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N. LERNER
J. NOURRIGAT

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CENTRE
DE
MATHEMATIQUES
Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE
F-91128 PALAISEAU Cedex (France)
Tél. (1) 69.41.82.00
Télex ECOLEX 601.596 F

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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

LOWER BOUNDS FOR PSEUDO-DIFFERENTIAL OPERATORS

N. LERNER (*) and J. NOURRIGAT

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The present paper is a summary of a work to be published. We are interested in lowerbounds for pseudo-differential operators. Our guideline will be one of Fefferman and Phong's conjecture (see §7 in [6] and [2]) : if $a(x, D_x)$ is a second order operator, its lower bound will be given by some average of its symbol on canonical images of the unit cube in the phase space. Namely, we wish to prove roughly, in some cases :

$$a(x, D_x) \geq \inf_{\chi \in \Phi} \int\int_{\chi(Q_0)} a(x, \xi) dx d\xi \ ,$$

as an operator, where Q_0 is the unit cube of $\mathbf{R}^{2n}$, Φ a family of canonical transformations to be specified. The inequality above gives a connection between the geometry of the symbol $a(x, \xi)$ and the spectral properties of its quantization $a(x, D_x)$. Many papers were devoted to these questions. The classical sharp Gårding inequality was first proved by Hörmander [10] : $a(x, \xi)$ first order ≥ 0 implies $a(x, D_x)$ semi-bounded from below.

We refer to ([12] section 18.1 or [1]) for a proof of this inequality, yielding also the case of systems, previously studied by Lax and Nirenberg [13]. In his paper on the Weyl calculus [11], Hörmander proved an inequality with a "gain" of 6/5 derivatives. Namely, if $a(x, \xi)$ is a symbol of order 6/5 such that

$$a(x, \xi) + \frac{1}{2} \quad \text{trace} \quad _{+}a \geq 0 \ ,$$

then $a(x, D_x)$ is semi-bounded from below. There, $\text{trace}_{+}a$ is a positive quantity related to the Hessian of the symbol introduced by Melin [14]. On the other hand, Fefferman and Phong proved a two-derivatives inequality [3] for non-negative symbols : $a(x, \xi)$ second order ≥ 0 implies $a(x, D_x)$ semi-bounded from below (see also the proof in [12], section 18.6). Moreover, these authors discussed the conjecture stated above for non negative symbols of order $2 - \varepsilon$, $\varepsilon > 0$ (see [6]).

The present work is concerned with various cases involving symbols which can take large negative values.

I. Preliminary remarks

In this section, we recall through examples some of the features of Fefferman-Phong's view of the uncertainty principle (see [3]-[7] and [2]). One achievement of quantum mechanics is the interpretation of the stability of the hydrogen atom, namely the fact that for $n = 3$, one has

$$-\frac{h^2}{8\pi^2 m}\Delta - \frac{e^2}{|x|} + \frac{2m\pi^2 e^4}{h^2} \geq 0 \tag{1.1}$$

as an operator. A direct proof of (1.1) is easily obtained from Hardy's inequality and the use of polar coordinates. In this case, "volume-counting" is enough, sinse the symplectic volume of the set where the symbol of (1.1) is negative is finite. However this is not the case for the following example, studied by Weder [16] and Herbst [19] ; in three dimensions, one has

$$(-\Delta)^{1/2} - \frac{2}{\pi}\frac{1}{|x|} \geq 0\ . \tag{1.2}$$

This non-local pseudo-differential operator is a quantization of a relativistic hamiltonian.

The scaling properties of $(-\Delta)^{1/2} - k|x|^{-1}$ show that it is either non-negative or unbounded below. The operator (1.2) cannot be understood from a "volume-countring" point of view since the volume of the negativity vet of its symbol is infinite. If we examine the symbol $|\xi| - k|x|^{-1}(n \geq 2)$, we note that if k is negative and small enough its averages on symplectic cubes $\{(x,\xi), |x| \leq \delta, |\xi| \leq \delta^{-1}\}$ will be non-negative. As a matter of fact, this operator is analogous to $-\Delta - k|x|^{-2}(n \geq 3)$ (see [2]). In the latter case the critical constant is $k_c(n) = (\frac{n-2}{2})^2$; namely if $k \leq (\frac{n-2}{2})^2, -\Delta - k|x|^{-2} \geq 0$ and unbounded below if $k > (\frac{n-2}{2})^2$. The critical constants for operators of type (1.2) can be found in the work of Herbst [9].

Our general policy will be as follows : given a symbol $a(x,\xi)$, find a family of canonical transformations Φ, tailored on the geometry of a, as restricted as possible, so that the lower bound of $a(x, D_x)$ is given by $\inf_{\chi\in\Phi} \int\int_{\chi(Q_0)} a(x,\xi)dxd\xi$, where Q_0 is the unit cube of $\mathbf{R}^{2n}$. The next section provides a case in which it can be done explicitly.

2. The Schrödinger Equation with Magnetic Potential

We are interested in the following operator

$$P = \sum_{j=1}^{n} (D_j - A_j(x))^2 + V(x) \ , \tag{2.1}$$

where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, and $A_1, \cdots, A_n, V$ are real polynomials of degree $< m$ (Note that V is **not** assumed to be non-negative). We set-up

$$p(x,\xi) = \sum_{j=1}^{n} (\xi_j - A_j(x))^2 + V(x) \ , \tag{2.2}$$

the Weyl symbol of the operator P.

We denote by Φ_m the group of canonical transformations of $\mathbf{R}^{2n}$ of the following type :

$$(y,\eta) \mapsto (x_0 + \lambda y, \lambda^{-1}\eta + \nabla\phi(y)) \ , \tag{2.3}$$

where $x_0 \in \mathbf{R}^n, \lambda > 0$ and ϕ is a real polynomial of degree $\leq m$.

Theorem 2.1.— *For each integer m, there exists $\delta_m > 0$ such that the following property holds. If $A_1, \cdots, A_n, V$ are real polynomials of degree $< m$ and if the symbol $p(x,\xi)$ given by (2.2) satisfies*

$$\int\int_{\max(|y|,|\eta|)\leq\delta_m} (p \circ \chi)(y,\eta) dy d\eta \geq 0 \ , \tag{2.4}$$

for any $\chi \in \Phi_m$ defined in (2.3), Then the operator P given by (2.1) is non-negative.

In other words, whenever (2.4) is satisfied, we have

$$\sum_{j=1}^{n} \|(D_j - A_j)u\|^2_{L^2(\mathbf{R}^n)} + \int V(x)|u(x)|^2 dx \geq 0 \ , \quad \text{for any} \quad u \in C_0^\infty(\mathbf{R}^n) \ .$$

Let's note that the magnetic potential is a one-form $A = \sum_{j=1}^n A_j dx_j$ and that

$$\text{curl } A = dA = \sum_{1\leq k<j\leq n} (\frac{\partial A_j}{\partial x_k}) dx_k \wedge dx_j \ .$$

Here, the quotient norm of A modulo exact forms is equivalent to the norm of curl A.

The first step in the proof is the

Lemma 2.2.— *For any m, there exists $C_m > 0$ so that (2.4) implies, for any $x_0 \in \mathbf{R}^n$ and any $R > 0$*

$$\delta R^{-2} + C_m \sup_{|x-x_0|\leq R} |B(x)|^2 R^2 + \frac{1}{R^n}\int_{|x-x_0|\leq R} V(x)dx \geq 0 \ , \tag{2.5}$$

where $B =$ curl A.

Next, we introduce, for x given in $\mathbf{R}^n$,

$$\psi_x(R) = \sup_{|y-x|\leq R} (R^2|B(y)|^2 + V(y)) - \inf_{|y-x|\leq R} V(y) \tag{2.6}$$

Assuming that V is not constant, $R \mapsto \psi_x(R)$ is continuous strictly increasing from 0 to ∞ with R. We define then, for $\lambda > 1$ to be chosen later, and for $x \in \mathbf{R}^n$, $R(x)$ to be the unique $R \in (0, \infty)$ such that

$$\psi_x(R(x)) = \lambda^2 R(x)^{-2} \ . \tag{2.7}$$

It is then possible to prove that the metric $\frac{dx^2}{R(x)^2}$ is slowly varying. More precisely, we have the

Lemma 2.3.— *For $x_1, x_2 \in \mathbf{R}^n, |x_1 - x_2| \leq \frac{1}{4}R(x_1)$ implies $2^{-1} \leq R(x_1)R(x_2)^{-1} \leq 2$.*

Using the lemma 2.3, we can construct a partition of unity and the localized problems as well as the commutator terms can be handled using the lemma 2.2 and the result for $V \geq 0$ proved in Mohamed-Nourrigat [15] (see also-Helffer-Nourrigat [8]).

3. Pseudo-differential operators

First of all, we intend to show that the non negativity of **averages** of a symbol on special boxes of volume I ensures that the Calderón-Zygmund procedure used by Fefferman and Phong ([3]-[7]) leads to the same trilogy. These authors proved that if a is a non-negative symbol of order 2, it is possible to find a pseudo-differential calculus such that a, still second-order, will be microlocally either elliptic or bounded or non-degenerate i.e. of the form $\tau^2 + V(t, x, \xi)$, where V is a pseudo-differential potential. This decomposition is still valid if we assume only non-negativity for some averages of the symbol.

a. The proper class of a symbol

We'll call Hörmander metric on $\mathbf{R}^{2n}$ a slowly varying, σ-temperate metric G so that $G \leq G^\sigma$ (see section 18.5 in [12]). For each $X \in \mathbf{R}^{2n}$, G_X is a positive definite quadratic form on $\mathbf{R}^{2n}$, such that the three following properties are satisfied.

(3.1) There exists $C > 0$ such that, for any $X, Y, T \in \mathbf{R}^{2n}$ $G_X(Y - X) \leq C^{-1}$ implies $C^{-1}G_Y(T) \leq G_X(T) \leq CG_Y(T)$.

(3.2) For any $X, T \in \mathbf{R}^{2n}$, $G_X(T) \leq G_X^\sigma(T)$, where $G_X^\sigma(T) = \sup_{G_X(U)=1} \sigma(T, U)^2$, σ the symplectic form on $\mathbf{R}^{2n}$. There exists $C > 0$, N such that, for any X, Y, T in $\mathbf{R}^{2n}$

$$G_X(T) \leq CG_Y(T)(1 + G_X^\sigma(X - Y))^N \ . \tag{3.3}$$

Let's also define the reciprocal Planck function

$$\Lambda_G(X) = \inf_T \left(\frac{G_X^\sigma(T)}{G_X(T)}\right)^{1/2}. \tag{3.4}$$

A function $a \in C^\infty(\mathbf{R}^{2n})$ belongs to $S^m(G)$ if for any k, there exists C_k such that

$$|a^{(k)}(X)T^k| \leq C_k \Lambda_G(X)^m G_X(T)^{k/2}, \tag{3.5}$$

any X, T in $\mathbf{R}^n$.

The semi-norms of a are the best constants

$$\gamma_{kG}(a) = \sup_{\substack{X,T \in \mathbf{R}^{2n} \\ G_X(T)=1}} |a^{(k)}(X)T^k| \Lambda_G(X)^{-m} .$$

For a given $a \in S^2(G)$, we consider

$$g_X = \Lambda_G(X)\lambda(X)^{-1}G_X , \tag{3.6}$$

with

$$\lambda^2(X) = \max_{0 \leq k \leq 3}(1, \|a^{(k)}(X)\|_{G_X}^{\frac{4}{4-k}} \Lambda_G(X)^{-\frac{2k}{4-k}}), \tag{3.7}$$

where $\|a^{(k)}(X)\|_{G_X} = \sup_{T, G_X(T)=1} |a^{(k)}(X)T^k|$.

The next proposition nummarizes the properties of a Calderon-Zygmund decomposition of a symbol

Proposition 3.1.—

(1) The metric g defined by (3.6) is an Hörmander metric i.e. satisfies (3.1), (3.2), (3.3). The constants in (3.1) for g depend only on the constants in (3.1) for G and on $(\gamma_{kG}(a))_{0 \leq k \leq 4}$.

(2) We have $\lambda_g(X) = \lambda(X)$ according to (3.4) and (3.7).

(3) The symbol $a \in S^2(g)$ and

$$\gamma_{kg}(a) \leq 1 \quad \textit{if} \quad k \leq 3 ,$$

$$\gamma_{kg}(a) \leq \gamma_{kG}(a) \max_{0 \leq \ell \leq 3}(1, (\gamma_{\ell G}(a))^{\frac{k-4}{\ell-4}}) \quad \textit{if} \quad k \geq 4 .$$

(4) We have $\lambda(X) \leq \Lambda(X) \max_{0 \leq k \leq 3}(1, (\gamma_{kG}(a))^{\frac{2}{4-k}})$.

Note that, in this proposition, a is any symbol in $S^2(G)$ without any non-negativity assumption. The proof is essentially standard and will not be given here.

b Rescaling the non-negativity assumption

An important step in the proof is the following lemma, analogous to lemma 18.6.9 in [12].

Lemma 3.2.— *Let δ, ε be given positive numbers and a aC^∞ function on $|x| \le 1$ in $\mathbf{R}^n$, so that*

(i) $|a^{(k)}(X)| \le 1 \quad , \quad 0 \le k \le 4$

(ii) $\max_{0\le k \le 3} |a^{(k)}(0)| \ge \delta$,

(iii) The averages of a on balls of radius ε are non-negative.

Then there exists $r(\delta), \varepsilon(\delta)$ and $\omega(\delta)$ positive so that if (i), (ii) and (iii) are satisfied with $\varepsilon \le \varepsilon(\delta)$ we have, on $|X| \le r(\delta)$,

(1) Either $a(X) \ge \omega(\delta)$

(2) or $a(X) = a(\alpha(X'), X') + e_0(X)(X_1 - \alpha(X'))^2$ with $\varepsilon_0(X) \ge \omega(\delta)$ and

$$\max_{|x|\le r(\delta)} (|e_0^{(k)}(X)| + |\alpha^{(k)}(X)|) \le C(k) F(\max_{|x|\le r(\delta), \ell \le k+2} |a^{(\ell)}(X)|) .$$

c. Egorov theorem

An other important ingredient is the "Sharp Egorov Principle" proved by Fefferman and Phong in [6].

Theorem 3.3.—

Let g be a quadratic form on $\mathbf{R}^{2n}$ such that $g = \lambda^{-1}\Gamma$, where $\lambda \ge 1$ and Γ is a quadratic form such that $\Gamma = \Gamma^\sigma$ (see 3.2). Let $a \in S^2(g)$ real valued supported in Q, a g-ball of radius 1. Let χ be a canonical transformation of $\mathbf{R}^{2n}$ such that

$$|\chi^{(k)}(X)|_\Gamma \le \gamma_k(\chi)\lambda^{\frac{1}{2}-\frac{k}{2}}$$

$$|\chi'(X)^{-1}|_\Gamma \le \gamma_1(\chi) .$$

Then, there exists a Fourier Integral Operator U, bounded on $L^2(\mathbf{R}^n)$ and $r \in S^0(g)$ so that

$$a^w = U^*(a \circ \chi)^w U + r^w$$

Using these results, we are reduced to study $\tau^2 + V(t, x, \xi)$ where V is a pseudo-differential operator whose symbol can take (large) negative values.

d. A one dimensional result

Let's take $a \in S^2(G)$, where G is an Hörmander metric so that $G_X = \Lambda(X)^{-1}\Gamma_X$, where $\Gamma_X = \Gamma_X^\sigma$ (This is the case for the classical ψ dos in $S^2_{1,0}$ or $(S^{2(p-\delta)}_{p,\delta})$. We'll denote by $g_X = \lambda(X)^{-1}\Gamma_X$ the proper conformal metric of a, defined in proposition 3.1. Our first assumption is

$$\int_{\Gamma_Y(X-Y)\le 1} a(X)dX \ge 0 \quad \text{for any} \quad Y \in \mathbf{R}^{2n}. \tag{3.8}$$

Then, we introduce a family Φ of canonical transformations : $\chi \in \Phi$ is C^∞, canonical and satisfies

$$\begin{cases} |\chi^{(k)}(X)|_{\Gamma_X} \le C_k(\chi)\lambda(X)^{\frac{1}{2}-\frac{1}{2}} \\ g_X(\chi(X) - X) \le \delta_0 \end{cases} \tag{3.9}$$

Theorem 3.4 (n=1).—

These exists $\omega_0 > 0$ such that, if (3.8) is satisfied, for $a \in S^2(G)$, and so that

$$\int_Q (a \circ \chi)(X)dX + \omega_0 \inf_Q (a \circ \chi) \geq 0 \tag{3.10}$$

for any symplectic cube Q and any $\chi \in \Phi$, then

$$a^w + C \geq 0 \ ,$$

where C depends only on a finite fixed number of semi-norms of a.

Here a symplectic cube denote any $g = \{(t, \tau), \max(|t|, (|\tau|)) \leq 1\}$ where t, τ are linear symplectic coordinates in $\mathbf{R}^2$.

REFERENCES

[1] A. CORDOBA, C. FEFFERMAN : Wave packets and Fourier integral operators, Comm. P.D.E., 3, 11, 1978, 979-1005.

[2] C. FEFFERMAN : The uncertainty principle, Bull. AMS, 9, 129-206, 1983.

[3] C. FEFFERMAN, D.H. PHONG : On positivity of pseudo-differential operators Proc. Natl. Ac. Sc. USA 75, 4673-4674, 1978.

[4] C. FEFFERMAN, D.H. PHONG : On the lowest eigenvalue of a pseudo-differential operator, Proc. Natl. Ac. Sc. USA 76, 6055-6056, 1979.

[5] C. FEFFERMAN, D.H. PHONG : On the asymptotic eigenvalue distribution, Proc. Natl. Ac. Sc. USA 77, 5622-5625, 1980.

[6] C. FEFFERMAN, D.H. PHONG : The uncertainty principle and sharp Garding inequalities, CPAM, 34, 285-331, 1981.

[7] C. FEFFERMAN, D.H. PHONG : Symplectic geometry and positivity of pseudo-differential operators, Proc. Natl. Ac. Sc. USA 79, 710-713, 1982.

[8] B. HELFFER, J. NOURRIGAT : Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs, Progress in Math. 58, Birkhauser 1985.

[9] I. HERBST : Spectral theory of the operator $(p^2 + m^2)^{1/2} - ze^2/r$, Comm. in Math. Phys. 53, 285-294, 1977 ; Errata ibidem 55, 316, 1977.

[10] L. HÖRMANDER : Pseudo-differential operators and non-elliptic boundary problems, Ann. of Math, 83, 129-209, 1966.

[11] L. HÖRMANDER : The Weyl calculus of pseudo-differential operators, CPAM, 32, 359-443, 1979.

[12] L. HÖRMANDER : The analysis of linear partial differential operators, four volumes, Berlin, Springer 1985.

[13] P.D. LAX, L. NIRENBERG : On stability for difference schemes : a sharp form of Garding's inequality, CPAM 19, 473-492, 1966.

[14] A. MELIN, Lower bounds for pseudo-differential operators, Ark. Mat. 9, 117-140, 1971.

[15] A. MOHAMED, J. NOURRIGAT : Encadrement du $N(\lambda)$ pour un opérateur de Schrödinger avec des champs électromagnétiques, to appear J. Math. Pures Appl.

[16] R. WEDER : Spectral analysis of pseudo-differential operators, J. Funct. Anal. 20, 319-337, 1975.

Nicolas LERNER
IRMAR
Département de Mathématiques
Université de Rennes I
Campus de Beaulieu
35042 RENNES CEDEX
and Purdue University

Jean NOURRIGAT
U.E.R. de Sciences Exactes
et Naturelles
Département de Mathématiques
Moulin de la Housse
51062 REIMS Cedex