Séminaire Équations aux dérivées partielles – École Polytechnique

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Asymptotic solutions of equations with slowly varying coefficients

Séminaire Équations aux dérivées partielles (Polytechnique) (1989-1990), exp. nº 21, p. 1-20

<http://www.numdam.org/item?id=SEDP_1989-1990____A23_0>

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Séminaire 1988-1989

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

ASYMPTOTIC SOLUTIONS OF EQUATIONS WITH SLOWLY VARYING COEFFICIENTS

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23 Mai 1989

ASYMPTOTIC SOLUTIONS OF EQUATIONS WITH SLOWLY VARYING COEFFICIENTS

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An important problem which has been investigated for the last years is the propagation and decay problem for bell-shaped and shock wave-like initial conditions for equations of mathematical physics. Such problems arise when we consider equations with constant coefficients such as Korteweg de-Vries (KdV) equation, Kolmogorov, Petrovsky and Piskunov (KPP) equation, Sine-Gordon equation etc. We want to understand how we can correctly generalize these results in the case of equations with variable coefficients and to notice certain general properties which are characteristical for asymptotics of both linear and nonlinear problems. This approach allows to look at certain well-known effects from a principally new point of view.

First we consider the Cauchy problem for the KdV equation

$$\frac{\partial u}{\partial t} + \alpha_1 u \frac{\partial u}{\partial x} + \alpha_2 \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in R^1, \quad u|_{t=0} = \Phi(x), \tag{1}$$

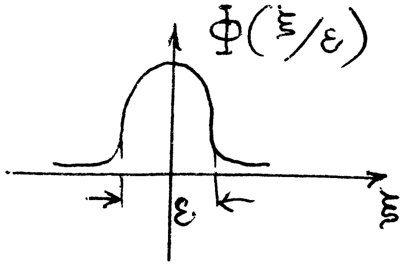
where $\Phi(x)$ is a smooth rapidly decreasing function. The exact solution of this problem for an arbitrary function $\Phi(x)$ is, of course, unknown. However, as it is well-known, in the case α_1, α_2 = const the solution can be defined for large t. Namely, the Cauchy problem solution (1) can be represented for $t \to \infty$ (accurate to the summands tending to zero as $t \to \infty$)

in the form of solitons sum, the quantity and velocities of which depend on the function $\Phi(x)$ and can be calculated. This result cannot be generalized in the case when α_1 and α_2 are not constants, but arbitrary smooth functions. However, a class of variable coefficients exists for which we can generalize the Theorem on the Cauchy problem solution asymptotics for $t \to \infty$. Such coefficients are called slowly varying coefficients. Let $0 < \varepsilon << 1$ be a small parameter. The coefficients α_i are called slowly varying if $\alpha_i = \alpha_i(\varepsilon x)$, $i=1,2$; $\alpha_i(\xi) \in C^\infty$. It is evident that the coefficients $\alpha_i = \alpha_i(\varepsilon x)$ vary by a value $O(\varepsilon)$ on any compact $K \subset R^1_x$ ($x \in K$) independent of ε. This fact explains the name given to this class of coefficients. The equation with slowly varying coefficients can be rewritten in the form of an equation with a small parameter at the derivatives. We set $\varepsilon x = \xi$, $\varepsilon t = t'$. Then problem (1) with slowly varying coefficients can be rewritten in the form

$$\frac{\partial u}{\partial t'} + \alpha_1(\xi) u \frac{\partial u}{\partial \xi} + \varepsilon^2 \alpha_2(\xi) \frac{\partial^3 u}{\partial \xi^3} = 0$$
$$u\big|_{t=0} = \Phi\left(\frac{\xi}{\varepsilon}\right). \qquad (2)$$

We note that $t = \frac{t'}{\varepsilon}$ since $t \to \infty$ for $\varepsilon \to 0, t' > 0$. Thus, the <u>first</u> important fact is that the problem considering the behaviour of problem (1) solution can be reduced as $t \to \infty$ to a problem dealing with the asymptotics construction for problem (2) solution as $\varepsilon \to 0$. The KdV equation in the form (2) is one of the problems for which the Whitham method [2] of asymptotical solutions construction is developed.

We also note that since the function $\Phi(x)$ decreases rapidly, the initial value of problem (2) is small for $\xi > 0$, $\varepsilon \to 0$. We can draw this function as follows



We recall the methods of the Cauchy problem solutions asymptotics construction for linear equations which can be regarded as models of problem (2).

As the first example we consider the wave equation (denoting again the "slowly " variable εx by x, and εt by t)

$$\Box_c u = u_{tt} - c^2(x) u_{xx} = 0,$$
$$u_{t=0} = \Phi\left(\frac{x}{\varepsilon}\right), \quad u'_t\Big|_{t=0} = 0. \tag{3}$$

We assume that $c(x) \in C^\infty$, $c(x) \geq \delta > 0$.

A function satisfying (3) accurate to $O(\varepsilon^M)$ and having the form

$$u = f^0\left(\frac{S(x,t)}{\varepsilon}, x, t\right) + \varepsilon f^1\left(\frac{S(x,t)}{\varepsilon}, x, t\right) + \dots + \varepsilon^N f^N\left(\frac{S(x,t)}{\varepsilon}, x, t\right), \tag{4}$$

where $S(x,t) \in C^\infty$, $f^j(\tau, x, t) \in C^\infty$, $f^0(\tau, x, t) \to 0$ for $\tau \to \pm\infty$, is called the one-phase asymptotical solution of problem (3).

We substitute the function defined by (4) into equation (3) and group the summands which contain equal powers of the parameter ε. Then we ontain

$$\Box_c u = \Big\{ \varepsilon^{-2} \left(S_t^2 - c^2(x) S_x^2 \right) \left(f^0_{\tau\tau} + \varepsilon f^1_{\tau\tau} + \dots \right) + 2\varepsilon^{-1} \left[S_t \frac{\partial}{\partial t} - c^2(x) S_x \frac{\partial}{\partial x} + \frac{1}{2} \left(S_{tt} - c^2 S_{xx} \right) \right] \times \left(f^0_\tau + \varepsilon f^1_\tau + \dots \right) + \Box_c \left(f^0 + \varepsilon f^1 + \dots \right) \Big\}_{\tau = \frac{S(x,t)}{\varepsilon}} = 0 \tag{5}$$

Here $f^i = f^i(\tau, x, t)$.

By equating to zero the coefficient at ε^{-2} in (5) we obtain the equation of characteristics for the function $S(x,t)$

$$S_t \pm c(x) S_x = 0 \,, \quad S\Big|_{t=0} = x \,. \tag{6}$$

By equating to zero the coefficients at ε^{-1} in (5), we obtain the transport equation

$$\hat{\Pi} f^0_0 = 0 \,, \quad f^0(\tau, x, 0) = \Phi(x),$$

where $\hat{\Pi}$ is the differential operator of the first order

$$\hat{\Pi} = S_t \frac{\partial}{\partial t} - c^2(x) S_x \frac{\partial}{\partial x} + \frac{1}{2} \Box_c S$$

By equating to zero the coefficients at ε^{j}, $j > -1$, we similarly obtain equations for the functions f^j, $j > 0$,

$$\Pi f^j_\tau + F_j = 0,$$

where $F_j = F_j(\tau, x, t)$ are functions known on the j-th step and which can be calculated explicitly due to (5). For example, $F_1 = \Box_c f^0$.

Equation (6) can easily be integrated. Let $X^{\pm}(x_0, t)$ be a solution of the problem

$$\dot{X}^{\pm} = \pm c(X^{\pm}), \quad X^{\pm}\big|_{t=0} = x_0 .$$

We assume that the equations $X^{\pm}(x_0, t) = x$ are solvable with respect to x_0 and denote their smooth solutions by $x_0^{\pm} = x_0^{\pm}(x,t)$. Then problem (6) has two solutions $S^{\pm}(x,t)$,

$$S^{\pm}(x,t) = x_0^{\pm}(x,t),$$

the Jacobians

$$J^{\pm} = \frac{\partial X^{+}}{\partial x_0}(x,t)$$

do not vanish and the following statement holds.

THEOREM 1. The asymptotical solution of problem (3) has the form

$$u = \frac{1}{2}\sum_{\pm}\sqrt{\frac{c(x)}{c(x_0^{+}(x,t))}}\;\frac{\Phi\left(\frac{S^{\pm}(x,t)}{\varepsilon}\right)}{\sqrt{\frac{\partial X^{\pm}}{\partial x_0}(x,t)}} + O(\varepsilon)$$

We note that according to this formula the dependence of the solution on the "rapid" variable $S^{\pm}/\varepsilon$ for $t > 0$ is the same as at the initial moment, and therefore it is rather arbitrary.

For equations with dispersion this property does not hold. As an example of a linear equation with dispersion we consider the Cauchy problem for the linearized KdV equation

$$u_t + \varepsilon^2 \rho(x) u_{xxx} = 0, \quad u\big|_{t=0} = \Phi\left(\frac{x}{\varepsilon}\right), \qquad (7)$$

where $\rho(x) \in C^\infty$, the function $\Phi(\xi)$ is the same as above. In this case the substitution of the form (4) leads to an ordinary differential equation (so called standard equation)

$$S_t \frac{\partial f^0}{\partial \tau} + S_x^3 \rho \frac{\partial^3 f^0}{\partial \tau^3} = 0$$

from which we can define the dependence of the function f^0 on the variable τ. The solution of this equation bounded with respect to τ with all its derivatives has the form

$$f^0 = A(x,t)\cos\left(\tau + \theta(x,t)\right), \; A, \theta \in C^\infty.$$

The function $S(x,t)$ satisfies the Hamilton-Jacobi equation

$$S_t - \rho S_x^3 = 0.$$

Thus the function u has the form of WKB asymptotics

$$u = A(x,t)\cos\left(\frac{S(x,t)}{\varepsilon} + \theta(x,t)\right) + O(\varepsilon). \qquad (8)$$

Evidently, such a function does not satisfy the initial condition (7), where $\Phi(\xi)$ is an arbitrary decreasing function. However, since the equation is linear, the Cauchy problem solution can be represented as a superposition of such functions. Further, it turns out that for $t \geq \delta > 0$ the Cauchy problem solution is close to a certain function of the form (8). This fact may be regarded as an analogue of the facts when the Cauchy problem solution for the KdV equation is represented as a superposition of n solitons and the Cauchy problem solution of the KPP equation is represented as a self-similar wave.

In this paper we give only the final result for problem (7). Let $X(p_0, t)$, $P(p_0, t)$ be solutions of a system of ordinary differential equations for $p_0 \in R$, $t \in [0, T]$

$$\dot{X} = 3P^2 a(X), \qquad X|_{t=0} = 0,$$
$$\dot{P} = \frac{\partial a}{\partial x}(X) P^3, \qquad P|_{t=0} = p_0 .$$

We assume that for $0 < \delta \le t \le T$ the Jacobian

$$J = \frac{\partial X(p_0, t)}{\partial p_0}$$

does not vanish and denote by $p_0(x, t)$ the solution of the equation

$$X(p_0, t) = x .$$

We introduce the phase

$$S(x, t) = 2p_0^2(x, t)\rho(0)t .$$

THEOREM 2. For $0 < \delta \le t \le T$ the solution of problem (7) has the form

$$u = \sqrt{2\pi\varepsilon}\, \mathrm{Re}\left(\frac{\tilde{\Phi}(p_0(x,t))}{\sqrt{J(p_0(x,t),t)}} \cdot \sqrt{\frac{\rho(x)}{\rho(0)}}\, e^{\frac{i}{\varepsilon} S(x,t)} \right) + O(\varepsilon)$$

where $\tilde{\Phi}(\eta) = \frac{1}{\sqrt{2\pi}} \int \exp\{-i\eta\xi\} \Phi(\xi) d\xi .$

In this case the asymptotical solution has the form (4) outside the neighbourhood $t = 0$ (this case differs from (4) since we expand with respect to half-integer powers of the parameter ε , though it is not essential).

Thus, the asymptotical solution of problem (7) with arbitrary initial conditions does not have the form (4), but it is close to this form for $t \geq \delta > 0$, $\varepsilon \to 0$.

How to derive the formula given in Theorem 2 we shall demonstrate by using the well-known linear Schrödinger equation. We consider the Cauchy problem

$$i h \frac{\partial \psi}{\partial t} = -\frac{h^2}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi,$$

$$\psi\big|_{t=0} = \Phi\left(\frac{x-\xi}{\varepsilon}\right) \tag{9}$$

In accordance with current notations we denote the small parameter by h, the function $\Phi = \Phi(\xi)$ is the same as above. Evidently, the equality holds

$$\psi\big|_{t=0} = \Phi\left(\frac{x-\xi}{h}\right) = \frac{1}{\sqrt{2\pi}} \int e^{ip(x-\xi)/h}\, \tilde{\Phi}(p)\, dp,$$

where $\tilde{\Phi}(p)$ is the Fourier transform of the function $\Phi(\xi)$. Thus, like in the WKB method the solution of problem (9) can naturally be represented for $t \to 0$ in the form [3],

$$\psi = \int \frac{1}{\sqrt{\mathcal{D}}} \exp\{i\varphi(p,x,t,\xi)/h\}\tilde{\Phi}(p)\,dp + O(h), \tag{10}$$

where $\mathcal{D}$ is the Jacobian.

It is easy to see that the function $\varphi(p,x,t,\xi)$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + V(x) = 0,$$

$$\varphi\big|_{t=0} = p(x-\xi)$$

We shall use the stationary phase method for $t \geq \delta > 0$ to the integral in formula (10). For this purpose we calculate the coordinate p of the stationary point (which depends on the parameters x, t, ξ) by means of the equation

$$\frac{\partial \varphi}{\partial p} = 0 ,$$

We have $p = p_0(x,t,\xi)$ and

$$\tilde{\psi} = \frac{\sqrt{2\pi h}}{\sqrt{J}} \exp\{ i\varphi(p_0, x, t, \xi)/h \} \tilde{\Phi}(p_0) + O(h) \tag{11}$$

where $J = \mathcal{D} \cdot \left. \frac{\partial^2 \varphi}{\partial p^2} \right|_{p = p_0(x,t,\xi)}$

The functions in (11) can be calculated by another method. We consider the boundary problem

$$\ddot{X} = -\frac{\partial V(X)}{\partial x} \quad ; \quad X(0) = \xi , \; X(t) = x , \tag{12}$$

and assume that it has the single solution $X(x,t,\xi)$.

We introduce the function $S(x,\xi,t)$ which is called an action

$$S(x,\xi,t) = \int_{\xi}^{x} L\, dt$$

where L is the Lagrangian, $L = p^2 - V(x)$. We recall that the integral in the formula for action is calculated along the extremal trajectory which, in this case, is the solution of the boundary problem (12). The following equalities hold

$$\varphi(p_0, x, t, \xi) = S(x, \xi, t),$$

$$J = \left| \frac{\partial^2 S}{\partial x \partial \xi} \right|^{-1} , \quad p_0 = -\frac{\partial S}{\partial \xi} ,$$

and equality (11) can be rewritten in the form

$$\psi = \frac{\sqrt{2\pi h}}{\sqrt{J}} \exp\left\{\frac{i}{h} S(x,t,\xi)\right\} \tilde{\Phi}\left(-\frac{\partial S}{\partial \xi}\right) + O(h). \qquad (13)$$

In the case when the bpundary problem

$$\ddot{X} = -\frac{\partial V}{\partial x}(X), \quad X(0) = \xi, \; X(t) = x.$$

has n solutions $X_1(x,t,\xi)$,...., $X_n(x,t,\xi)$ and the Jacobians J_1 ,...., J_n, constructed with respect to actions corresponding to the extremal trajectories X_1 ,...., X_n, do not vanish, then the asymptotics of problem (9) solution has the form [4] for $t \geq \delta > 0$

$$\psi = \sqrt{2\pi h} \sum_{k=1}^{w} e^{\frac{i\pi}{4}\gamma_k} \frac{1}{\sqrt{J_k}} e^{i S_k(x,\xi,t)/h} \tilde{\Phi}\left(-\frac{\partial S_k}{\partial \xi}\right) + O(h) \qquad (14)$$

where γ_k is the Morse index of the trajectory X_k.

Thus, the number of phases ("rapid" variables) in the asymptotical solution depends on the number of solutions of boundary problem (12). In particular, if $V(x) = x^4$, then problem (9) solution cannot be represented in the form (14) for any $n < \infty$ as $t \geq \delta > 0$. The solutions of the form (14) are superpositions of n waves with the phases S_1 ,...., S_n. A well-known analogue of the n-phase solution for nonlinear equations is the n-soliton solution of KdV equation.

So, now we shall consider nonlinear equations. For nonlinear equations with variable coefficients the Whitham method is the nonlinear analogue of the WKB-method, it allows to construct certain special solutions which are self-similar solutions. Since the superposition principle does not work in the nonlinear case, it is impossible to construct using these special solutions the solution of a rather general Cauchy problem. However we have a remarkable property of the equations mentioned above, namely, the asymptotical solutions of the Cauchy problem tend to these special solutions as $t \geq \delta > 0$. Thus, the wave profile which is arbitrary at first can be "improved" by a nonlinear equation and accepts the profile described by the standard equation as it was in the linear case (see Theorem 2).

We give such a result for the KdV equation;

$$v_t + \rho_1(x) v v_x + h^2 \rho(x) v_{xxx} = 0,$$

$$v\big|_{t=0} = \Phi\left(\frac{x}{h}\right),$$

$$\rho_1, \rho_2 > 0, \ \rho_x > 0, \ \Phi(\tau) \in C_0^\infty .$$

The appropriate asymptotical solutions have the form of solitons with variable amplitudes and velocities; outside the collision points they have the form [5]

$$v_{sol}^{(n)} = \sum_{j=1}^{n} \frac{A_j(t,h)}{\operatorname{ch}^2\left(\beta_j(t,h)\dfrac{x-\varphi_j(t,h)}{h} + \theta_j(t,h)\right)},$$

where the functions A_j, φ_j satisfy the ordinary differential equations:

$$\frac{d\varphi_j}{dt} = \frac{2}{3} A_j \rho_1(\varphi_j),$$

$$\frac{dA_j}{dt} = \frac{2}{3} A_j \sqrt{\frac{\rho_1}{\rho}} \left\{ \frac{\partial}{\partial x} \sqrt{\frac{\rho_1^3}{\rho}} + \frac{2}{3} A_j \left(\frac{\rho}{\rho_1}\right)^{3/2} \times \right.$$

$$\left. \times \frac{\partial}{\partial x} \left(\rho_1^{21/10} \rho^{-11/10} \right) \right\} x = \varphi_j(t),$$

$$\beta_j = \sqrt{A_j \rho_1(\varphi_j)/12\rho(\varphi_j)}$$

The functions $\theta_j(t)$ satisfy equations of the second order which we do not give here because of their cumbersomeness. As was already mentioned, the remarkable fact is that the Cauchy problem solution for an arbitrary initial boundary layer function $\Phi(\tau)$, after a very small period of time, either turns into $v_{sol}^{(n)}$ or becomes equal to $O(h^\nu)$, $\nu > 0$.

Now we formulate the result. We assume, without loss of generality, that $\rho_1(0) = 0$, $\rho(0) = 1$. Then we consider the Sturm-Louiville problem:

$$y_{\tau\tau} + \Phi(\tau) y = -\lambda y, \quad \tau \in R.$$

Since Φ is finite, this problem can have only a finite number of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We denote by $\tilde{\varphi}_j$, $\tilde{A}_j$ the solutions of the system of ordinary differential equations mentioned above which satisfy the conditions:

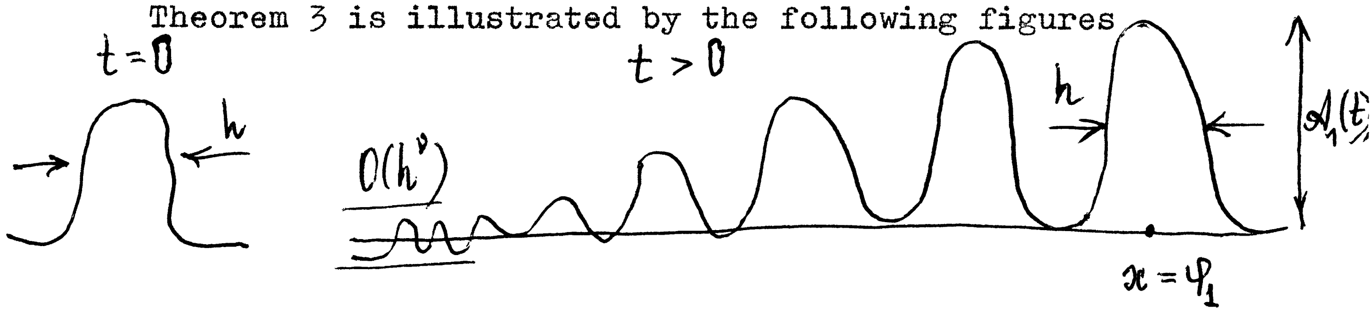
$$\tilde{\varphi}_j \Big|_{t=0} = 0, \quad \tilde{A}_j \Big|_{t=0} = 2\lambda_j^2.$$

THEOREM 3. For $t > \delta > 0$, $x > \delta > 0$ the following equalities hold

$$v = v_{sol}^{(n)} + O(h^{\nu}), \quad \nu > 0,$$
$$\varphi_j(t,h) = \tilde{\varphi}_j(t) + \left(h \ln \frac{1}{h}\right) \psi_j(t),$$
$$A_j(t,h) = \tilde{A}_j(t) + \left(h \ln \frac{1}{h}\right) a_j(t),$$

where ψ_j, A_j are defined by a linear system of ordinary differential equations, which we do not give here, since they are very cumbersome.

Theorem 3 is illustrated by the following figures



When $\rho_1, \rho_2 =$ const, Theorem 3 turns into the well known result by A.B.Shabat.

So far we considered equations with small dispersion. Now we shall consider nonlinear equation with small viscosity. These equations describe quite different processes. Here the Cauchy problem solution also turns into a single solution.

As an example, we consider the Kolmogorov-Petrovski--Piskunov equation with variable coefficients

$$\varepsilon \frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - \gamma(x,t) F(u) = 0 \quad , \quad F(u)\Big|_{\substack{u=0\\u=1}} = 0$$
$$\gamma(x,t) \in C^{\infty}, \ 0 < \delta_1 < \gamma < \delta_2 \ , \ \delta_i = \text{const} ,$$
$$F'(0) = 1.$$

First of all we note that the single phase solution of this equation has the form $u = \chi\left(\frac{S}{\varepsilon}, x, t\right)$. The standard equation in this case has the form

$$b\chi' - \chi'' - F(\chi) = 0, \quad b = const \geq 2.$$

It turns out that the asymptotics of a solution with arbitrary initial conditions which have a finite derivative can be expressed in terms of the single phase solution.

Namely, let the initial condition have the form

$$u\Big|_{t=0} = \psi\left(\frac{x}{\varepsilon}\right), \; x \in R^1, \; 0 \leq \psi \leq 1, \; \psi' \in C_0^\infty .$$

For any such initial condition the solution becomes (as $t > 0$) a singel phase one whose front propagates according to the law [6], [7].

$$x = -\varphi(t), \quad \frac{d\varphi}{dt} = 2\sqrt{\gamma(-\varphi, t)}, \quad \varphi(0) = 0.$$

The wave profile has the form

$$u(x,t,\varepsilon) - \chi\left(\frac{\sqrt{\gamma(-\varphi,t)}\,(x+\varphi) + \sqrt{\varepsilon}\, f(t,\varepsilon)}{\varepsilon}\right) \xrightarrow[\varepsilon \to 0]{} 0$$

where $|f| < 1$, χ is the solution of the standard equation for $b = 2$, satisfying the conditions $\chi(0) = 1/2$, $\chi(-\infty) = 0$, $\chi(\infty) = 1$.

Now we consider an equation in two-dimensional space

$$\varepsilon \frac{\partial u}{\partial t} - \varepsilon^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \gamma(x,y,t) F(u) = 0$$

with the same initial conditions.

The solution of this problem is also a single phase solution which is an inner boundary layer of the second type. The boundary layer Γ_t front surface is the levet surface of the function $J = J(x,y,z)$ which is defined by the equation

$$|\nabla_{x,y,z}\tilde{\nu}| = \left(2\sqrt{\gamma(x,y,-\tilde{\nu})}\right)^{-1}, \quad \tilde{\nu}\big|_{x=0} = 0. \qquad (15)$$

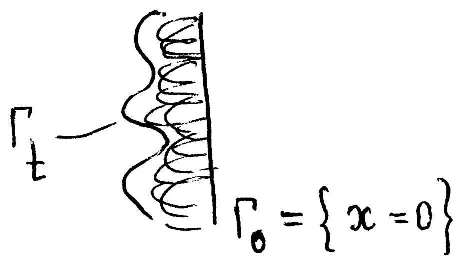
The front form is the following [2]

$$u - \chi\left(\frac{\sqrt{\gamma(x,y,-\tilde{\nu})}\,(t+\tilde{\nu}) + \sqrt{\varepsilon}\, f_n}{\varepsilon}\right) \underset{\varepsilon \to 0}{\longrightarrow} 0$$

the function χ the same as in above.

For $\gamma = \gamma(x,y,z)$ equational (15) is the eikonal equation which is well known in geometrical optics and the function $2\sqrt{\gamma}$ plays the role of the index of refraction.

In order to construct the wave front Γ_t, we can use the Huygens principle, which says that the position of the wave front at time t is the envelope of the wave fronts coming out from point sources on the front at the initial instant of time.



In the case of a variable index of refraction, the initial plane front of wave will bend.

If the derivative $\frac{\partial\psi}{\partial z}(z)$ of the initial condition is not finite in z, then by finding the asymptotics of the function Ψ for $z \to \pm\infty$ we see that the Cauchy problem solution for $t \to 0$, $\varepsilon \to 0$ turns into the single phase solution which is associated with the solution of the standard equation for $b > 2$.

For example, if

$$\psi(z) \sim e^{l_1 z} \quad \text{i.e.} \quad \Psi'/\Psi \to l_1(-1),\ z \to -\infty,\ l_1 = -\frac{b}{2} - \sqrt{\frac{b^2}{4} - |F'(0)|}$$

$$\psi(z) \sim 1 - e^{-l_2 z} \quad \text{i.e.} \quad \frac{\partial \Psi}{\partial z} / (1-\psi) \to l_2,\ z \to \infty,\ l_2 = \frac{b}{2} + \sqrt{\frac{b^2}{4} + |F'(1)|}$$

where $b > 2$ is an arbitrary number, then in relation above the function $\chi(\eta)$ should be changed to the solution $\chi_1(\eta)$ of the following differential equation

$$b\chi_1' - \chi_1'' - F(\chi_1) = 0,\ \chi_1(0) = 1/2,\ \chi_1(-\infty) = 0,$$
$$\chi_1(\infty) = 1.$$

The front motion law will have in this case the form

$$\frac{d\varphi^0}{dt} = b\sqrt{\gamma(-\varphi^0,\ t)}\ , \quad \varphi^0(0) = 0.$$

If the solution consists of peaks, as it was in the case of KdV equation, then these peaks scatter, as a rule, and the solution is, in essence, a sum of single phase solutions. Even if the peaks collide, then their interaction lasts for a short time, after which the solution is again a sum of peaks (elastic collision of peaks).

In the case when we have small diffusion in the equation of reaction-diffusion, the situation varies sharply. The front's collision becomes non-elastic.

Now we consider an example of fronts collision in the case of a parabolic equation with cubic nonlinearity, which was used in the theory of combustion by Zeldovich and Frank-Kamenetski, and which is also used for describing signals in a long line without inductance [7]

$$\varepsilon \frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - \gamma(x,t)u(1-u^2) = 0.$$

We assume that the initial condition is the sum of two fronts of (generally speaking) arbitrary form with given exponential asymptotics at infinity

$$u|_{t=0} = \Psi_1\left(\frac{x+\xi}{\varepsilon}\right) + \Psi_2\left(\frac{x-\xi}{\varepsilon}\right), \quad \xi > 0.$$

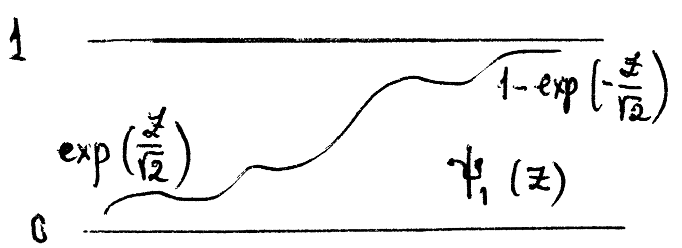
$$0 \le \Psi_1 \le 1, \quad \Psi_1 \to 1, \quad \Psi'_{1z}/(1-\Psi_1) \to 1/\sqrt{2}, \quad z \to \infty,$$

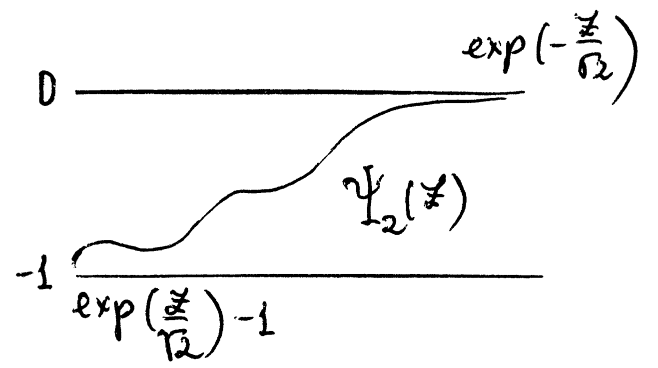
$$\Psi_1 \to 0, \quad \Psi'_{1z}/\Psi_1 \to 1/\sqrt{2}, \quad z \to -\infty,$$

$$-1 \le \Psi_2 \le 0, \quad \Psi_2 \to -1, \quad \Psi'_{2z}/(1+\Psi_2) \to 1/\sqrt{2},$$

$$z \to -\infty,$$

$$\Psi_2 \to 0, \quad \Psi'_{2z}/\Psi_2 \to -1/\sqrt{2}, \quad z \to \infty.$$





Obviously, it is impossible to use the single phase asymptotics only.

The two-phase asymptotical solution of this equation has the form

$$u = f_0\left(\frac{S_1}{\varepsilon}, \frac{S_2}{\varepsilon}, x, t\right) + \varepsilon f_1\left(\frac{S_1}{\varepsilon}, \frac{S_2}{\varepsilon}, x, t\right) + \ldots .$$

By substituting this function into the equation and equating to zero the sum of summands independent of ε, we obtain

$$\left(\frac{\partial S_1}{\partial t}\frac{\partial}{\partial \tau_1} + \frac{\partial S_2}{\partial t}\frac{\partial}{\partial \tau_2}\right) f_0 - \left(\frac{\partial S_1}{\partial x}\frac{\partial}{\partial \tau_1} + \frac{\partial S_2}{\partial x}\frac{\partial}{\partial \tau_2}\right)^2 f_0 -$$

$$- \gamma f_0 \left(1 - f_0^2\right) = 0.$$

Then we change the variables, by setting

$$\frac{1}{\gamma}\left(\frac{\partial S_1}{\partial t}\frac{\partial}{\partial \tau_1} + \frac{\partial S_2}{\partial t}\frac{\partial}{\partial \tau_2}\right) = \frac{\partial}{\partial \xi},$$

$$\frac{1}{\sqrt{\gamma}}\left(\frac{\partial S_1}{\partial x}\frac{\partial}{\partial \tau_1} + \frac{\partial S_2}{\partial x}\frac{\partial}{\partial \tau_2}\right) = \frac{\partial}{\partial \eta}.$$

We denote the function f_0 in the variables ξ, η by $F_0 = F_0(\xi, \eta, x, t)$. Then for F_0 we obtain the equation

$$\frac{\partial F_0}{\partial \xi} - \frac{\partial^2 F_0}{\partial \eta^2} - F_0\left(1 - F_0^2\right) = 0,$$

i.e. we obtain the initial equation, but with constant coefficients. The solution of this equation has the form

$$F_0' = \frac{e^{\frac{1}{\sqrt{2}}\eta + \frac{3}{2}\xi + C_1} - e^{-\frac{1}{\sqrt{2}}\eta + \frac{3}{2}\xi + C_2}}{1 + e^{\frac{1}{\sqrt{2}}\eta + \frac{3}{2}\xi + C_1} + e^{-\frac{1}{\sqrt{2}}\eta + \frac{3}{2}\xi + C_2}}$$

where $C_i = C_i(x,t)$ are arbitrary smooth functions.

We consider the case when the phase S_1 is stationary and S_{2t} is equal to zero. Returning to the initial variables τ_1, τ_2, we obtain

$$f_0 = \left\{ e^{\frac{\tau_1}{\Delta}\left(\frac{3}{2}\gamma S_{2x} - \sqrt{\frac{\gamma}{2}} S_{2t}\right) + \frac{\tau_2}{\Delta}\left(\sqrt{\frac{\gamma}{2}} S_{1t} - \frac{3}{2}\gamma S_{1x}\right) + C_1} - e^{\frac{\tau_1}{\Delta}\left(\sqrt{\frac{\gamma}{2}} S_{2t} + \frac{3}{2}\gamma S_{2x}\right) + \frac{\tau_2}{\Delta}\left(\sqrt{\frac{\gamma}{2}} S_{1t} + \frac{3\gamma}{2} S_{1x}\right) + C_2} \right\} \Big/ \left\{ 1 + e^{\frac{\tau_1}{\Delta}\left(\frac{3}{2}\gamma S_{2x} - \sqrt{\frac{\gamma}{2}} S_{2t}\right)} \times e^{\frac{\tau_2}{\Delta}\left(\sqrt{\frac{\gamma}{2}} S_{1t} - \frac{3}{2}\gamma S_{1x}\right) + C_1} + e^{\frac{\tau_1}{\Delta}\left(\sqrt{\frac{\gamma}{2}} S_{2t} + \frac{3}{2}\gamma S_{2x}\right) + \frac{\tau_2}{\Delta}\left(\sqrt{\frac{\gamma}{2}} S_{1t} + \frac{3}{2}\gamma S_{1x}\right) + C_2} \right\}$$

$$\Delta = S_{1t} S_{2x} - S_{2t} S_{1x}.$$

The solution of the initial Cauchy problem turns into into the two phase solution constructed before for $t > 0$, $\varepsilon \to 0$

$$u(x,t,\varepsilon) - f_0\left(\frac{S_1}{\varepsilon}, \frac{S_2}{\varepsilon}, x, t\right) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where

$$S_1 = \sqrt{\gamma(-\varphi_1, t)}\,(x + \varphi_1) + \sqrt{\varepsilon} f_1(t, \varepsilon),$$

$$\frac{d\varphi_1}{dt} = -3\sqrt{\gamma(-\varphi_1, t)/2}, \quad \varphi_1(0) = -\xi,$$

$$S_2 = \sqrt{\gamma(-\varphi_2, t)}\,(x + \varphi_2) + \sqrt{\varepsilon} f_2(t, \varepsilon),$$

$$\frac{d\varphi_2}{dt} = 3\sqrt{\gamma(-\varphi_2, t)/2}, \quad \varphi_2(0) = -\xi, \quad |\varphi_{1,2}| < 1.$$

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