-1}\|\text{div } v(t)\|^2 =$$

$$= (f, v(t)) \le \|f\|_{-1}\|\partial_x v(t)\|,$$

$$(9_2) \qquad \|\partial_t v(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\partial_x v(t)\|^2 + \frac{1}{2\varepsilon}\frac{d}{dt}\|\text{div } v(t)\|^2 =$$

$$= -(v(t).\nabla v(t), \partial_t v(t)) - \frac{1}{2}(v(t)\text{div}(t), \partial_t v(t)) + (f, \partial_t v(t)) \le$$

$$\le \|v(t)\|_{4,\Omega}\|\partial_x v(t)\|\|\partial_t v(t)\|_{4,\Omega} + \frac{1}{2}\|v(t)\|_{4,\Omega}\|\text{div } v(t)\|\|\partial_t v(t)\|_{4,\Omega} +$$

$$+\|f\|_{-1}\|\partial^2_{xt}v(t)\| \ ,$$

$$(9_3) \qquad \frac{1}{2}\frac{d}{dt}\|\partial_t v(t)\|^2 + \nu\|\partial^2_{xt}v(t)\|^2 + \varepsilon^{-1}\|\mathrm{div}\ \partial_t v(t)\|^2 =$$

$$= -(\partial_t v(t).\nabla v(t) + \frac{1}{2}v(t)\mathrm{div}\ \partial_t v(t), \partial_t v(t)) - (v(t).\nabla\partial_t v(t) +$$

$$+\frac{1}{2}\partial_t v(t)\mathrm{div}\ v(t), \partial_t v(t)) = -(\partial_t v(t).\nabla v(t) + \frac{1}{2}v(t)\mathrm{div}\ \partial_t v(t),$$

$$\partial_t v(t)) \le \|\partial_t v(t)\|^2_{4,\Omega}\|\partial_x v(t)\| + \frac{1}{2}\|v(t)\|_{4,\Omega} \cdot$$

$$\|\mathrm{div}\ \partial_t v(t)\| \ \|\partial_t v(t)\|_{4,\Omega}$$

## 4 Difference schemes

We have consider difference schemes suggested by us in 50th and 60th (see, for example, [7], [8], [4]) and have found that some of them are globaly stable and have MIGBAs lying near $\mathfrak{M}$. Let us discribe here on of them. Take $h \in (0, h_0]$ and $\tau \in (0, \tau_0]$, with some $h_0$ and $\tau_0$, and the mesh $\mathbf{R}^2_h \ : \ x = (kh) = (k_1 h, k_2 h), (k_1, k_2) \in \mathbf{N} \times \mathbf{N}$ in $\mathbf{R}^2$. Let $\omega_{kh} = \{x = (x_1, x_2) \in \mathbf{R}^2 | x_j \in (k_j h, (k_j + 1)h), j = 1, 2\}$ ; $\overline{\Omega}_h$ - the set $\bigcup_{\omega_{kh}\subset\Omega} \overline{\omega}_{kh} \subset \mathbf{R}^2$ ; $\mathbf{S}_h = \partial\overline{\Omega}_h$ and $\Omega_h = \overline{\Omega}_h \setminus \mathbf{S}_h \subset \mathbf{R}^2$. We shall use the same notations $\overline{\Omega}_h, \mathbf{S}_h$ and $\Omega_h$ for the sets of points $x = (kh) \in \mathbf{R}^2_h$ belonging to $\overline{\Omega}_h, \mathbf{S}_h$ and $\Omega_h$ corresponding by. Introduce also notations :

$$v_{x_i}(x,t) = h^{-1}[v(x + he^i, t) - v(x,t)], v_{\bar{x}_i}(x,t) = h^{-1}[v(x,t) - v(x - he^i, t)] \ ,$$

$$v_{\bar{t}}(x,t) = \tau^{-1}[v(x,t) - v(x, t-\tau)], \overset{\pm i}{v}(x,t) = v(x \pm he^i, t), \overset{\pm 0}{v}(x,t) = v(x, t \pm \tau),$$

where $e^i$ is the ort along the axis $x_i$. Take the following difference scheme (see [4], p. 238):

$$(10_1) \qquad v_{i\bar{t}} - \nu v_{ix_k\bar{x}_k} + \frac{1}{2}\overset{-0,+k}{v_k}\, v_{ix_k} + \frac{1}{2}\overset{-0}{v_k}\, v_{i\bar{x}_k} = -p_{\bar{x}_i} + f^h_i, i = 1, 2,$$

$$(10_2) \qquad v_{kx_k} = 0,$$

$$(10_3) \qquad v|_{\mathbf{S}_h} = 0,$$

$$(10_4) \qquad v|_{t=0} = \varphi^h.$$

the equations $(10_1)$ have to be fulfilled in the points $(x,t)$ with $t = \ell\tau,\ \ell = 1,2,\cdots$, and $x \in \Omega_h$ ; the equations $(10_2)$- in $(x,t)$ with $t = \ell\tau,\ \ell = 1,2,\cdots$, and $x = \Omega'_h \equiv \Omega_h \cup S'_h$, where $S'_h$ is a part of $S_h$ which we get replacing the points of $\Omega_h$ by vectors $-he^i,\ i = 1,2$ ; the equations $(10_3)$ - in $(x,t)$ with $t = \ell\tau,\ \ell = 1,2,\cdots$ and $x \in S_h$, the equations $(10_4)$ - in $(x,0)$ with $x \in \bar{\Omega}_h$. In $(10_1)$ $f^h$ is a mesh-function on $\Omega_h$ and $\varphi^h$ is a mesh-function on $\bar{\Omega}_h$ satisfying the equations

$$\varphi^h|_{S_h} = 0 \text{ and } \varphi^h_{kx_k} = 0 \text{ in } x \in \Omega'_h.$$

We add to $(10_k)$ the equations

$$(10_5) \qquad \sum_{\Omega'_k} p = 0 \text{ for } t = \ell\tau,\ \ell = 1,2,\cdots$$

It was proved in [8] (see also [4]) that the system $(10_k)$, $k = 1,\cdots,5$, is uniquely solvable and its solutions $v^{h,\tau}$ converge when $h = \mu\tau \to 0$ ($\mu$ is a fixed positive number) to the solution $v$ of the problem $(1_k)$ on the finite intervals $[0,T]$ of $t$-axis if $f,\varphi$ and $\partial\Omega$ are smooth enough and $f^h,\varphi^h$ approximate $f$ and $\varphi$ in a properly way. Now we have proved that the scheme $(10_k)$ is globally stable. More precisely : introduce the linear set of mesh-functions $u^h$ : $H^h = \{u^h|_{\bar{\Omega}_h} | u^h|_{S_h} = 0,\ u^h_{kx_k} = 0$ in the poins of $\Omega'_h\}$ and consider $H^h$ as Euclidian space with the norm

$$(11) \qquad ||u^h||_{\Omega_h} \equiv (h^2 \sum_{\bar{\Omega}_h} (u^h)^2)^{1/2}.$$

For a fixed $f^h \in H^h$ the solution operators $V_t^{h,\tau}$ : $\varphi^h \to v^{h,\tau}(t,\varphi^h),\ t = \ell\tau,\ \ell \in \mathbf{N}^+$, form a discrete semi-group in $H^h$. It has MIGBA $\mathfrak{M}^{h,\tau}$ and there is a common majorant $\phi_4$ for all $\mathfrak{M}^{h,\tau}$ with $h = \mu\tau,\ \tau \in (0,\tau_0]$ :

$$\sup_{\varphi^h \in \mathfrak{M}^{h,\tau}} \sup_{\ell \in \mathbf{N}} \{||\varphi^h_x||_{\Omega_h},\ ||v_{\bar{t}}^{h,\tau}(\ell\tau,\varphi^h)||_{\Omega_h}\} \le \phi_4(||f^h||_{\Omega_h}, \nu^{-1}, \mu, \tau_0)$$

Denote by $\tilde{u}^h$ the piece-wise constant interpolation of $u^h \in H^h$, i.e. $\tilde{u}^h \in L^2(\Omega, \mathbf{R}^2), \tilde{u}^h(x) = u^h(kh)$ for $x \in \omega_{kh}, \subset \Omega \tilde{u}^h(x) = 0$ for $x \in \Omega \setminus \Omega_h$. Let, for example, $\tau = \tau_k = \tau_0 2^{-k}, k = 0,1,2,\cdots, h = h_k = \mu\tau_k, \mathfrak{M}^{h_k,\tau_k} \equiv \mathfrak{M}^k_\mu$ and $\widetilde{\mathfrak{M}}^k_\mu$ - the set of $\tilde{\varphi}^{h_k}$ for all $\varphi^{h_k} \in \mathfrak{M}^k_\mu$. The following statement is true :

**Theorem 4.**— *Let $\Omega$ be a.b.d. in $\mathbf{R}^2$ and $f \in L^2(\Omega\ ;\ \mathbf{R}^2)$. For any $\varepsilon > 0$ exists a number $n = n(\varepsilon,\mu)$ such that*

$$\tilde{\mathfrak{M}}^k_\mu \subset 0_\varepsilon(\mathfrak{M}) \text{ for } k \ge n(\varepsilon,\mu).$$

*Here $0_\varepsilon(\mathfrak{M})$ is $\varepsilon$-vicinity of $\mathfrak{M}$ in the space $L^2(\Omega, \mathbf{R}^2)$* ∎.

Analogous results hold for the systems of $\mathcal{ODE}$ :

$$(11_1) \qquad \partial_t v_i(t) - \nu v_{ix_k\bar{x}_k}(t) + \frac{1^{+k}}{2} v_k(t) v_{ix_k}(t) + \frac{1}{2} v_k(t) v_{i\bar{x}_k}(t) = -p_{\bar{x}_i}(t) + f_i, \; i = 1,2,$$

$$(11_2) \qquad v_{kx_k}(t) = 0, \; v(t)|_{S_h} = 0, \; v(0) = \varphi^h, \; t \in \mathbf{R}^+.$$

The basic relations which we use for the solutions of problem $(10_k)$ resemble the relations $(7_k)$ and for the solutions of problem $(11_k)$- the relations $(5_k)$.

The results analogous to results described above are true for :

1) the Navier-Stokes equations with the periodic or non homogeneous boundary conditions : $v|_{\partial\Omega} = a|_{\partial\Omega}$ with $a(x) =$ rot $b(x)$, $b \in W_2^2(\Omega)$, if $\partial\Omega$ is a piece-smooth curve.

2) the termo-convection and magneto-hydrodynamical systems for viscous incompressible fluids in $b.d.\Omega \subset \mathbf{R}^2$.

3) the modifications of the three dimensional Navier-Stokes equations in $b.d.$ $\Omega \subset \mathbf{R}^3$ which were suggested by us in [9] [10] (see also addendum in [4], second russian edition).

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