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EQUATIONS AUX DERIVEES PARTIELLES

ON SPECTRAL PROPERTIES OF ADIABATICALLY PERTURBED SCHROEDINGER OPERATORS WITH PERIODIC POTENTIALS

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Abstract : We describe the asymptotic structure of the spectrum and the resonances of the equation

$$-\psi_{xx} + V(\varepsilon x, x)\psi = E\psi ,$$

where $V(\xi, x)$ is a periodic function of the second variable and ε is a small parameter.

1. Introduction

The standard constructions of the semiclassical approach can be applied to the equations of the following form

$$\mathcal{L}(\varepsilon x, -i\partial_x)\psi = 0 \ , \ x \in \mathbf{R}^d \ . \tag{1.1}$$

Generally, the symbol $\mathcal{L}(\xi, p)$ here is operator-valued. Equations (1.1) admit formal asymptotic solutions of the form

$$\psi = \exp \frac{i}{\varepsilon}\ \theta(\varepsilon x) \sum_{n\geq 0} \varepsilon^n \psi_n(\varepsilon x) \ . \tag{1.2}$$

It is known also that this class of formal solutions can be enlarged essentially since equations (1.1) reproduce their structure after the Fourier transformation with respect to some subset of the variables $x_1, \cdots, x_d$.

Solutions (1.2) can be described in terms of the corresponding classical Hamiltonian system with the Hamiltonian $\mathcal{L}(\xi, p)$ and with the standard symplectic structure. The literature devoted to the general ideas of the semiclassical approach is extensive and sufficiently well known.

In recent years there have appeared a series of works devoted to the asymptotic investigations of the equations of the following more general form

$$\mathcal{L}(\varepsilon x,\ x\ , -i\partial x)\psi = 0 \ . \tag{1.3}$$

The symbol $\mathcal{L}(\xi, x, p)$ was considered as a periodic function of the second variable x [1-3]. Such equations have various applications and arise quite naturally in solid state physics. From a mathematical point of view equations (1.3) can be treated as natural generalizations of usual semiclassical equations (1.1).

The main idea of the generalization is quite simple. Consider a function $\psi(\xi, x)$ which is a solution of the equation

$$\mathcal{L}(\xi\ , x\ , -i\partial_x - i\varepsilon\partial_\xi)\psi = 0 \ , \tag{1.4}$$

then the function $\psi_0(x,\varepsilon) = \psi(\varepsilon x, x)$ obeys equation (1.3). But equation (1.4) is an equation of form (1.1) with the operator-valued symbol

$$\mathcal{L}(\xi, x, -i\partial_x - p) \ .$$

This idea has been indicated already in [4], in [1-3] it was shown that the idea leads to essentially new asymptotic consequences.

Spectral applications of the semiclassical approach have absolutely different characters in cases $d = 1$ and $d > 1$. We shall consider here spectral properties of ordinary Schroedinger differential equations

$$-\psi_{xx} + V(\varepsilon x, x)\psi = E\psi \tag{1.5}$$

with the potentials $V(\xi, x)$ which are real smooth functions, periodic with respect to the second variable x. A part of the results has been already described in [2,5-9].

This work has been written during my stay in the University Paris VII.

I want to express my gratitude to A. Boutet de Monvel who was my host and was very helpful in course of this visit.

2. Classification of spectral problems

One is interested in the spectral properties of equation (1.5) in $L_2(\mathbf{R})$-space. Assume that the period of the function $x \mapsto V(\xi, x)$ is constant and is equal to a, $a > 0$.

It is known that the simpler equation

$$-\psi_{xx} + V(\varepsilon x)\psi = E\psi \tag{2.1}$$

has definite spectral properties only if the potential $V(\xi)$ has more or less definite asymptotic behavior at infinity. One can distinguish three definite types of this asymptotic behavior :

i) $$V(\xi) \to +\infty \ , \ \xi \to \infty \ ,$$

ii) $$V(\xi) \to \text{const} \ , \ \xi \to \infty \ ,$$

iii) $$V(\xi) \to -\infty \ , \ \xi \to \infty \ .$$

In all the cases one has to assume also that there is some regularity in the asymptotic behavior of V at infinity. The simplest way to express this regularity is to introduce some estimates of the derivatives of V at infinity. We are not going to use such estimates explicity, so we shall not write down them here. In all the cases they lead to a semiclassical asymptotic description of the solution of (2.1), when $x \to \infty$, uniformly with respect to ε *. These estimates are well known in cases i) - ii). As for iii) it is worth to note that, of course, one has to demand

$$V(\xi) \geq -V_0|\xi|^{\alpha} \ , \alpha < 2 \ .$$

Some details in case iii) can be found in [10,11].

The main condition demands that the potential $V(\xi, x)$ admits the asymptotic behaviour

$$V(\xi, x) \sim p_{\pm}(x) + V_{\pm}(\xi) \ , \ \xi \to \pm\infty \ , \tag{2.2}$$

with the remainders which are quickly decreasing functions at infinity.

In accordance with three cases i)-iii) one can introduce the following six situations :

A :	$V_-(\xi) \to +\infty$	as	$\xi \to -\infty$,
	$V_+(\xi) \to +\infty$	as	$\xi \to +\infty$;
B :	$V_-(\xi) \to +\infty$	as	$\xi \to -\infty$,
	$V_+(\xi) \to 0$	as	$\xi \to +\infty$;
C :	$V_-(\xi) \to C_-$	as	$\xi \to -\infty$,
	$V_+(\xi) \to C_+$	as	$\xi \to +\infty \ , C_- > C_+$;
D :	$V_-(\xi) \to +\infty$	as	$\xi \to -\infty$,
	$V_+(\xi) \to -\infty$	as	$\xi \to +\infty$;
E :	$V_-(\xi) \to 0$	as	$\xi \to -\infty$,
	$V_+(\xi) \to -\infty$	as	$\xi \to +\infty$;
F :	$V_-(\xi) \to -\infty$	as	$\xi \to -\infty$,
	$V_+(\xi) \to -\infty$	as	$\xi \to +\infty$.

Asymptotically the spectral properties of equation (1.5) depend on the spectral properties of the set of the equations

$$-\psi_{xx} + V(\xi, x)\psi = E\psi \tag{2.3}$$

with the purely periodic potentials $x \mapsto V(\xi, x)$. For each ξ we can introduce the corresponding Bloch solutions ψ and the dispersion function $\mathcal{E}$:

$$\psi(x, \kappa, \xi) = e^{i\kappa x}\varphi(x, \kappa, \xi), \tag{2.4}$$

$$E = \mathcal{E}(\kappa, \xi) \ .$$

The quasi-momentum κ here is a free parameter which belongs to the complex plane **C** or, more exactly, to some Riemann surface, see details in [2 , 7]. The dispersion function

* In fact, this description is given by the same formulas as in the case $\varepsilon \to 0$.

$\mathcal{E}(\kappa,\xi)$ obeys the symmetry relations : $\mathcal{E}(\bar{\kappa},\xi) = \overline{\mathcal{E}(\kappa,\xi)}, \mathcal{E}(-\bar{\kappa},\xi) = \overline{\mathcal{E}(\kappa,\xi)}$. It maps monotonically the intervals $[(\ell-1)\frac{\pi}{a}+0 \ , \ \ell\frac{\pi}{a}-0] \ , \ell = 1,2,\cdots$, on the spectral intervals $\Delta_\ell(\xi) = [E_{m-1}(\xi) \ , E_m(\xi)] \ , \ m = 2\ell - 1$, of equation (2.3).

Note that the functions $E_m(\xi)$ can be described explicitely if

$$V(\xi,x) = p(x) + V_0(\xi) \ .$$

In this case

$$\mathcal{E}(\xi,\kappa) = \mathcal{E}_0(\kappa) + V_0(\xi) \ ,$$

where $\mathcal{E}_0$ is the dispersion function of the equation

$$-\psi_{xx} + p(x)\psi = E\psi \ . \tag{2.5}$$

So

$$E_m(\xi) = E_{0m} + V_0(\xi) \ ,$$

where $[E_{0,2\ell-2}, E_{0,2\ell-1}]$ are the spectral intervals of (2.5).

In all cases A-F the continuous spectrum of (1.5) depends only on limit potentials (2.2), so it depends on the limit behavior of the spectral intervals as $\xi \to -\infty \ , +\infty$. But the discrete spectrum (and also the resonances on the continuous spectrum) depends on total behavior of the functions $\xi \mapsto E_m(\xi) \ , \ m = 0,1,\cdots$ Assume for simplicity of the narrative that all these functions are strictly monotonically decreasing functions. The spectral role of isolated critical points of these functions will be demonstrated by cases A,F. Assume, again for simplicity, that in these two cases each function $\xi \mapsto E_m(\xi)$ is a strictly monotonous function excepting a single non-degenerated critical point ξ_m^0.

3. Isoenergy curve

The initial object of all asymptotic constructions is the so called isoenergy curve :

$$\mathcal{E}(\kappa,\xi) = E \ , \tag{3.1}$$

E is a fixed value of the parameter E from (1.5). The isoenergy curve is considered as a curve in the "plane" (κ,ξ), κ and ξ belonging to the mentioned Riemann surface and $\mathbf{R}$ respectively. Friequently instead of the Riemann surface one will consider the "complex cylinder" on which the points κ and $\kappa + (2\pi/a)\ell \ , \ \ell \in \mathbf{Z}$, are treated as identical ones.

To understand a structure of the isoenergy curve introduce in the plane $(\mathcal{E} \ , \xi)$ admitted and forbidden strips corresponding to the spectral intervals $\Delta_\ell(\xi)$ and the gaps $\bar{\Delta}_\ell(\xi) = (E_{m+1}(\xi), E_{m+2}(\xi)) \ , m = 2\ell - 1$, between them. The boundaries of these strips are given by the formulas

$$\mathcal{E} = E_m(\xi) \ . \tag{3.2}$$

Now consider a fixed level $\mathcal{E} = E$. It intersects curves (3.2) in some set of points $z_i, i \in I$. Their projections ξ_i in the ξ-axis are called the turning points.

Each finite interval $[\xi_i, \xi_{i+1}]$ between two turning points is covered by a closed finite branch γ of the isoenergy curve (on the cylinder) which consists of two regular branches γ_+, γ_- :

$$\gamma_+ \ : \kappa = \tau(\xi) \ , \ \gamma_- \ : \kappa = -\tau(\xi). \tag{3.3}$$

If the segment $[z_i, z_{i+1}]$ belongs to an admitted strip, the branch γ is real. If $[z_i, z_{i+1}]$ belongs to an forbidden strip, γ is complex and Re $\kappa = (\pi/a)\ell$, $\ell \in \mathbf{Z}$, on γ. The branch of the isoenergy curve covering the infinite interval $(-\infty \ , \xi_i]$ or $[\xi_i, \infty)$ can be disjoint at infinity.

Each regular branch, γ_+ or γ_-, generates the formal asymptotic solution of equation (1.5)

$$f = \exp i\phi \ . \sum_{n \geq 0} \varepsilon^n f_n(\varepsilon x, x) \ , \tag{3.4}$$

where $x \mapsto f_n(\xi, x)$ are periodic functions and

$$\phi = \int^{\xi} \frac{1}{\varepsilon} \kappa d\xi + \omega \ , \ \kappa = \pm\tau(\xi) \ , \tag{3.5}$$

$$\omega = -\int_0^a d\varphi(x, \kappa, \xi) \cdot \overline{\varphi(x, \kappa, \xi)} \cdot dx \ , \tag{3.6}$$

d is the differential with respect to κ and ξ, φ is supposed to be normalized. The formal solution f can be used to describe the asymptotic behavior of an exact solution of equation (1.5) as $\varepsilon \to 0$ (or as $x \to \pm\infty$).

Above small vicinities of the turning points the solutions f have to be changed by more complicated ones, the details can be found in [1,2].

Since the set of the turning points can be infinite, sometimes the distance between two closest turning points in not bounded below. This situation can be treated as a case of the double turning point [2,12].

At last, note that the formal solutions f are quickly oscillating functions of $\xi = \varepsilon x$, if ξ belongs to the interval $[\xi_i, \xi_{i+1}]$ covered by a real branch of the isoenergy curve. In opposite case, when $[\xi_i, \xi_{i+1}]$ is covered by a complex branch, these solutions are quickly decreasing or quickly increasing functions of ξ.

4. Quantization conditions

Consider an interval $[\xi_i, \xi_{i+1}]$ covered by a real closed branch γ of the isoenergy curve. On this interval one can consider two quickly oscillating formal solutions f_+ and f_-. Using the formal solutions in vicinities of the turning points ξ_i, ξ_{i+1}, one can continue the solutions f_+, f_- on the intervals $[\xi_{i-1}, \xi_i]$ and $[\xi_{i+1}, \xi_{i+2}]$ covered by the complex branches. It is possible to find conditions which lead to a linear combination $b_+ f_+ + b_- f_-$ decreasing when ξ goes away from $[\xi_i, \xi_{i+1}]$. These so called quantization conditions have the following form

$$\int_\gamma \frac{1}{\varepsilon}\kappa d\xi + \omega + \sum_{n\geq 1} \varepsilon^n \omega_n = 2\pi n + \pi\ /2 \text{ ind } \gamma\ , \tag{4.1}$$

where (ω_n) is some sequence of 1-form, ind γ is the Maslov index of γ, and $n \in \mathbf{Z}$.

The quantization conditions can be considered as conditions on the spectral parameter E, which determine asymptotically some discrete set of values of the parameter. This set is a set of eigenvalues or a set of resonances of equation (1.5).

One can treat the resonances by two different ways. If the potential $V(\xi, x)$ is an analytical function with respect to both variables in some strips $|\text{Im } x| < \alpha$, $|\text{Im}\xi| < \beta$, one can consider the resonances as poles of the analytical continuation of the kernel of the resolvent

$$\left[-\frac{d^2}{dx^2} + V(\varepsilon x, x) - E\right]^{-1} .$$

If the analytical continuation of the resolvent kernel is impossible, the resonance can be consider as a point of very charp increasing of the so called function of the spectral shift defined on the E-axis. In the eigenvalue this function has a jump.

If a real branch of the isoenergy curve is infinite, the corresponding quantization conditions lose their sense. The corresponding solution $b_+ f_+ + b_- f_-$decreasing after its continuation through a vicinity of the only turning point describes the scattering process and corresponds to the given value E of the continuous spectrum.

The following important principle of superposition is true. Consider a fixed spectral interval $\Delta_\ell(\xi)$ and the corresponding admitted strip. It generates some set of the turning points and some set of the real branches of the isoenergy curve. They give their contribution to the description of the spectrum. The total asymptotical picture of the spectrum can be obtained as the simple unification of the contributions of the single strips.

What is the influence of the interaction between different real branches of the isoenergy curve through complex branches ? The value of the tunnelling through a complex branch γ is given by the factor

$$\exp \frac{i}{\varepsilon} \oint_\gamma \kappa d\xi\ . \tag{4.2}$$

If this factor has the order of 1 (in fact, it is the situation of a double turning point), it is impossible to split two corresponding real branches separated by γ ; the interaction between them is too strong. One can meet this situation if the set of turning points is infinite or if the parameter E is sufficiently large and the potentials V_+ or V_- tend to infinity sufficiently quickly. We are not going to consider these cases. Moreover, it will be assumed (for simplicity of the narrative) that the potential $V(\xi, x)$ generates for each ξ only a finite number N of the finite spectral intervals such that the $(N+1)$-th spectral interval is infinite.

So in any cases all factors (4.2) will be very small, in fact, exponentially small. The interaction between different real branches will displace all spectral points generated by single strips on exponentially small distances. The interaction between two bounded real branches (through some chain of the tunnelling) leads to the displacement of the corresponding spectral points only along the real E-axis. But the interaction between a bounded and an unbounded real branches leads to the exponentially small displacement of the spectral points into the complex E-plane and gives a possibility to obtain the imaginary parts of the resonances, see [7].

5. Asymptotic behavior of the spectrum in case A

In case A : $V_\pm(\xi) \to +\infty$ as $\xi \pm \to \infty$. The spectrum of equation (1.5) is simple and purely discrete.

Consider a fixed spectral interval $\Delta_\ell(\xi) = [E_{m-1}(\xi), E_m(\xi)], m = 2\ell - 1$, of equation (2.3) and the corresponding admitted strip in the plane $(\xi, \mathcal{E})$.

If $E < \min E_{m-1}$, this strip does not generate any turning points and the corresponding isoenergy curve does not contain any real branches. Therefore it is clear that the contribution of the strip to the spectrum of equation (1.5) is empty asymptotically.

If $E \in (\min E_{m-1}, \min E_m)$, the strip generates two turning points $\xi_{m-1}^- < \xi_{m-1}^+$ and there exists a closed finite real branch of the isoenergy curve which covers the interval $[\xi_{m-1}^-, \xi_{m-1}^+]$. Outside of this interval the isoenergy curve is complex. The quantization conditions lead to the asymptotic description of the discrete spectrum which lies in the considered interval of E-axis.

At last, if $E > \min E_m$, the strip generates four turning points $\xi_{m-1}^- < \xi_m^- < \xi_m^+ < \xi_{m-1}^+$. They bound two closed finite real branches of the isoenergy curve which cover the intervals $[\xi_{m-1}^-, \xi_m^-]$, $[\xi_m^+, \xi_{m-1}^+]$ respectively. Each of them gives its contribution to the discrete spectrum independently.

The picture is quite different for the last spectral interval Δ_{N+1}. If $E < \min E_{2N}$, the corresponding spectrum is absent asymptotically. If $E > \min E_{2N}$, there exist two turning points ξ_{2N}^-, ξ_{2N}^+ and an only closed finite real branch of the isoenergy curve covering the interval between them. So the last strip always generates a single branch of the discrete spectrum.

The cases $E = \min E_{m-1}$ and $E = \min E_m$ are transitional, they generate the double turning point : $\xi_{m-1} = \xi_{m-1}^- = \xi_{m-1}^+$ and two simple turning points ξ_{m-1}^-, ξ_{m-1}^+ and the double turning point $\xi_m = \xi_m^- = \xi_m^+$ respectively. In some vicinities of the double turning points the standard quantization condition should be replaced by a little more complicated condition.

All spectrum of equation (1.5) can be described essentially as the simple unification of the contributions of the separate strips. However, one should remember that this description of the spectrum is not uniform with respect to the spectral parameter E as $E \to +\infty$.

6. Asymptotic behavior of the spectrum in case B

In case B : $V_-(\xi) \to +\infty$ as $\xi \to -\infty$ and $V_+(\xi) \to 0$ as $\xi \to +\infty$. The spectrum of equation (1.5) is again simple. The continuous spectrum coincides with the continuous spectrum of the equation

$$-\psi_{xx} + p_+\psi = E\psi \ , \tag{6.1}$$

i.e. it consists of the set of the spectral intervals $\Delta_\ell^+ = [E_{m-1}^+, E_m^+]$ of equation (6.1). The spectral gaps of equation (6.1) can contain points of the discrete spectrum of (1.5). Consider a fixed spectral interval $\Delta_\ell(\xi) = [E_{m-1}(\xi), E_m(\xi)]$ of equation (2.3) and the corresponding admitted strip.

If $E < E_{m-1}^+$, this strip does not generate any turning points and the isoenergy curve does not contain any real branch. So the contribution of the strip to the spectrum is empty.

If $E \in (E_{m-1}^+, E_m^+)$, the strip generates one turning point ξ_{m-1}. It appears at infinity when $E = E_{m-1}^+$ and moves monotonically along the ξ-axis while E increases. The interval $[\xi_{m-1}, \infty)$ is covered by a disclosed infinite real branch of the isoenergy curve, so the interval $[E_{m-1}^+, E_m^+]$ belongs to the continuous spectrum of equation (1.5).

The corresponding formal asymptotic solutions of (1.5) can be used to obtain the asymptotic representation as $\varepsilon \to 0$, and also as $x \to +\infty$, of the exact solution ψ of equation (1.5), which describes the scattering process. This unique solution ψ can be characterized by the condition $\psi \to 0$ as $\xi \to -\infty$. Its support coincides asymptotically with the half-axis $[\xi_{m-1}, \infty)$. It defines naturally the reflection coefficient $r(\kappa), |r(\kappa)| = 1$, and the semiclassical formulas for ψ lead to the asymptotic description of $r(\kappa)$ as $\varepsilon \to 0$.

If $E_m^+ < E$, there are two turning points $\xi_{m-1} < \xi_m$. The character of the dependence of ξ_m on E is the same as the character of the dependence of ξ_{m-1}. The interval $[\xi_{m-1}, \xi_m]$ is covered by a closed finite real branch of the isoenergy curve. The quantization conditions give asymptotic formulas for the points of discrete spectrum (or for the resonances). The last strip generates only the continuous spectrum on the half-axis $[E_{2N}^+, \infty)$

It is worth to say some words about the transitional values E_m^+ of the spectral parameter. If $E \to E_m^+ + 0$, then the turning point ξ_m tends to $+\infty$. However, for sufficiently large ξ_m the point ξ_m can lose the property of a turning point. Indeed, it loses this property if $V_+(\xi)$ vanishes quicker than $|\xi|^{-\alpha}$, $\alpha > 2$. In the opposite case, when V_+ vanishes slower than $|\xi|^{-\alpha}, \alpha < 2$, the turning point ξ_m can be treated as a standard turning point for the arbitrary large values of ξ_m. In the first, non standard case, the interval of E around E_m^+, which is impossible to treat semiclassically, is quite small.

All spectrum is given by the unification of the contributions of the separate strips. One has to introduce an essential correction if E belongs to an interval of the continuous spectrum. The tunnelling interaction between a bounded real branch and the unbounded real branch of the isoenergy curve leads to the displacement of the discrete points, generated by the bounded branch and the quantization conditions, into the complex E-plane, such that they become resonances. So the quantization conditions characterize only their real parts. To find their exponentially small imaginary parts one has to consider the chain of tunnelling between the corresponding finite branch of the isoenergy curve and the infinite branch related with the scattering.

7 Asymptotic behavior of the spectrum in case C

In this case $V_-(\xi) \to C_-$ as $\xi \to -\infty$ and $V_+(\xi) \to C_+$ as $\xi \to +\infty$, $C_+ < C_-$. Introduce the spectral intervals Δ_ℓ^- and Δ_ℓ^+ of the equations

$$-\psi_{xx} + (p_\mp(x) + C_\mp)\psi = E\psi \ .$$

The continuous spectrum of equation (1.5) coincides with the unification $\cup \Delta_\ell^- \cup \Delta_{\ell'}^+$ of all these intervals. In particular its multiplicity is equal to 2 on all non-empty intersections $\Delta_\ell^- \cap \Delta_{\ell'}^+$. Besides, equation (1.5) can have discrete spectrum in the gaps of the continuous spectrum.

Fix a spectral interval $\Delta_\ell(\xi)$, it tends to Δ_ℓ^- as $\xi \to -\infty$ and to Δ_ℓ^+ as $\xi \to +\infty$. One can meet two different possibilities : $\Delta_\ell^- \cap \Delta_\ell^+ = \phi$ and $\Delta_\ell^- \cap \Delta_\ell^+ \neq \phi$.

Consider the first possibility, moreover let $E_{m-1}^- > E_m^+$.

If $E < E_{m-1}^+$, the corresponding strip does not generate any turning points and the isoenergy curve does not contain any real branches. Therefore the spectrum of equation (1.5), related with the ℓ-th strip, is empty asymptotically.

If $E \in \Delta_\ell^+$, there exists one turning point ξ_{m-1}, and the interval $[\xi_{m-1}, \infty)$ is covered by an unbounded real branch of the isoenergy curve. This situation is completely similar to the analogous situation in case B, so one can conclude that the interval Δ_ℓ^+ generates a part of the continuous spectrum of equation (1.5).

If $E \in (E_m^+, E_{m-1}^-)$, there exist two turning points $\xi_{m-1} < \xi_m$ and the interval $[\xi_{m-1}, \xi_m]$ is covered by a bounded real branch of the isoenergy curve. This situation is again similar to the analogous situation in case B, so the interval $[E_m^+, E_{m-1}^-]$ is covered by a discrete set of the spectral points and the quantization conditions give their asymptotic description. These points are eigenvalues or the real parts of resonances depending on the interaction between the different strips.

If $E \in \Delta_\ell^-$, there exists one turning point ξ_m and the interval $(-\infty, \xi_m]$ is covered by an unbounded real branch of the isoenergy curve. As in the case $E \in \Delta_\ell^+$ it leads to the simple continuous spectrum on the interval Δ_ℓ^- and to the asymptotic description of the corresponding scattering picture.

If $E > E_m^-$, turning points are absent, the isoenergy curve does not contain any real branch, therefore this set of the E-axis is free from the spectrum (and resonances).

Consider now the second possibility, $\Delta_\ell^- \cap \Delta_\ell^+ \neq \phi$, moreover let $E_{m-1}^- < E_m^+$.

Now one can distinguish the following situations : $E < E_{m-1}^+, E \in (E_{m-1}^+, E_{m-1}^-), E \in (E_{m-1}^-, E_m^+), E \in (E_m^+, E_m^-), E > E_m^-$. The 1-st, 2-nd, 4-th and 5-th situations are similar to the corresponding situations of the preceding first possibility excluding the character of the transitional points $E = E_{m-1}^-$ and $E = E_m^+$. But the 3-rd situation is different.

If $E \in (E_{m-1}^-, E_m^+)$, turning points are absent and all ξ-axis is covered by two disjoint infinite real regular branches of the isoenergy curve.

The formal solutions generated by these two branches gives the possibility to introduce the exact solutions of equation (1.5) with the prescribed asymptotic behavior as $x \to +\infty$ or $x \to -\infty$. Therefore one can consider two pairs of the solutions with the standard asymptotic behavior either as $x \to +\infty$ or $x \to -\infty$. These two pairs can be connected by means of the natural 2×2 unitary scattering matrix. The semiclassical constructions generate asymptotic description of all these solutions and the scattering matrix. Under the natural choice of the solutions the scattering matrix turns out to be diagonal up to small terms of the order $0(\varepsilon^\infty)$.

The last spectral interval Δ_{N+1} generates only continuous spectrum : the simple continuous spectrum on the interval $[E_{2N}^+, E_{2N}^-]$ and the double continuous spectrum on the half-axis $[E_{2N}^-, \infty)$.

The general properties of the transitional points are the same as in case B.

All spectrum of equation (1.5) asymptotically is given, in accordance with the principle of superposition, by the unification of the contributions of the separate strip. It is necessary to add the same remark as in the case B about the transformation of the discrete spectral points to the resonances. The continuous spectrums generated by different strips can coincide partially. Nevertheless they can be treated independently up to terms of the order $0(\varepsilon^\infty)$.

8. Asymptotic behavior of the spectrum in case D

In case : D $V(\xi) \to +\infty$ as $\xi \to -\infty$ and $V_+(\xi) \to -\infty$ as $\xi \to +\infty$. The spectrum of equation (1.5) is simple purely continuous on all axis.

Fix again some spectral interval $\Delta_\ell(\xi)$ and consider the corresponding admitted strip.

For all values of E this strip generates two turning points $\xi_{m-1} < \xi_m$ and the interval $[\xi_{m-1}, \xi_m]$ is covered by a closed finite real branch of the isoenergy curve, which leads together with the quantization conditions to the infinite set of the asymptotic spectral points distributed on all E-axis.

The last admitted strip generates only one turning point ξ_{2N}. The infinite interval $[\xi_{2N}, \infty)$ is covered by a disclosed infinite real branch of the isoenergy curve. It means that all E-axis belongs to the continuous spectrum of equation (1.5). One can introduce naturally the corresponding solution ψ of the scattering problem and the reflection coefficient $r(\kappa)$. The considered real branch of the isoenergy curve gives a possibility to find the asymptotic representation of ψ as $\varepsilon \to 0$ and also as $x \to \infty$. As a result one has the asymptotic representation of r.

The interaction between a bounded real branch of the isoenergy curve and the unbounded branch shifts all spectral points, generated by the bounded branch, from the real spectral axis into the complex plane, so they become resonances. Therefore one has exactly N branchs of the resonances in the complex plane. They again have the exponentially small imaginary parts.

In the special case

$$V(\xi, x) = p(x) - \xi \ , \tag{8.1}$$

equation (1.5) can be treated as the Schroedinger equation which characterizes the motion of the quantum electron in a crystal placed in the homogenious electric field. In this case the mentioned series of resonances are periodic :

$$E_n(\varepsilon) \sim n\varepsilon a + \sum_{m\geq 0} \varepsilon^m E_m \ , \ n \in \mathbf{Z} \ . \tag{8.2}$$

They are called the Stark ladders. The Stark ladders were considered with the details in [6,7]. Under special assumptions case (8.1) was investigated also in [13,14] and in other works ; the list of references can be found in the indicated publications.

9. Asymptotic behavior of the spectrum in case E

In case E : $V_-(\xi) \to 0$ as $\xi \to -\infty$ and $V_+(\xi) \to -\infty$ as $\xi \to +\infty$. The spectrum of equation (1.5) is purely continuous on all E-axis and double on the spectral intervals Δ_ℓ^- of the equation

$$-\psi_{xx} + p_-(x)\psi = E\psi \ .$$

Fix a spectral interval $\Delta_\ell(\xi)$ and the corresponding strip.

If $E < E_{m-1}^-$, there are two turning points $\xi_{m-1} < \xi_m$ with the interval $[\xi_{m-1}, \xi_m]$ between them covered by a closed finite real branch of the isoenergy curve. The quantization conditions generate the asymptotic description of the discrete set of the spectral points. In fact, they are the real parts of resonances.

If $E \in \Delta_e^-$, there is only one turning point ξ_m and the interval $(-\infty, \xi_m]$ is covered by a disclosed infinite real branch of the isoenergy curve. It leads to a simple branch of the continuous spectrum, and the semiclassical constructions can be used to obtain the asymptotic description of the corresponding scattering process.

If $E > E_m^-$, turning points and real branches of the isoenergy curve are absent, so the contribution to the spectrum is empty asymptotically.

For the last spectral interval the situation is different again. If $E < E_{2N}^-$ one has an only turning point and the scattering picture generating the simple continuous spectrum.

If $E > E_{2N}^-$, one has no turning points, the all ξ-axis is covered by two disjoint infinite regular branches of the isoenergy curve and the corresponding scattering picture with 2×2 scattering matrix generates the double continuous spectrum.

After the unification of the contributions of all strips the discrete spectral points transform to resonances, such that there exist exactly N branches of resonances with the real parts distributed on the half-axis $E < E_{2\ell-2}^-$, $\ell = 1, \cdots, N$, in accordance with the quantization conditions.

10. Asymptotic behavior of the spectrum in case F

In case F $V_{\pm}(\xi) \to -\infty$ as $\xi \to \pm\infty$. The spectrum of equation (1.5) is purely continuous of the multiplicity 2.

Consider a fixed spectral interval $\Delta_\ell(\xi)$ of equation (2.3).

One has to distinguish three following cases : $E < \max E_{m-1}, E \in (\max E_{m-1}, \max E_m), E$ $\max E_m$.

If $E < \max E_{m-1}$, there are four turning points $\xi_m^- < \xi_{m-1}^- < \xi_{m-1}^+ < \xi_m^+$, the intervals $[\xi_m^-, \xi_{m-1}^-]$ and $[\xi_{m-1}^+, \xi_m^+]$ are covered by closed finite branches of the isoenergy curve and these branches generate the asymptotic description of the discrete sets of spectral points in the considered interval of E-axis. It is clear that these points are real parts of resonances.

On principle, the picture is the same as $E \in (\max E_{m-1}, \max E_m)$, but in this case there are only two turning points $\xi_m^- < \xi_m^+$, only one real branch of the isoenergy curve and only one set of spectral points generated by the quantization conditions.

If $E > \max E_m$, the contribution of the strip to the spectrum is empty asymptotically.

The last spectral interval generates the double continuous spectrum on all E-axis.

If $E < \max E_{2N}$, there are two turning points $\xi_{2N}^- < \xi_{2N}^+$ and the intervals $(-\infty, \xi_{2N}^-], [\xi_{2N}^+, \infty)$ are covered by a disclosed infinite real branches of the isoenergy curve.

If $E > \max E_{2N}$, there are no turning points at all and all ξ-axis is covered by two disjoint regular infinite real branches of the isoenergy curve.

In both the cases it is possible to consider two pairs of the solutions with the standard asymptotic behavior either as $x \to +\infty$ or as $x \to -\infty$, and the 2×2 unitary scattering matrix. Asymptotically, as $\varepsilon \to 0$, it becomes unti-diagonal if $E < \max E_{2N}$ and diagonal, if $E > \max E_{2N}$.

In case F all transitional points are double turning points.

The interaction between different strips leads to the displacement of the discrete spectral points into the complex E-plane, such that there appear exactly N branches of the resonances which real parts are distributed along the half-axises $(-\infty, \max E_{2\ell-1}), \ell = 1, \cdots, N$, in accordance with the quantization conditions. Along the finite intervals $(\max E_{2\ell-2}, \max E_{2\ell-1})$ these branches of resonances are simple, along the infinite intervals $(-\infty, \max E_{2\ell-2})$ they are double in the clear sense.

References

[1] Buslaev, V.S., On adiabatic perturbation of a periodic potential, Teoreticheskaya i matematichskaya fizika **58**, N 2, (1984), 233-243.

[2] Buslaev, V.S., Semiclassical approximation for equations with periodic coefficients, Russian Mathem. Surveys **42** : (1987), 97-125.

[3] Guillot, J.C. ; Ralston, J. and Trubowitz, E., Semi-classical asymptotics in solid state physics, Commun. Math. Phys. **116**, (1988), 401-415.

[4] Maslov V.P., Nonstandard characteristics in asymptotic problems, Russian Mathem. Surveys **38** : 6, (1983), 3-36.

[5] Buslaev, V.S., Dmitrieva, L.A., Spectral properties of the Bloch electrons in external fields, Preprint University of Berkeley PAM-477, (1989).

[6] Buslaev, V.S., Dmitrieva, L.A., Bloch electrons in the external electric field, in Schroedinger operators, standard and non-standard (eds P. Exner and P. Seba), World Scientific, Singapore, (1989), 103-129.

[7] Buslaev, V.S., Dmitrieva, L.A., A Bloch electron in an external field, Leningrad Math. J. **1** : 2, (1990), 287-320.

[8] Buslaev, V.S., Spectral properties of the operators $\mathcal{H}\psi = -\psi_{xx} + p(x)\psi + V(\varepsilon x)\psi$, p is periodic, Operator Theory : Advances and Applications **46**, (1990), 85-107.

[9] Buslaev, V.S., Spectral properties of adiabatically perturbed differential operators with periodic coefficients, in Recent Development in Quantum Mechanics, Mathematical Physics Studies **12** (eds A. Boutet de Monvel, P. Dita, G. Nenciu, R. Purice), Kluwer Academic Publishers, (1991), 97-112.

[10] Buslaeva, M.V., Asymptotical dynamics and spectral analysis of the one-dimensional Schroedinger operators with accelerating potentials, in Problems in Mathematical Physics **11**, (ed M. Birman), Leningrad University, (1986), 67-77.

[11] Buslaeva, M.V., Asymptotical dynamics and spectral analysis of the Schroedinger operator with weakly accelerating potentials, Zapiski Nauchnih Seminarov LOMI **164** (ed L. Faddeev), Leningrad, Nauka, (1987), 10-19.

[12] Buslaev, V.S., Dmitrieva, L.A., Adiabatic perturbation of a periodic potential. II, Teoreticheskaya i matematicheskaya fizika **73**, (1987), 430-442.

[13] Bentosela, F., Stark-Wannier resonant states, preprint, Centre de Physique Theorique Marseille, CPT-90/ P.2256, (1990), 11pp

[14] Combes, J.M., Hislop P.D., Quantum tunnelling, non trapping and the existence of Stark ladder resonances, J. Phys. A : Math. Gen. **23**, (1990), 1-11.

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