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Thomas C. Sideris

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F-91128 PALAISEAU Cedex (FRANCE)
Tél. (1) 69 33 40 91
Fax (1) 69 33 30 19 ; Télex 601.596 F

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EQUATIONS AUX DERIVEES PARTIELLES

THE LIFESPAN OF 3D COMPRESSIBLE FLOW

Thomas C. SIDERIS

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The Lifespan of 3D Compressible Flow

Thomas C. Sideris
Department of Mathematics
University of California
Santa Barbara, CA 93106
USA

Outline:

1. Existence. Lower bounds for the lifespan.
2. Formation of singularities. Upper bound on the lifespan.
3. The incompressible limit.
4. Relationship between the lifespan of compressible and incompressible flow.

Existence

The compressible Euler equations are given by

$$\begin{aligned} \partial_t \, \rho + \nabla \cdot \rho \, u &= 0 \\ \rho \, (\partial_t \, u + u \cdot \nabla u) + \nabla p &= 0 \\ p = \rho^\gamma \quad \gamma &> 1 \end{aligned}$$

where $\rho > 0$ is the density, $u \in \mathbb{R}^3$ is the velocity, and p is the pressure of an ideal, compressible, isentropic gas or fluid.

Consider the initial value problem for initial values which are a small perturbation of a constant state

$$\rho(0, x) = \bar{\rho} + \varepsilon \rho_0(x), \quad u(0, x) = \varepsilon u_0(x)$$

with

$$\bar{\rho} = const. > 0 \quad \rho_0, u_0 \in \mathcal{S}(\mathbb{R}^3).$$

The dependence on ε of the corresponding solution will be suppressed. The *lifespan* of a solution to this initial value problem is the largest time T_ε for which the solution exists and is C^1 on the strip $[0, T_\varepsilon) \times \mathbb{R}^3$.

Local existence of a regular solution follows since the equations can be rewritten in symmetric hyperbolic form, [5]. Define the new variables,

$$\xi(t,x) = \frac{2}{\gamma - 1}\left[\left(\frac{\rho(t/\bar{c}, x)}{\bar{\rho}}\right)^{\frac{\gamma-1}{2}} - 1\right] \quad \text{and} \quad v(t,x) = \frac{u(t/\bar{c}, x)}{\bar{c}},$$

where $\bar{c} = \sqrt{\gamma}\bar{\rho}^{\frac{\gamma-1}{2}}$ is the sound speed. The new equations are

$$\partial_t\, \xi + \nabla \cdot v + v \cdot \nabla\xi + \tfrac{\gamma-1}{2}\, \xi\, \nabla \cdot v \;=\; 0 \tag{1}$$

$$\partial_t\, v + \nabla\, \xi + v \cdot \nabla v + \tfrac{\gamma-1}{2}\, \xi\, \nabla\xi \;=\; 0, \tag{2}$$

with initial conditions of order ε

$$\xi(0,x) = \varepsilon\, \xi_0^\varepsilon(x), \quad v(0,x) = \varepsilon\, v_0^\varepsilon(x). \tag{3}$$

ξ_0^ε , $v_0^\varepsilon \in \mathcal{S}(\mathbb{R}^3)$ are uniformly bounded, and the sound speed is now equal to unity.

Letting

$$w = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \xi \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The system has the form

$$P(\partial)w \equiv \left[\mathbf{I}\, \partial_t + \sum_{j=1}^{3} B_j\, \partial_j\right] w = F(w),$$

where the B's are symmetric constant matrices and the nonlinearity $F(w)$ is quadratic.

The operator $P(\partial)$ is called the linear acoustical operator. It is invariant under translations, rotations, and changes of scale. The translations are generated by the usual partial differential operators:

$$\partial_0 = \partial_t, \quad \partial_j, \quad j = 1, 2, 3,$$

The spatial rotations are generated by:

$$\Omega_j = \omega_j \, \mathbf{I} + U_j, \quad j = 1, 2, 3,$$

with

$$\omega_1 = x_2\partial_3 - x_3\partial_3, \quad \omega_2 = x_3\partial_1 - x_1\partial_3, \quad \omega_3 = x_1\partial_2 - x_2\partial_1,$$

and

$$U_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The appearance of the matrices U is due to the fact that the dependent and independent variables transform simultaneously under rotations. Changes in scale are generated by the operator:

$$S = t\partial_t + \sum_{j=1}^{3} x_j\partial_j.$$

The invariance of $P(\partial)$ under these families of transformations leads to the following commutation relations:

$$[P(\partial), \partial_\mu] = 0, \quad [P(\partial), \Omega_j] = 0, \quad [P(\partial), S] = P(\partial).$$

Denote this collection of operators by

$$\mathbf{G} = \{\partial_\mu, \Omega_j, S\} = \{G_i\}_{i=0}^{7},$$

and define the *generalized energy norms* by

$$E_k[w(t)] = \sum_{|\alpha|\leq k} \|G^\alpha w(t,\cdot)\|_{L^2}.$$

Theorem 1: *(Local existence, [6].) There exists a solution*

$$w(t,x) \in C^1([0,T_\varepsilon) \times \mathbb{R}^3)$$

where

$$T_\varepsilon > C/\varepsilon.$$

For every $k \geq 4$, the energy satisfies the uniform bound

$$E_k(t) \leq 2E_k(0) = O(\varepsilon),$$

for $0 \leq t \leq C/\varepsilon$.

Moreover, if the inital data are compactly supported, then

$$supp\; w(t,\cdot) \subset \{x \in \mathbb{R}^3 : |x| \leq R + t\}$$

where R is the radius of the support of the initial data.

This result is proved by deriving a standard energy estimate for the derivatives $G^\alpha w$, and applying the classical Sobolev inequality.

The lower bound on the lifespan can be improved by considering irrotational initial velocity. To get this improvement, perform energy estimates with the generators of the Lorentz group, i.e. add the Lorentz rotations to **G**.

$$\mathbf{\Gamma} = \mathbf{G} \cup \{L_j\} = \{\Gamma_i\}_{i=1}^{10} \quad L_j = (t\partial_j + x_j\partial_t)\mathbf{I} - B_j,$$

where the matrices B are those appearing in the definition of $P(\partial)$.

The Euler equations are not invariant under the Lorentz rotations, and therefore, the L_j do not commute with $P(\partial)$. Nevertheless, $[P(\partial), L_j]w$ is a vector which contains only a component of $\nabla \times u$, the vorticity. It measures the amount by which the L_j fail to commute with $P(\partial)$. It is typical that for quadratically nonlinear Lorentz invariant hyperbolic equations in three space dimensions that local solutions are *almost global*, [2]. Irrotational flow means that the vorticity vanishes for all time, thus one expects a stronger local existence result in this case since the L_j now commute with the linear operator $P(\partial)$.

This result [7] is obtained through the use of another energy norm:

$$\hat{E}_k(t) = \sum_{|\alpha| \leq k} \|\Gamma^\alpha w(t, \cdot)\|_{L^2}.$$

Theorem 2: *If* $\nabla \times v_0 = 0$, *then the lifespan has the lower bound* $T_\varepsilon > \exp(C/\varepsilon) - 1$. *The energy has the uniform stability estimate*

$$\hat{E}_k(t) \leq 2\hat{E}_k(0) = O(\varepsilon)$$

for $0 \leq t \leq \exp(C/\varepsilon) - 1$, $k \geq 4$.

The key is to obtain the energy estimate:

$$\hat{E}_k(t) \leq \hat{E}_k(0) \exp\left(C \sum_{|\alpha| \leq 2} \int_0^t \|\Gamma^\alpha w(s, \cdot)\|_{L^\infty} ds \right).$$

Then apply the following Sobolev–type inequality.

Lemma 1: (Klainerman [3]) *Let* $\phi(t,x)$ *be an arbitrary smooth function. Then for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$,

$$|\phi(t,x)| \leq \frac{C}{(1+t)} \sum_{|\alpha| \leq 2} \|\Gamma^\alpha \phi(t, \cdot)\|_{L^2}$$

where the constant C *is independent of* ϕ.

It is the decay provided by this inequality, not present in the usual Sobolev inequality, which leads to almost global exstence. Its use is possible due to the inclusion of all the generators Γ into the energy norm.

Think of the flow as small amplitude waves superimposed on an underlying incompressible flow. If irrotational, then the incompressible part is zero, hence the enhanced lifespan. Practically nothing is known for incompressible flow in three dimensions. Such flows certainly do not decay in time (based on what is known in 2D). Numerical evidence show that they grow rapidly [1], and they are suspected to blow up in finite time. In general, the incompressible flow interacts with the acoustical waves. The acoustical waves radiate to infinity, and at far distances they do not interact with the incompressible flow, as the following result illustrates, [7].

Theorem 3: *Consider initial velocity of the form*

$$v_0 = v_{01} + v_{02}$$

with

$$\nabla \times v_{01} = 0 \qquad \text{and} \qquad \text{supp}\, v_{02} \subset \{|x| \leq R\}.$$

Then, on the set

$$\{|x| \geq R + C_1[\varepsilon \log(1+t)]^{2/3} + t\}$$

the solution w with data $\varepsilon\xi_0, \varepsilon v_0$ (with lifespan $\geq C/\varepsilon$) agrees with the irrotational solution w_1 with data $\varepsilon\xi_0, \varepsilon v_{01}$ (and with lifespan $T_\varepsilon \geq \exp(C/\varepsilon) - 1$). Thus w can be extended for large times in an exterior domain.

Formation of singularities

The following result [6] shows that the almost global existence result is sharp.

Theorem 4: *Suppose that the initial data* $w_0 = (\varepsilon\xi_0, \varepsilon v_0)$ *is supported in* $\{|x| \leq R\}$ *and satisfies* $\xi_0(x) > 0, x \cdot v_0(x) > 0$. *on some annulus* $\{R_1 < |x| < R\}$. *Then there is fixed constant* $C_0 > 0$ *such that the* C^1 *local solution* w *cannot be extended to the region*

$$\{(t,x) \in \mathbb{R}^4 : R_1 + t \leq |x| \leq R + t, 0 \leq t \leq T\}$$

for $T > \exp(C_0/\varepsilon^2)$. *In particular, the solution does not exist past time* $T = \exp(C_0/\varepsilon^2)$.

This is proved by obtaining a differential inequality for the function

$$F(t) = \int_0^t (t-\tau) \int_{R_1+t}^{R+t} r^{-1} \int_{|x|>r} |x|^{-1}(|x| - r)^2 \xi(t,x)\,dx\,dr\,d\tau$$

of the form

$$F''(t) \geq C_1[(R+t)^3 \log(R+t)]^{-1} F^2(t), \quad C_1 > 0,$$

together with the lower bound

$$F(t) \geq C_2(R+t)\log(R+t), \quad C_2 > 0.$$

The particular form of the function $F(t)$ is not important for this exposition except to illustrate that the analysis is performed near the front of the disturbance. Thus, singularities do actually appear at the front (after a long time) even if they may have appeared at a much earlier time far behind the front.

The Incompressible Limit

Let w be a solution of (1),(2),(3) with (by theorem 1)

$$E_4(t) \leq 2E_4(0) = O(\varepsilon) \quad 0 \leq t \leq T_0/\varepsilon.$$

If w is rescaled according to

$$\hat{\xi}(t) = \xi(t/\varepsilon), \quad \hat{v}(t) = \varepsilon^{-1} v(t/\varepsilon), \quad t \leq T_0.$$

then $\hat{w} = (\hat{\xi}, \hat{v})$ is a C^1 solution of:

$$\partial_t \hat{\xi} + \nabla \cdot \hat{v} + \hat{v} \cdot \nabla \hat{\xi} + \tfrac{\gamma-1}{2} \hat{\xi} \nabla \cdot \hat{v} = 0$$
$$\partial_t \hat{v} + \varepsilon^{-2} \nabla \hat{\xi} + \hat{v} \cdot \nabla \hat{v} + \varepsilon^{-2} \tfrac{\gamma-1}{2} \hat{\xi} \nabla \hat{\xi} = 0,$$

defined for $t \leq T_0$. The sound speed is now proportional to ε^{-2}, and the initial data is

$$\hat{\xi}(0, x) = \varepsilon\, \xi_0^\varepsilon(x), \quad \hat{v}(0, x) = v_0^\varepsilon(x).$$

The uniform bound for $\|w(t, \cdot)\|_{H^4}$ yields

$$\|\hat{\xi}(t)\|_{H^4} = O(\varepsilon) \quad \|\hat{v}(t)\|_{H^4} = O(1), \quad t \leq T_0.$$

This bound was obtained in [4] working directly with the equations for $\hat{w}$, and it can be used to show that if the initial data is fixed and incompressible, i.e.

$$\hat{\xi}_0^\varepsilon = 0 \quad \text{and} \quad \nabla \cdot v_0^\varepsilon = \nabla \cdot v_0 = 0,$$

then as $\varepsilon \to 0$, the solution $\hat{\xi}, \hat{v}$ converges to a solution of the incompressible Euler equations

$$\begin{aligned} \nabla \cdot v^\infty &= 0 \\ \partial_t v^\infty + v^\infty \cdot \nabla v^\infty + \nabla p &= 0 \end{aligned}$$

with initial data

$$v^\infty(0, x) = v_0(x)$$

uniformly on $[0, T_0) \times \mathbb{R}^3$.

The uniform bound for $\|Sw(t,\cdot)\|_{H^3}$ and a Sobolev inequality lead to

$$\|\partial_t\hat{\xi}(t)\|_{L^\infty} = O(\varepsilon/t), \quad \|\partial_t\hat{v}(t)\|_{L^\infty} = O(1/t), \quad t \le T_0.$$

This estimate was discovered and used in [8] to show that the compressible flow converges uniformly on intervals of the form $[\delta, T_0]$ to a solution of the incompressible Euler equation, for arbitrary initial data $\hat{w}(0)$. The data satisfied by the incompressible limit is the projection of $\hat{w}(0)$ onto the subspace of incompressible data.

Relationship between the lifespan of compressible and incompressible flow

Any improvement in the life span of w leads to global solutions of the incompressible Euler equation.

Theorem 5: *Suppose that for every $\varepsilon > 0$, there exists a C^1 solution $w(t)$ of the compressible Euler equations on $[0, T_\varepsilon] \times \mathbb{R}^3$ such that $E_4[w(t)] = O(\varepsilon)$. Assume that the initial data is incompressible, i.e.*

$$\xi_0^\varepsilon = 0, \quad v_0^\varepsilon = v_0^\infty, \quad \nabla \cdot v_0^\infty = 0.$$

If $\liminf_{\varepsilon\to 0} \varepsilon T_\varepsilon = \infty$, then the incompressible Euler equations have a global C^1 solution.

For an arbitrarily large time T the preceding argument works for sufficiently small ε. Construct a smooth solution of the incompressible Euler equation with the given initial data defined on $[0, T]$.

Conclusion: Blow-up for incompressible Euler implies that the lifespan C/ε is most likely sharp for compressible flow.

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