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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

SPECTRAL ASYMPTOTICS WITH HIGHLY ACCURATE REMAINDER ESTIMATES

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I would like to consider local and microlocal semiclassical spectral asymptotics with highly accurate remainder estimates and present their applications to asymptotics of an eigenvalue counting function. It is easy enough to derive asymptotics with respect to other parameters (e.g. with respect to a spectral parameter) starting from semiclassical asymptotics ; however inverse considerations are complicated and may be even impossible. I start from completely (micro)local results ; then I consider the case of non-periodic Hamiltonian flow and present Weylian asymptotics with more accurate remainder estimate than a standard one ; some of these results were obtained in cooperation with A. Kachalkina ; then I consider the case of periodic Hamiltonian flow and present Weylian and non-Weylian asymptotics with very accurate remainder estimates.

1. Let us consider an h-pseudo-differential operator $A = a^w(x, hD)$ in $X = \mathbf{R}^d$ with the Weylian symbol $a(x,\xi)$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C\gamma^{-|\alpha|}\rho^{-|\beta|} \quad \forall \alpha,\beta \ : |\alpha|+|\beta| \le K \tag{1.1}$$

where here and below parameters ρ,γ satisfy condition

$$\rho \in (0,1], \gamma \in (0,1], \rho\gamma \ge h^{1-\delta}, h \in (0,1], \tag{1.2}$$

s and $\delta > 0$ are arbitrary, $K = K(d,\delta,s)$ is large enough. We assume that

(1.3) $a(x,\xi)$ is a Hermitian $D \times D$-matrix for every (x,ξ) ; then

(1.4) A is a self-adjoint operator in $L^2(X,\mathbf{H})$ with $\mathbf{H} = \mathbf{C}^D$.

Let $E(\tau)$ be a spectral projector of A ; I am interested in semiclassical asymptotics of

$$Tr(Q_1 E(\tau) Q_2) = tr \int (Q_{1x}\ {}^tQ_{2y} e)(x,x,\tau)dx$$

where Q_1 and Q_2 are h-pseudo-differential operators ${}^tQ_2$ means dual operator, $e(x,y,\tau)$ is a Schwartz kernel of $E(\tau)$ and Tr and tr mean an operator and a matrix traces respectively.

I start from assertions of microlocal character. The first result is a trivial enough

Theorem 1.— *Let conditions (1.1)-(1.3) be fulfilled and Ω, Ω_1 be domains in $T^*\mathbf{R}^d$ such that*

$$\mathrm{diam}_{(x,\xi)}\Omega \le c \ , \ \Omega_1 \subset \Omega \ , \ \mathrm{dist}(\Omega_1, \partial\Omega) \ge c^{-1}. \tag{1.5}$$

Moreover, let $Q_i = q_i^w(x,hD)$ where

(1.6) Supp $q_i \subset \Omega_1$ and q_i satisfy (1.1) with $\rho = \gamma = 1$ $(i = 1,2)$.

Then

(i) *Following rough estimates take place :*

$$|Q_{1x}{}^tQ_{2y} e(x,y,\tau)| \le Ch^{-d} \quad \forall x,y,\tau, \tag{1.7}$$

(1.7)′ $$|Tr\ Q_1 E(\tau) Q_z| \leq Ch^{-d} \quad \forall \tau$$

with $C = C(d, D, \delta, s, c)$.

(ii) *Moreover, if*

(1.8) $$|(\tau - a(x,\xi))v| \geq c^{-1}|v| \quad \forall v \in \mathbf{H} \quad \forall (x,\xi) \in \Omega$$

for every $\tau \in [\tau_1, \tau_2]$ *then*

(1.9) $$|Q_{1x}{}^t Q_{2y} e(x,y,\tau_1,\tau_2)| \leq Ch^s,$$

(1.9)′ $$|Tr\ Q_1 E(\tau_1,\tau_2) Q_z| \leq Ch^s$$

where $e(x,y,\tau_1,\tau_2)$ *is a Schwartz kernel of* $E(\tau_1,\tau_2) \doteq E(\tau_2) - E(\tau_1)$.

(iii) *Moreover if*

(1.10) symbol $a(x,\xi)$ *satisfies in* Ω *condition (1.1) with* $\rho = \gamma = 1$

then (1.9) and (1.9)' remain true if we replace (1.8) by a weaker condition

(1.8)′ $$|(\tau - a(x,\xi))v| \geq Ch|v| \quad \forall v \in \mathbf{H}\ \forall (x,\xi) \in \Omega\ .$$

(iv) *Moreover, if (1.10) is fulfilled and*

(1.11) There exist a symbol $a_0(x,\xi)$ *and a real valued function* $\lambda(x,\xi)$ *such that* $h^{-1}(a-a_0)$ *satisfy (1.1) with* $\rho = \gamma = 1$ *in* Ω *and*

$$\mathrm{Spec}\ \mathrm{a}_0(\mathrm{x},\xi) \cap [\tau - \mathrm{c}^{-1}, \tau + \mathrm{c}^{-1}] = \{\lambda(\mathrm{x},\xi)\} \cap [\tau - \mathrm{c}^{-1}, \tau + \mathrm{c}^{-1}]$$

for $\tau = \tau_1$ *and* $\tau = \tau_2$ *then (1.9), (1.9)' remain true if we replace (1.8)' by a weaker condition*

(1.8)″ $$|(\tau - a(x,\xi))v| \geq c^{-1}h^{2-\delta}|v| \quad \forall v \in \mathbf{H} \quad \forall (x,\xi) \in \Omega\ .$$

More sophisticated is the following

Theorem 2.— *Let conditions (1.1)-(1.3), (1.5), (1.6), (1.10) be fulfilled and*

(1.12) $$\varphi \in C_0^k([-1,1]), |D^k\varphi| \leq c \quad \forall k \leq K\ .$$

Then

(i) *For every* $L \in [h^{1/2-\delta}, 1]$ *the following complete asymptotics*

$$|\int \varphi((\tau' - \tau)/L)(Tr\ Q_1 E(\tau') Q_2 - \sum_{n=0}^{M} \kappa_n(\tau') h^{n-d}) d\tau'| \leq Ch^s$$

has place with $C = C(d,\delta,s), \kappa_n \in \mathcal{S}'(\mathbf{R})$ *where*

$$\kappa_0(\tau) = (2\pi)^{-d} \int\int q_1(x,\xi) q_2(x,\xi) n(x,\xi,\tau) dx d\xi, \tag{1.14}$$

$n(x,\xi,\tau)$ *is an eigenvalue counting function for* $a(x,\xi)$ *and we assume here that*

$$q_j (j = 1,2) \quad \text{are scalar symbols.} \tag{1.15}$$

(ii) *Moreover if (1.11) is fulfilled then (1.13) remains true for every* $L \in [h^{1-\delta}, 1]$.

(iii) *On the other hand if (1.8)' is fulfilled for some* $\tau = \tau^0 \in \mathbf{R}$ *then one can calculate number values of* $\kappa_n(\tau^0)$ *and*

$$|Tr\ Q_1 E(\tau^0) Q_2 - \sum_{n=0}^{M} \kappa_n(\tau^0) h^{n-d}| \le C h^s. \tag{1.16}$$

(iv) *Moreover if (1.11) is fulfilled then one can replace in the previous assertion (1.8)' by (1.8)".*

Now I am able to formulate main results of this section :

Theorem 3.— *Let conditions (1.1)-(1.3), (1.5), (1.6), (1.10) be fulfilled and for* $\tau = \tau^0$

(1.17) For every $(x,\xi) \in \Omega$ *there exists a vector field* $\mathcal{T} \in T_{(x,\xi)}(T^*\mathbf{R}^d)$ *such that* $|\mathcal{T}| \le 1$ *and*

$$\langle (\mathcal{T}a)(x,\xi)v, v\rangle \ge c^{-1}|v|^2 - c|(\tau - a(x,\xi))v|^2 \quad \forall v \in \mathbf{H} \quad \forall (x,\xi) \in \Omega\ .$$

Then $\kappa_n(\tau)$ *are regular functions near* $\tau = \tau^0$ *and*

$$R_0 = |Tr\ Q_1 E(\tau^0) Q_2 - \kappa_0(\tau^0) h^{-d}| \le C h^{1-d}. \tag{1.18}$$

Theorem 4.— *Let conditions (1.1)-(1.3), (1.5), (1.6), (1.10), (1.11) be fulfilled and*

(1.19)$_r$ For every $(x,\xi) \in \Omega$ *either*

$$|\lambda(x,\xi) - \tau^0| + |d\lambda(x,\xi)| \ge c^{-1}$$

or (Hess λ) (x,ξ) *has* r *eigenvalues counting their multiplicities* $f_1, \cdots, f_2$ *with* $|f_j| \ge c^{-1} \forall_j$.
Then

(i) *For* $r > 2$ *estimate (1.18) remains true and*

$$R_0 \le C h^{1/2-d} \quad \text{for} \quad r = 1, \tag{1.20$_1$}$$

$$R_0 \le C h^{1-d}(1 + |\ln h|) \quad \text{for} \quad r = 2 \tag{1.20$_2$.}$$

(ii) *If* $(1.19)_2^+$ *is fulfilled (this means that* f_1 *and* f_2 *have the same sign in* $(1.19)_2$*) then (1.18) remains true and moreover*

$$R_0 \le C h^{1-d}\left(\int_{\Sigma_{\tau^0}} d\mu_{\tau^0} + h^\beta\right) \tag{1.21}$$

where $\Sigma_\tau = \{(x,\xi) \in \Omega, \tau \text{ is an eigenvalue of } a(x,\xi)\}, d\mu_\tau = \partial_\tau n(x,\xi,\tau)$ *is a natural density at* Σ_τ, $\beta = \beta(d,\delta) > 0$ *is small enough.*

(iii) *If* $d = 1$ *and* $(1.19)_2$ *is fulfilled then (1.18) remains true.*

2. In order to obtain more precise remainder estimates it is necessary to make certain assumptions of a global nature. First we consider the case of non-periodic Hamiltonian flow ; these investigations were inspired by results of A. Volovoi.

Let $\Lambda_1, \cdots, \Lambda_n$ be closed subsets of Σ_{τ^0} and $1 < T_1 \leq \cdots \leq T_n$ numbers such that

(2.1) $T_n \leq \rho_0^{-1}$ where here and below $\rho_0 = h^\sigma, \gamma_0 = h^{1/2-\delta}, \sigma = \sigma(d,\delta) > 0$ is small enough,

(2.2) For every point (x,ξ) of the γ_0-neighbourhood of Λ_j

$$\mathrm{Spec}\ a_0(x,\xi) \cap [\tau^0 - \rho_0, \tau^0 + \rho_0] = \{\lambda_j(x,\xi)\} \cap [\tau^0 - \rho_0, \tau^0 + \rho_0]$$

where $h^{-1}(a - a_0)$ satisfies (0.1) with $\rho = \gamma = \rho_0$ in Ω,

(2.3) Through every point $(\bar{x},\bar{\xi})$ of the γ_0-neighbourhood of Λ_j there passes a Hamiltonian trajectory $(x(t),\xi(t))$ of λ_j with $(x(0),\xi(0)) = (\bar{x},\bar{\xi})$ and either $t \in [0,T_j]$ or $t \in [-T_j, 0]$ along which

(2.4) Conditions (1.1) with $\rho = \gamma = \rho_0$ and (2.2) remain true,

$$|D(x(t),\xi(t))/D(\bar{x},\bar{\xi})| \leq \rho_0^{-1} \tag{2.5}$$

where the left-hand expression is a norm of the Jacobi matrix,

$$\mathrm{dist}((x(t),\xi(t)),(\bar{x},\bar{\xi})) \geq \gamma_0|t|\ ; \tag{2.6}$$

here (2.6) is a uniform and refined non-periodicity condition.

Our main result here is

Theorem 5.— *Let conditions (1.1)-(1.3), (1.5), (1.6), (1.10) and either (1.17) or (1.11), $(1.19)_2^+$ be fulfilled. Moreover let closed sets $\Lambda_1, \cdots, \Lambda_n$ and numbers $1 \leq T_1 \leq \cdots \leq T_n$ satisfy (2.1)-(2.6). Then*

$$\begin{aligned} R_1 &= |Tr\ Q_1 E(\tau^0) Q_2 - \kappa_0(\tau^0) h^{-d} - \kappa_1(\tau^0) h^{1-d}| \\ &\leq C h^{1-d} \left(\sum_{j=0}^{n} T_j^{-1} \int_{\Lambda_j} d\mu_{\tau^0} + h^\beta \right) \end{aligned} \tag{2.7}$$

where $T_0 = 1, \Lambda_0 = \Sigma_{\tau^0} \setminus (\Lambda_1 \cup \cdots \cup \Lambda_n)$ and $\Sigma_\tau, d\mu_\tau$ are introduced in theorem 4.

3. Variants and modifications.

In this section some modifications of theorem 1-5 for h-differential operators are presented. The emphasis is that the assertions of theorems 1-5 remain true with no condition of a global character excluding self-adjointness of an operator and in the modification of theorem also conditions along Hamiltonian trajectories are added. In the second part of this section there is treated the case when the presence of the boundary of the domain is essential and in the modification of theorem 5 Hamiltonian trajectories with reflection at the boundary are considered. In this part only a Schrödinger operator is treated ; however theorem 9 has a wide generalization.

So let X be a domain in $\mathbf{R}^d$ and

$$A = \sum_{|\alpha|\le m} a_\alpha(x,h)(hD)^\alpha = a^w(x,hD) \tag{3.1}$$

an h-differential operator with the Weylian symbol $a(x,\xi)$; we assume that

$$|D_x^\beta a_\alpha| \le c \quad \forall\alpha : |\alpha| \le m \quad \forall\beta : |\beta| \le K \tag{3.2}$$

in $B(0,1)$ where $B(y,z) = \{x : |x-y| < r\}$.

Theorem 6.— *Let conditions (3.1), (3.2), (1.4) be fulfilled and*

$$X \supset B(0,1). \tag{3.3}$$

Moreover, let conditions (1.5), (1.6) be fulfilled and $Q_1 = q_1^w(x,hD)\psi_1(x)$, $Q_2 = \psi_2(x)q_2^w(x,hD)$ *where*

$$\operatorname{supp}\psi_j \subset \mathrm{B}(0,1/2), |\mathrm{D}^\alpha\psi_j| \le \mathrm{c} \quad \forall\alpha \ : |\alpha| \le \mathrm{K}, \tag{3.4}$$

$$|D_x^\alpha D_\xi^\beta q_j| \le c \quad \forall\alpha \ : |\alpha| \le K \quad \forall\beta : |\beta| \le K\ . \tag{3.5}$$

On the other hand conditions (1.1), (1.2), (1.10) are no longer assumed to be fulfilled. Then

(i) *All the assertions of theorems 1-5 remain true ;*

(ii) *Moreover, if A is elliptic in $B(0,1)$ i.e.*

$$|a(x,\xi)v| \ge c^{-1}|\xi|^m|v| \quad \forall v \in \mathbf{H} \quad \forall\xi : |\xi| \ge C \tag{3.6}$$

and all the conditions (excluding (1.6)) are fulfilled in $\Omega = T^*B(0,1)$ then all these assertions remain true with $Q_j = \psi_j$ and with $E(\tau)$ replaced by

$$L^{-1}\int \varphi_1((\tau'-\tau)/L)(E(\tau)-E(\tau'))d\tau'$$

with φ_1, L of the same nature as in theorem 2 ; certainly coefficients $\kappa_n(\tau)$ should be changed by the same way ;

(iii) *Moreover if A is positively elliptic in $B(0,1)$ i.e.*

$$\langle a(x,\xi)v,v\rangle \geq c^{-1}|\xi|^m|v|^2 \quad \forall v\in \mathbf{H} \quad \forall \xi : |\xi|\geq c \tag{3.7}$$

and all the conditions (excluding (1.6)) are fulfilled in $\Omega = T^*B(0,1)$ then all the assertions of theorems 1-5 remain true with $Q_j = \psi_j$ and with no modification of $E(\tau)$ and $\kappa_n(\tau)$; moreover in this case

$$|e(x,y,\tau)| \leq Ch^s(|\tau|+1)^{-s} \tag{3.8}$$

$$\forall x,y\in B(0,1/2) \quad \forall \tau \leq -C \ .$$

Theorem 7.— *Let conditions (3.1)-(3.5), (1.4)-(1.6) and either (1.17) or (1.11), $(1.19)_2^+$ be fulfilled and let closed sets $\Lambda_1,\cdots,\Lambda_n$ and numbers $1 < T_1 \leq \cdots \leq T_n$ satisfy conditions (2.1)-(2.6) where it is also assumed that along Hamiltonian trajectories*

$$\mathrm{dist}(\mathrm{x(t)},\partial \mathrm{X}) \geq \gamma_0 \ . \tag{3.9}$$

Then estimate (2.7) remains true. Moreover, assertions (ii), (iii) of the previous theorem remain true in this new situation.

On the other hand let us consider a Schrödinger operator

$$A = \Sigma(hD_j - V_j)g^{jk}(hD_k - V_k) + V \tag{3.10}$$

with real-valued functions $g^{jk} = g^{kj}, V_j, V$ such that

$$\Sigma g^{jk}\xi_j\xi_k \geq c^{-1}|\xi|^2 \quad \forall \xi \in \mathbf{R}^d \quad \forall x\in B(0,1) \tag{3.11}$$

Then (1.11), (3.7) are fulfilled and $(1.19)_2^+$ is fulfilled automatically provided $d\geq 2$.

Theorem 8.— *Let $d \geq 2$ and conditions (3.10), (3.2), (1.4), (3.4), (3.11) be fulfilled and*

$$X\cap B(0,1) = \{\varphi(x)\geq 0\}\cap B(0,1) \quad \textit{where} \quad |D^\alpha \varphi|\leq C, |d\varphi|\geq c^{-1} \quad \textit{in} \quad B(0,1). \tag{3.12}$$

Moreover let us assume that

(3.13) *On $\partial X\cap B(0,1)$ either Dirichlet or Neumann condition*

$$\Sigma g^{jk}(\partial_j\varphi)(hD_k - V_k)u = 0$$

is given.

Then all the conclusions of theorems 1-4 remain true with $Q_j = \psi_j$ and (3.8) also remains true.

Theorem 9.— *Let all the conditions of the previous theorem be fulfilled and let closed sets $\Lambda_1 \cdots, \Lambda_n$ and numbers $1 < T_1 \leq \cdots \leq T_n$ satisfy (2.1)-(2.6) ; here we consider Hamiltonian trajectories with reflection at ∂X and we also assume that along these trajectories*

$$\Sigma g^{jk}\eta_j\eta_k \geq C^{-1}|\eta|^2\rho_0 \quad \forall\eta \in \mathbf{R}^d \tag{3.14}$$

and in every point of these trajectories either (3.9) is fulfilled or

$$X \cap B(x(t),\gamma_0) = \{\varphi(x) \geq 0\} \cap B(x(t),\gamma_0) \quad \text{where} \quad |D^\alpha\varphi| \leq \rho_0^{-1} \quad \forall\alpha : |\alpha| \leq K , \tag{3.15}$$

$$|\Sigma g^{jk}(\xi_j - v_j)\partial_k\varphi| \geq \rho_0 \tag{3.16}$$

and on $\partial X \cap B(x(t),\gamma_0)$ either Dirichlet or Neumann boundary condition is given ; here φ and the type of the boundary condition depend on the point of trajectory.

Then estimate (2.7) remains true with $Q_j = \psi_j$.

Theorem 8 in fact is proved for more general operators and boundary value problems ; theorem 9 is proved for slightly more general boundary condition. Moreover under naturally modificated conditions both these theorems are also proved for a Dirac operator uniformly with respect to mass $m \in [0,\infty)$; in this case we assume that (3.3) is fulfilled and replace $E(\tau)$ by $E(\tau) - E(\tau')$ with finite τ, τ'.

4. Applications

We discuss here only applications of theorem 9.

(i) Let X be either polyhedral (or polygonal for $d = 2$) domain or a ball or a planar elliptic domain and $A = -\Delta +$ *lower order terms* with the Dirichlet boundary condition at $Y_0 \subset \partial X$ and the Neumann boundary condition at $Y = \partial X \setminus Y_0$ (This operator is defined by a quadratic form). Let us assume that

$$\text{mes}_{\partial X}\{x \in \partial X, \text{dist}(x, \bar{Y}_0 \cap \bar{Y}) < \varepsilon\} = 0(\varepsilon^\delta) \tag{4.1}$$

as $\varepsilon \to 0$ where $\bar{Y}$ and $\bar{Y}_0$ are closures of Y_0 and Y_1, $\text{mes}_{\partial X}$ denotes $(d-1)$-dimensional Lebesgue measure at $\partial X, \delta > 0$. Then

$$N(\lambda) = \kappa_0\lambda^{d/2} + (\kappa_1 + 0(\lambda^{-\beta}))\lambda^{(d-1)/2} \quad \text{as} \quad \lambda \to +\infty \tag{4.2}$$

where $N(\lambda)$ is an eigenvalue counting function, $\kappa_0 = (2\pi)^{-d}\omega_d \text{ vol } X, \kappa_1 = \frac{1}{4}(2\pi)^{1-d}\omega_{d-1}$ $(\text{mes}_{\partial X}Y - \text{mes}_{\partial X}Y_0), \omega_d$ is a volume of $B(0,1)$ in $\mathbf{R}^d$, $\beta = \beta(d,\delta) > 0$.

It is very likely that for a general planar domain with smooth strictly convex boundary the remainder term is $0(\lambda^{(d-1)/2}/\ln\lambda)$ but it is not proved.

(ii) Let $X = \mathbf{R}^d$, A is given by (3.10) with $h = 1$ where $g^{jk} = g^{kj}, V_j, V$ are real-valued smooth functions, (3.11) is fulfilled in $\mathbf{R}^d$ and

$$|D^\alpha(g^{jk} - \delta_{jk})| = O(|x|^{-|\alpha|-\delta}), \tag{4.3}$$

$$|D^\alpha V_j| = O(|x|^{m-|\alpha|-\delta}),$$
$$|D^\alpha(V - V^0)| = O(|x|^{2m-|\alpha|-\delta})$$
$$\forall\alpha : |\alpha| \le K \quad \text{as} \quad |x| \to \infty$$

with $\delta > 0, V^0 = \zeta|x|^{2m}$.

Then for $m > 0, m \ne 1, \zeta > 0$

$$N(\lambda) = N^w(\lambda) + O(\lambda^{q(d-1)-\sigma}) \quad \text{as} \quad \lambda \to +\infty \tag{4.4}$$

and for $-1 < m < 0, m \ne -1/2, \zeta < 0$

$$N(\lambda) = N^w(\lambda) + O(\lambda^{q(d-1)+\sigma}) \quad \text{as} \quad \lambda \to -0 \tag{4.4$'$}$$

where in both cases

$$N^w(\lambda) = (2\pi)^{-d}\omega_d \int (\lambda - V)_+^{d/2}\sqrt{g}\; dx \tag{4.5}$$

is a Weylian expression, $g = \det(g^{jk})^{-1}, N^w(\lambda) \asymp \lambda^{qd}, g = (m+1)/2m, \sigma = \sigma(d,\delta,m) > 0$ and the best known before remainder estimate is $o(\lambda^{q(d-1)})$.

Here quadratic and Coulomb potentials V^0 are forbidden because for these potentials all the trajectories are periodic (these trajectories are ellipses with the center in the origin and Keplerian ellipses respectively).

5. Let us consider now a case of periodic Hamiltonian trajectories. We assume that

(5.1) $A = a^w(x, hD)$ is a scalar operator, (1.1), (1.5) are fulfilled and in Ω a, a_0 and $h^{-1}(a - a_0)$ satisfy (1.1) with $\rho = \gamma = 1$,

$$|da| \ge C^{-1} \quad \text{in} \quad \Omega\,, \tag{5.2}$$

$$\phi_t(\Omega) = \Omega \quad \forall t \in \mathbf{R}\,, \tag{5.3}$$

$$\forall (x,\xi) \in \Omega\ \exists T = T(x,\xi) > 0 \quad \text{such that} \quad \phi_T(x,\xi) = (x,\xi) \tag{5.4}$$

where ϕ_t is a Hamiltonian flow generated by $a_0(x,\xi)$.

It is well-known from the Hamiltonian mechanics that in this case $T(x,\xi) = T_0(a_0(x,\xi))$ for a generic point $(x,\xi) \in \Omega$ where T_0 is a smooth function (however the existence of subperiodic points with $T_0(x,\xi)/T(x,\xi) \in \mathbf{Z} \setminus \{1\}$ is possible) ; if we replace A by operator $\widetilde{A} = f(A)$ then we can take $\widetilde{a}_0 = f(a_0)$ and in this case $\widetilde{T}_0(\widetilde{a}_0) = T_0(a_0)/f'(a_0)$

where f' denotes the derivative of f and hence for an appropriate function f we obtain $T_0 \equiv 1$; so without a loss of generality we can replace (5.4) by a condition

$$\text{(5.4)*} \qquad \phi_1(x,\xi) = (x,\xi) \quad \forall (x,\xi) \in \Omega \ .$$

It is also well-known that in this case

$$\text{(5.5)} \qquad \exp ih^{-1}A \equiv \exp iB \quad \text{in} \quad \Omega_1$$

where

(5.6) $B = b^w(x, hD, h)$ is a self-adjoint operator and $b(x,\xi,h)$ satisfies (1.1) with $\rho = \gamma = 1$ for $\alpha \neq 0$,

$$\text{(5.7)} \qquad b = \int_0^1 h^{-1}(a - a_0) \circ \phi_t dt + \theta_0 h^{-1} + \theta_1 + O(h),$$

an action $\theta_0 = \int_0^1 (a - a_\xi \xi) \circ \phi_t \, dt$ is a constant in Ω and θ_1 is the Maslov' constant. Here $P \equiv P'$ denotes that norms of operators $(P - P')Q$ and $Q(P - P')$ are less than Ch^s for every operator $Q = q^w(x, hD)$ satisfying (1.1) and (1.6).

However it is possible that symbol $b(x,\xi,h)$ is uniformly small in Ω and hence let us assume that

$$(5.5)' \qquad \exp ih^{-1}A \equiv \exp i\eta B \quad \text{in} \quad \Omega \quad \text{where} \quad B \quad \text{satisfies (5.6) and} \quad \eta \in [h^\ell, 1]$$

On the other hand if we take $A = a_0^w(x, hD) + \mu a_1^w(x, hD)$ with a_1 satisfying (1.1) with $\rho = \gamma = 1$ and $\mu \in [h, h^\delta]$ then it is easy to prove that for this operator condition (5.5)' is fulfilled with $\eta = \mu h^{-1}$ and with

$$b = \int_0^1 a_1 \circ \phi_t dt + \mu^{-1}(\theta_0 + \theta_1 h) + O(\mu) \ .$$

Hence let us assume that $\eta \in [h^\ell, h^{-\delta}]$ and replace (5.1) by

(5.1)' $A = a^w(x, hD)$ is a scalar operator, (1.1), (1.5) are fulfilled and in Ω a, a_0 and $h^{-1}(\eta + 1)^{-1}(a - a_0)$ satisfy (1.1) with $\rho = \gamma = 1$.

First of all we consider gaps in the semi-classical approximation to the spectrum :

Theorem 10.— *Let conditions (1.4), (5.1)', (5.2), (5.3), (5.4)*, (5.5)', (1.5) and (1.6) be fulfilled. Moreover let condition*

$$\text{(5.8)} \qquad |\tau/h - \eta b - 2\pi m| \geq \eta h^{2-\delta} \quad \forall (x,\xi) \in \Omega \quad \forall m \in \mathbf{Z}$$

be fulfilled for every $\tau \in [\tau_1, \tau_2]$. Then (1.9), (1.9)' hold.

The most important and sophisticated is the following

Theorem 11.— *Let conditions (1.4), (5.1)', (5.2), (5.3), (5.4)*, (5.5)', (1.5) and (1.6) be fulfilled and let $\Lambda_1, \cdots, \Lambda_n$ be closed subsets of Ω and $h(\eta+1) \le T_1 \le \cdots \le T_n \le 1$ numbers such that for $\tau = \tau^0$*

(5.9)$_\tau$ For every $(x,\xi) \in \Omega$ either $|db_\tau| \ge c^{-1}$ or Hess $b_\tau(x,\xi)$ has r eigenvalues $f_1, \cdots, f_r$ (counting their multiplicities) with $|f_j| \ge C^{-1}$ $\forall_j$

where b_τ is a restriction of b to $\Sigma_\tau = \{(x,\xi) \in \Omega, a_0(x,\xi) = \tau\}$ and

(5.10) For every $(x,\xi) \in \Omega_j$;

$$\operatorname{dist}(\psi_{t'}\phi_t(x,\xi),(x,\xi)) \ge \gamma_0 = h^{1/2-\delta} + C\eta h \quad \forall t \in (\varepsilon, 1-\varepsilon) \quad \forall t' : |t'| \le T_j$$

where $\psi_{t'}$ is a Hamiltonian flow generated by $b(x,\xi)$ and $\varepsilon = \varepsilon(d,\delta,c) > 0$ is small enough.

Then the following estimate holds for $\tau = \tau^0$:

$$R_2 = |Tr\ Q_1 E(\tau) Q_2 - \kappa_0(\tau)h^{-d} - \kappa_1(\tau)h^{1-d} - F_0(\tau,\tau/h)h^{1-d}| \le \tag{5.11}$$
$$\le C\sum_{j=0}^{n} h^{2-d}(\eta+1)T_j^{-1}\int_{\Sigma_\tau\cap\Lambda_j} dxd\xi : da_0 + C\gamma_0^\tau h^{1-d}$$

where $\Lambda_0 = \Omega \setminus (\Lambda_1 \cup \cdots \cup \Lambda_n), T_0 = h(\eta+1)$ and

$$F_0(\tau,\xi) = (2\pi)^{-d}\int_{\Sigma_\tau} f(\xi - \eta b)q_1q_2 dxd\xi : da_0 \tag{5.12}$$

with 2π-periodic function $f(\xi), f(\xi) = \pi - \xi$ at $[0,2\pi]$.

It is obvious that we can omit the last term in the right-hand expression in (5.11) provided either $r \ge 3$ or $\eta \ge h^{\tau/2-\delta-1}$. Moreover, for $\eta \ge h^{-\beta}$ with $\beta = 2/(r+2)$ the non-Weylian term $F_0(\tau,\tau/h)h^{1-d}$ is negligible and the same estimate remains true for R_1 ; here and before r is arbitrary provided $|db_\tau| \ge C^{-1}$ in Ω. On the other hand this non-Weylian term is very important for $\eta \le 1$ because in this case its oscillation at the interval $[\tau_1, \tau_2]$ with $\tau_2 - \tau_1 \asymp h$ is of the same order as oscillation of the principal term (moreover inside the gap their oscillations compensate one another).

Let us return back to spectral gaps :

Theorem 12.— *Let conditions (1.4), (5.1)', (5.2), (5.3), (5.4)*, (5.5)', (1.5) and (1.6) be fulfilled. Moreover let condition (5.10) be fulfilled for $j = 1, T_1 \ge h^{1-\delta}(\eta+1)$ and (5.8) be fulfilled for some τ. Then*

$$|Tr\ Q_1E(\tau)Q_2 - \sum_{n=0}^{N}\kappa_n(\tau)h^{n-d} - \sum_{n=0}^{N} F_n(\tau,\tau/h)h^{n+1-d}| \le \tag{5.13}$$
$$\le Ch^s + Ch^{1-d}\int_{\Sigma_\tau\cap\Lambda_0} dx\ d\xi : da_0$$

where $\Lambda_0 = \Omega \setminus \Lambda_1$,

$$F_k(\tau,\xi) = \int_{\Sigma_\tau} f_k(\zeta - \eta b, x, \xi) dxd\xi\ , \tag{5.14}$$

$f_k(\zeta,x,\xi)$ are 2π-periodic with respect to ζ functions, smooth at $[0,2\pi]\times\mathbf{R}^{2d}$ with jumps at $\zeta = 2\pi m (m \in \mathbf{Z})$, supported in $\mathbf{R}\times\Omega_1$.

Our investigation was inspired by papers of Yu. Safarov ; his assertions are more general but our results are more precise.

6 Variants and modifications.

(i) Let us assume that $ah^\sigma, a_0 h^\sigma, h^\sigma, h^{\sigma-1}(\eta+1)^{-1}(a-a_0), h^\sigma b, h^\sigma q_j$ satisfy in Ω (1.1) with $\rho=\gamma=h^\sigma$ and that supp $q_j \subset \Omega_1 \subset \Omega$, dist $(\Omega_1, \partial\Omega) \geq h^\sigma$; we no longer assume that it is true for $\sigma=0$ (look conditions (5.1)', (5.6), (1.5), (1.6)) ; moreover, let us replace in (5.2), $(5.9)_2$ ineqalities $|da| \geq C^{-1}, |db_\tau| \geq C^{-1}, |f_j| \geq C^{-1}$ by $|da| \geq h^\sigma, |db_\tau| \geq h^\sigma, |f_1| \geq h^\sigma$ respectively. Then for small enough $\sigma=\sigma(d,\delta,s)>0$ theorems 11-13 remain true with an additional factor $h^{-\delta}$ in the right-hand expressions of every estimate.

(ii) For h-differential operators our results remain true after modifications like in section 3 provided $\Omega \subset X \times \mathbf{R}^d$; so here we consider Hamiltonian flows without reflections.

7. Applications

(i) Let $\bar{A}=-\Delta$ be a positive Laplace-Beltrami operator on $X=\mathbf{S}^d, d\geq 2$. It is well-known that the spectrum of $\bar{A}$ consists of eigenvalues $\bar{\lambda}_n=n(n+d+1)$ of multiplicities $r(n)$ where here and below $r(n)$ is a polynomial of degree $d-1$ and $n \in \mathbf{Z}^+$. Hence $|\bar{\lambda}_{n+1}-\bar{\lambda}_n| \asymp \sqrt{\lambda_{n+1}}$.

Let us consider a perturbed operator

$$A=-\Delta+A' \tag{7.1}$$

with a differential operator A'. If the perturbation A' is either a first-order operator with small coefficients or a potential V then A has a cluster spectrum

$$\text{Spec } A=\{\lambda_{n,j}, j=1,\cdots,r(n), n\in\mathbf{Z}^+\} \tag{7.2}$$

with

$$|\lambda_{n,j}-\bar{\lambda}_n| \leq C_0\nu\sqrt{\lambda_{n+1}}, \tag{7.3$_1$}$$

$$|\lambda_{n,j}-\bar{\lambda}_n| \leq C_0\nu \tag{7.3$_0$}$$

where $C_0=C_0(d)$, index denotes an order of A' and ν is a C-norm of coefficients of A'.

Moreover calculating $\exp 2\pi i A$ one can improve these estimates for perturbations anti-commutating with the antipodal map ρ,

$$\rho A'=-\rho A' \ , \ (\rho u)(x)=u(-x) \ . \tag{7.4}$$

Namely, in this case

$$|\lambda_{n,j}-\bar{\lambda}_n| \leq C_0\nu^2, \tag{7.5$_1$}$$

$$(7.5)_0 \qquad |\lambda_{n,j} - \bar{\lambda}_n| \le C_0 \nu^2 \lambda_{n+1}^{-1}$$

where here ν is a C^K-norm of coefficients.

Let us consider a $(2d-2)$-dimensional manifold $\mathcal{K} \simeq S^*(\mathbf{S}^d)/\mathbf{S}^1$ of Hamiltonian trajectories on $S^*(\mathbf{S}^d)$ where S^*X is a sphere bundle over X and let us consider a function

$$b(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} a' \circ \phi_t dt$$

where a' is a principal symbol of A'.

Then theorem 12 implies that

$$(7.6) \qquad N(\lambda) = \kappa_0 \lambda^{d/2} + F(\sqrt{\lambda})\lambda^{(d-1)/2} + O(\lambda^{(d-2)/2+\sigma}) \quad \text{as} \quad \lambda \to +\infty$$

with an arbitrary $\sigma > 1/4, \sigma > 0, \sigma = 0$ provided

$$(7.7)_r \qquad db = 0 \Rightarrow \text{ rank Hess } b \ge r$$

with $r = 1, 2, 3$ respectively where

$$(7.8)_1 \qquad F(\xi) = (2\pi)^{-d} \int_{S^*(\mathbf{S}^d)} \bar{f}(\xi - \frac{d-1}{2} - b(x,\xi))d\mu,$$

$$(7.8)_0 \qquad F(\xi) = (2\pi)^{-d} \int_{S^*\mathbf{S}^d} \bar{f}(\xi - \frac{d-1}{2} + \frac{(d-1)^2}{8}\lambda^{-1/2} - b(x,\xi)\lambda^{-1/2})d\mu,$$

$$d\mu = dxd\xi : d|\xi|, \bar{f}(\xi) = f(2\pi\xi)/2\pi \ .$$

It is easy to prove that for a generic p-th order operator $A'(p = 0, 1)$ condition $(7.7)_r$ is fulfilled with $r = 3 (r = 2)$ provided $d \ge 3, d = 2$ respectively and hence for a generic p-th order operator $A'(p = 0, 1)$ asymptotic formula (7.6) has place with arbitrary $\sigma > 0 (\sigma = 0)$ provided $d = 2$ ($d \ge 3$ respectively).

It is easy to prove that these assertions remain true for a generic p-th order operator A' commutating with the antipodal map ρ.

On the other hand $b \equiv 0$ provided (7.4). However it is possible to define in this case symbol $b(x,\xi)$ positively homogeneous of degree $2p-3$ with respect to ξ such that

$$\exp 2\pi i((A + (d-1)^2/4)^{1/2} - (d-1)/2) \equiv \exp iB, B = b^w(x,D) + \ell.o.t.$$

and transfer above results to this case. One can obtain similar results for a general manifold with a periodic geodesic flow ; however in this case certain hypotheses concerning the behaviour of this flow near subperiodic points are assumed.

(ii) Let $\bar{A} = -\Delta + |x|^2$ in $\mathbf{R}^d, d \geq 2$. Then its spectrum consists of eigenvalues $\bar{\lambda}_n = 2n + d$ of multiplicities $r(n), n \in \mathbf{Z}^+$. Let us perturb $\bar{A}$ by an operator $A' = a'^w(x, D)$ where

$$a'(x,\xi) = \Sigma_j W_j(x)\xi_j + V(x), \quad a'(x,\xi) = W(x)$$

for $p = 0, 1$ respectively. Then for $p = 0$ and small enough $\overline{\lim}|V|$ as $|x| \to \infty$ spectrum has the cluster character.

On the other hand let us assume that

$$(7.9)_1 \qquad D^\alpha(W_j - W_j^0) = O(|x|^{m-1-|\alpha|-\delta}),$$

$$(7.9)_2 \qquad D^\alpha(V - V^0) = O(|x|^{m-|\alpha|-\delta}),$$

as $|x| \to \infty \quad \forall \alpha : |\alpha| \leq K$ where $m < 2, \delta > 0, K = K(d, m, \delta)$ and $W_j^0 = |x|^{m-1} w_j(x/|x|), V^0 = |x|^m v(x/|x|)$.

Then for $p = 0$ and generic $v \in C^K(\mathbf{S}^{d-1})$ and for $p = 1$ and generic $(v, w_1, \cdots, w_d) \in (C^K(\mathbf{S}^{d-1}))^{d+1}$

$$(7.10) \qquad N(\lambda) = N^w(\lambda) + F(\lambda)\lambda^{d-1} + O(\lambda^{d-1-\sigma}) \quad \text{as} \quad \lambda \to +\infty$$

where $\sigma = \sigma(d, m, \delta) > 0$ is small enough,

$$(7.11) \qquad F(\lambda) = (2\pi)^{-d} \int_\Sigma \bar{f}(\lambda - \lambda^{m/2} b(x,\xi)) d\mu,$$

$$\Sigma = \{(x,\xi), h(x,\xi) = 1\}, h = |x|^2 + |\xi|^2, d\mu = dx d\xi : dh,$$

$$(7.12) \qquad b(x,\xi) = \frac{1}{\pi} \int_0^\pi a'_0 \circ \phi_t dt,$$

$a'_0 = \Sigma_j W_j^0(x)\xi_j + V^0(x), \phi_t$ is the Hamiltonian flow generated by h. Moreover, for $m > 0$ (7.10) remains true with $F(\lambda) = 0$.

(iii) Let $\bar{A} = -\Delta - |x|^{-1}$. Then its spectrum also consists of eigenvalues λ_n of multiplicities $r(n)$ with $\lambda_n \sim -1/2n^2, (\lambda_{n+1} - \lambda_n) \sim 1/n^3$ as $n \to \infty$. (Moreover, for odd d modulo finite number of eigenvalues $\lambda_n = -1/2n^2$).

Let us perturb $\bar{A}$ by the same operator A' as before and assume that

$$(7.9)'_1 \qquad D^\alpha(W_j - W_j^0) = O(|x|^{m+1/2-|\alpha|-\delta}),$$

$$(7.9)'_2 \qquad D^\alpha(V - V^0) = O(|x|^{m-|\alpha|-\delta}),$$

as $|x| \to \infty \quad \forall \alpha : |\alpha| \leq K$ where $m < -1, \delta > 0, \; K = K(d, m, \delta)$ and $W_j^0 = |x|^{m+1/2} w_j(x/|x|), V^0 = |x|^m v(x/|x|)$.

Then for $p = 0$ and generic v and for $p = 1$ and generic $(v, w_1, \cdots, w_d)$

$$(7.10)' \qquad N(\lambda) = N^w(\lambda) + F(|\lambda|^{-1/2})|\lambda|^{-(d-1)/2} + O(\lambda^{-(d-1)/2+\sigma}) \quad \text{as} \quad \lambda \to -0$$

where $\sigma = \sigma(d, m, \delta) > 0$ is small enough and $F(\zeta)$ is given by (7.11)-type formula with $\lambda^{m/2}$ replaced by $\lambda^{2m+3}, \Sigma = \{(x,\xi), h(x,\xi) = -1\}$ and $h = |\xi|^2 - |x|^{-1}$ and for $m > -3/2$ (7.10)' remains true with $F(\xi) = 0$.

Remark : We didn't treat in (ii), (iii) the highly sophisticated case of odd v and even $(w_1, \cdots, w_d)$; in this case $\delta > 0$ is no longer arbitrary small.

8. Idea of proofs

I discuss only proofs of the most sophisticated and refined theorems 5,11 (proofs of theorem 7,9 are based on the same ideas as proof of theorem 5 ; however proof of theorem 9 is more complicated because of the presence of the boundary). In order to prove these theorems it is necessary for us to treat a long-time propagation of singularities and apply obtained results to the investigation of singularities of the function

$$\sigma_Q(t) = Tr\ QU(t) = \int tr(Q_x u)(x,x,t)dx$$

where $u(x,y,t)$ is a Schwartz kernel of $U(t) = \exp ih^{-1}tA$.

Let us start from

Definition 13.— *Let v be a family of functions such that $\|v\| \le ch^{-n}$. Then v is s-negligible in the family of boxes $\Pi = \{(x,\xi) : |x_j - \bar{x}_j| \le \gamma_j, |\xi_j - \bar{\xi}_j| \le \rho_j\ \forall_j = 1,\cdots,d\}$ with $\rho_j\gamma_j \ge h^{1-\delta}\quad \forall_j$ (uncertainly principle) if and only if $\|q^w(x,hD)v\| \le Ch^s$ for every symbol q supported in Π and satisfying inequalities*

$$|D_x^\alpha D_\xi^\beta q| \le C\rho^{-\beta}\gamma^{-\alpha}\quad \forall \alpha,\beta : |\alpha| + |\beta| \le K, K = K(d,\delta,n,s)\ .$$

A standard definition with fixed Π i.e. with fixed $(\bar{x},\bar{\xi})$ and $\rho_1,\cdots,\gamma_d$ guids us to a notion of oscillatory front set which is too rough for our purposes.

In the proof of theorem 5 the crucial step is

Theorem 14.— *Let (1.1) - (1.6) be fulfilled with $\rho = \gamma = \gamma_0 = h^{1/2-\delta}$ and let $(\bar{x},\bar{\xi}) \in \Omega$ and through every point of γ_0-neighbourhood of $(\bar{x},\bar{\xi})$ there passes a Hamitlonian trajectory $\phi_t(x,\xi)$ generated by $-\lambda(x,\xi)$ with either $t \in [0,T]$ or $t \in [-T,0], 0 \le T \le h^{-\sigma}$ along which (1.1), (2.2) with $\lambda_j = \lambda$ and (2.5) are fulfilled with $\rho = \gamma = \rho_0 = h^\sigma$ with small enough $\sigma = \sigma(d,\delta,n,s) > 0$. Let $\|v\| \le Ch^{-n}$ and $v_{|t=0}$ and $(hD_t - a^w(x,hD))v$ be negligible in γ_0-neighbourhoods of $(\bar{x},\bar{\xi})$ and $\cup_t\phi_t(\bar{x},\bar{\xi})$ respectively. Then v is negligible in γ_1-neighbourhood of $\cup_t\phi_t(\bar{x},\bar{\xi})$ with $\gamma_1 = h^{1/2-\delta/2}$.*

Here $\rho_1,\cdots,\gamma_d$ must be equal because of a rotation along trajectories and hence uncertainty principle yields that $\rho_1 = \cdots = \gamma_d \ge h^{1/2-\delta}$; hence the elementary (the lowest distinguishable) distance is $\gamma_0 = h^{1/2-\delta}$.

The proof of this theorem I discuss only for operators with scalar principal symbols ; the general case can be reduced to this one. Let us consider operators $\bar{Q}$ and $Q(t) = U(t)\bar{Q}U(-t)$; then

$$(8.1)_{1,2}\qquad \frac{d}{dt}Q(t) = ih^{-1}[A,Q(t)]\ ,\ Q(0) = \bar{Q}\ .$$

Assuming that $Q(t)$ is an h-pseudo-differential operator we replace (8.1) by a sequence of recurrent Cauchy problems

$$(8.2)_1\qquad \partial_t q_0 - \{a_0,q_0\} - i[a^s,q_0] = 0 \text{ etc,}$$

$$(8.2)_2 \qquad q_n|_{t=0} = \bar{q}_n$$

where $q = \Sigma_n q_n h^n$ etc, a^s is the subprincipal symbol of A.

It is easy to prove in frames of theorem 14 that if $\bar{q}_n$ are supported in γ_0-neighbourhood Ω' of $(\bar{x}, \bar{\xi})$ and satisfy (1.1) with $\rho = \gamma = \gamma_0$ then solutions $q_n(t)$ of $(8.2)_{1,2}$ are supported in $U_t \phi_t(\Omega')$ and satisfy

$$|D_x^\alpha D_\xi^\beta q_n| \le C\gamma_0^{-2n-|\alpha|-|\beta|} h^{-\sigma L} \quad \forall \alpha, \beta : |\alpha| + |\beta| + 2n \le K/2$$

with $L = L(n, \alpha, \beta)$ and hence $q = \Sigma_{n \le k/\sigma} q_n h^n$ satisfies (1.1) with $\rho = \gamma = h^{1/2-\delta/2}$. Then $Q'(t) = q^w(t, x, hD, h)$ satisfies $(8.1)_{1,2}$ modulo negligible operators and then it is easy to prove that $Q(t) - Q'(t)$ also is a negligible operator. Theorem 14 is a simple consequence of this construction.

Finally we obtain from theorem 14 that $\sigma_Q(t)$ is negligible for $h^\sigma \le |t| \le T_j/2$ provided Q is supported in the γ_0-neighbourhood of Λ ; and all the conditions of theorem 5 are fulfilled ; similar assertions are proved also in frames of theorems 7,9.

In the case of periodic Hamiltonian flow we are able to treat a very long time propagation. First of all we prove that (5.5)' yields an equality

$$(8.3) \qquad \exp inh^{-1}A \equiv \exp in\eta B \quad \text{in} \quad \Omega_1 \quad \forall n \in \mathbf{Z}, |n| \le \varepsilon/\eta h$$

with a small enough constant ε. Then we prove that

$$(8.4) \qquad \exp ith^{-1}A \equiv \exp it_1 h^{-1}A \exp it_2 h^{-1} B \quad \text{in} \quad \Omega_2$$

$$\forall t : |t| \le \varepsilon/\eta h, t_2 = h\eta[t], t_1 = \{t\} = t - [t]$$

where $\Omega_2 \subset \Omega_1$, dist $(\Omega_2, \partial\Omega_1) \ge c^{-1}$.

Hence singularities of u propagate along bicharacteristics of a_0 which are drifting along bicharacteristics of b with the velocity ηh provided $\eta \le h^{-1/2-\sigma}$ and a slightly modified assertion remains true for $\eta \in [h^{-1/2-\sigma}, h^{\delta-1}]$.

Let us assume for a sake of simplicity that $|db_\tau| \ge c^{-1} \quad \forall (x,\xi) \in$ Supp Q. Then it follows from (8.4) that $\sigma_Q(t)$ is negligible $\forall t : h^{-\sigma}\eta^{-1} \le |t| \le \varepsilon/2\eta h, |\{t\}| \le \varepsilon$. Then $\sigma_Q(t)$ is negligible $\forall t : h^{-\sigma}\eta^{-1} \le |t| \le T/\eta h$ provided (5.10) is fulfilled on Supp Q with T_j replaced by $T, h(\eta+1) \le T \le \varepsilon^2$. Then for $\eta \ge h^{-\sigma}$ we obtain immediately for $Tr(Qe)(\tau)$ the Weylian asymptotics with a remainder estimate

$$Ch^{2-d}\eta T^{-1} \int_{\Omega' \cap \Sigma_\tau} dx d\xi : da_0 + Ch^s$$

where Ω' is γ_0-neighbourhood of Supp Q.

In the case $\eta \le h^{-\sigma}$ our considerations are more sophisticated. We observe that for $n \in \mathbf{Z}, |n| \le \eta^{-1}h^{-\sigma} \exp in\eta B$ is an h-pseudo-differential operator with a symbol satisfying

(1.1) with $\rho = \gamma = h^{-\sigma'}$ where $\sigma' \to +0$ as $\sigma \to +0$. Then using (8.4) we can calculate modulo $O(h^s)$

$$F_{t\to h^{-1}\tau}\chi_{T'}(t)\sigma_Q(t)$$

with $T' = \min(\eta^{-1}h^{-\sigma}, T/\eta h)$ and moreover with $T' = T/\eta h$ where $\chi \in C_0^K([-1,1]), \chi = 1$ at $[-1/2, 1/2], \chi_{T'}(t) = \chi(t/T')$. Then Tauberian arguments give an estimate

$$|Tr\ QE(\tau) - \kappa_0(\tau)h^{-d} - \kappa_1(\tau)h^{1-d} - F_0(\tau, \tau/h)h^{1-d}| \leq Ch^{2-d}T^{-1}M + Ch^s$$

where $M = \int_{\Omega'\cap\Sigma_\tau} dxd\xi : da_0$ and F_0 is connected with singularities of $\sigma_Q(t)$ at $t \in \mathbf{Z}\setminus 0$:

$$|\int_{-\infty}^{\tau}(F_{t\to h^{-1}\lambda}(\chi_{T'}(t) - \chi_\varepsilon(t))\sigma(t))d\lambda - h^{2-d}F_0(\tau, \tau/h)| \leq CMh^{3-d}.$$

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