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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

THE GLOBAL NONLINEAR STABILITY OF THE MINKOWSKI SPACE

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The aim of our work is to provide a proof of the nonlinear gravitational stability of the Minkowski space-time. More precisely it accomplishes the following goals:

1. It provides a constructive proof of global, smooth, nontrivial, solutions to the Einstein Vaccum Equations which look, in the large, like the Minkowski space-time. In particular, these solutions are free of black holes and singularities.

2. It provides a detailed description of the sense in which these solutions are close to the Minkowski space-time, in all directions, and gives a rigorous derivation of the laws of gravitational radiation proposed by Bondi.

3. It obtains these solutions as dynamic developments of all initial data sets, which are close, in a precise manner, to the initial data set of the Minkowski space-time, and thus establishes the global dynamic stability of the latter.

4. Though our results are established only for developments of initial data sets which are uniformly close to the trivial one, they should in fact be valid in the complement of the domain of influence of a sufficiently large compact subset of the initial manifold of any "strongly asymptotically flat "initial data set. We plan in fact to prove such a theorem in the future.

According to Einstein the underlying geometry of space-time is that given by a pair (**M**,**g**) where **M** is a 3+1 dimensional manifold and **g** is an Einstein metric on **M** , that is, a smooth, nondegenerate, 2-covariant tensor field with the property that at each point one can choose 3+1 vectors e_0, e_1, e_2, e_3 such that $\mathbf{g}(e_\alpha, e_\beta) = \eta_{\alpha\beta}$; $\alpha, \beta = 0, 1, 2, 3$ where η is the diagonal matrix with entries $-1, 1, 1, 1$. The Einstein metric divides the nonzero vectors X in the tangent space at each point into time-like, null or space-like vectors according to whether the quadratic form $< X, X >= \mathbf{g}_{\alpha\beta} X^\alpha X^\beta$ is, respectively, negative, zero or positive.

The set of null vectors form a double cone, called the null cone of the corresponding point. The set of time-like vectors form the interior of this cone. It has two connected components whose boundaries are the corresponding components of the null cone. The set of space-like vectors is the exterior of the null cone, a connected open set. Any physically meaningful space- time should be time orientable, that is, one can choose in a continuous fashion a future directed component of the set of time-like vectors. This allows us to specify the causal future and past of any point in space-time. More general, the causal future of a set $S \subset \mathbf{M}$, denoted by $J^+(S)$, is defined as the set of points q which can be reached by a future directed causal curve [1] which initiates at S. Similarly $J^-(S)$ consists of the set of all points q which can be reached, from S, by a past directed causal curve.

The boundaries of past and future sets of points in **M** are null geodesic cones, often called light cones. Their specification defines the *causal structure* of the space-time which, up to a conformal factor, uniquely determines the metric.

A hypersurface M in **M** is said to be space-like if its normal direction is time-like at every point on M. We denote by g the Riemannian metric induced by **g** on M. The covariant differentiation on the space-time **M** will be denoted by **D**, while that on M will be written with the symbols D or ∇. Similarly we denote by **R**, resp. R, the Riemann curvature tensors of **M**, resp. M. Recall that for any given vectorfields X,Y,Z on (**M**,**g**),

$$\mathbf{D}_X\mathbf{D}_Y - \mathbf{D}_Y\mathbf{D}_X Z = \mathbf{R}(X,Y)Z + \mathbf{D}_{[X,Y]}Z$$

[1] A differentiable curve $\lambda(t)$ whose tangent at every point is a future directed time -like or null vector

or, in components, relative to an arbitrary frame e_α, $\alpha = 0, 1, 2, 3$,

$$\mathbf{D}_\beta \mathbf{D}_\alpha Z^\gamma = \mathbf{D}_\alpha \mathbf{D}_\beta Z^\gamma + \mathbf{R}^\gamma{}_{\sigma\beta\alpha} Z^\sigma$$

The extrinsic curvature, or second fundamental form, of M in **M** will be denoted by k. Recall that , if T denotes the future directed unit normal to M we have,

$$k_{ij} = - < \mathbf{D}_{e_i} T, e_j > = < T, \mathbf{D}_{e_i} e_j >$$

with e_i, i=1,2,3, an arbitrary frame on M.

We will use the notation $\in_{\alpha\beta\gamma\delta}$ to express the components of the volume element $d\mu_{\mathbf{M}}$ relative to an arbitrary frame. Similarly, if e_i, i=1,2,3 is an arbitrary frame on M, then $\in_{ijk} = \in_{oijk}$ are the components of $d\mu_M$, the volume element of M, with respect to the frame $e_0 = T, e_1, e_2, e_3$.

The Riemann curvature tensor **R** of the space-time satisfies the following,

Bianchi Identities

$$\mathbf{D}_{[\epsilon} \mathbf{R}_{\alpha\beta]\gamma\delta} = \frac{1}{3}(\mathbf{D}_\epsilon \mathbf{R}_{\alpha\beta\gamma\delta} + \mathbf{D}_\alpha \mathbf{R}_{\beta\epsilon\gamma\delta} + \mathbf{D}_\beta \mathbf{R}_{\epsilon\alpha\gamma\delta}) = 0$$

The traceless part of the curvature tensor is

$$\begin{aligned} \mathbf{C}_{\alpha\beta\gamma\delta} &= \mathbf{R}_{\alpha\beta\gamma\delta} - \frac{1}{2}(\mathbf{g}_{\alpha\gamma}\mathbf{R}_{\beta\delta} + \mathbf{g}_{\beta\delta}\mathbf{R}_{\alpha\gamma} - \mathbf{g}_{\beta\gamma}\mathbf{R}_{\alpha\delta} - \mathbf{g}_{\alpha\delta}\mathbf{R}_{\beta\gamma}) \\ &\quad + \frac{1}{6}(\mathbf{g}_{\alpha\gamma}\mathbf{g}_{\beta\delta} - \mathbf{g}_{\alpha\delta}\mathbf{g}_{\beta\gamma})\mathbf{R} \end{aligned}$$

where the 2-tensor $\mathbf{R}_{\alpha\beta}$ and scalar **R** are respectively the Ricci tensor and the scalar curvature of the space-time. We call this the conformal curvature tensor of the space-time. We notice that the Riemann curvature tensor has twenty independent components while the conformal curvature and Ricci tensors have ten components each.

The conformal curvature tensor is a particular example of a Weyl tensor. These refers to arbitrary four tensors W which satisfy all the symmetry properties of the curvature tensor and in adition are traceless.

We say that such W's satisfy the Bianchi equation if, with respect to the covariant differentiation on **M**,

Bianchi Equation

$$\mathbf{D}_{[\epsilon} W_{\alpha\beta]\gamma\delta} = 0$$

For a Weyl tensorfield W the following definitions of left and right Hodge duals are equivalent;

$$^{\star}W_{\alpha\beta\gamma\delta} = \frac{1}{2} \in_{\alpha\beta\mu\nu} W^{\mu\nu}{}_{\gamma\delta}$$

$$W^{\star}_{\alpha\beta\gamma\delta} = W_{\alpha\beta}{}^{\mu\nu} \frac{1}{2} \in_{\mu\nu\gamma\delta}$$

where $\in^{\alpha\beta\gamma\delta}$ are the components of the volume element in **M**. One can easily check that, $^{\star}W = W^{\star}$ is also a Weyl tensorfield and $^{\star}(^{\star}W) = -W$. Given an arbitrary vectorfield X, we can define the *electric-magnetic decomposition* of W to be the pair of 2-tensors formed by contracting W with X according to the formulas,

$$ii_X(W)_{\alpha\beta} = W_{\mu\alpha\nu\beta} X^{\alpha} X^{\beta}$$

$$ii_X(^{\star}W)_{\alpha\beta} = {}^{\star}W_{\mu\alpha\nu\beta} X^{\alpha} X^{\beta}$$

These new tensors are both symmetric, traceless, and orthogonal to X. Moreover, they completely determine W, provided that X is not null (see [Ch-Kl]).

Given a vectorfield X and a Weyl field W, $\mathcal{L}_X W$ is not, in general, a Weyl field, since it fails to be traceless. To compensate for this we define its modified Lie derivative,

$$\begin{aligned} \hat{\mathcal{L}}_X W_{\alpha\beta\gamma\delta} &= \mathcal{L} W_{\alpha\beta\gamma\delta} - \frac{1}{2}\left(\pi^{\mu}{}_{\alpha} W_{\mu\beta\gamma\delta} + \pi^{\mu}{}_{\beta} W_{\alpha\mu\gamma\delta}\right) \\ &+ \pi^{\mu}{}_{\gamma} W_{\alpha\beta\mu\delta} + \pi^{\mu}{}_{\delta} W_{\alpha\beta\gamma\mu}\Big) + \frac{3}{8} \mathrm{tr}\pi W_{\alpha\beta\gamma\delta} \end{aligned}$$

where π is the deformation tensor of X i.e.,

$$\pi = \mathcal{L}_X \mathrm{g}$$

One can associate to the conformal curvature tensor or, more general, to any Weyl tensorfield W, a 4-tensor which is quadratic in W and plays precisely the same role, for solutions of the Bianchi equations, as the energy-momentum tensor of an electromagnetic field plays for solutions of the Maxwell equations.

Bell-Robinson Tensor

$$Q_{\alpha\beta\gamma\delta} = 1/2(W_{\alpha\mu\beta\nu}W_{\gamma}{}^{\mu}{}_{\delta}{}^{\nu} + {}^{\star}W_{\alpha\mu\beta\nu}\,{}^{\star}W_{\gamma}{}^{\mu}{}_{\delta}{}^{\nu})$$

Q is fully symmetric and traceless, moreover it satisfies the positive energy condition, namely Q(X,Y,X,Y) is positive whenever X,Y are future directed time-like vectors(see [Ch-Kl] , for a proof of the above properties of Q). Moreover,

$$\mathbf{D}^{\delta}Q_{\alpha\beta\gamma\delta} = 0$$

whenever W satisfies the Bianchi equations. This remarkable property of the Bianchi equations is intimately connected with its conformal properties. Indeed they are covariant under conformal isometries. That is, if $\phi : \mathbf{M} \longrightarrow \mathbf{M}$ is a conformal isometry of the space-time, i.e $\phi_{*}\mathbf{g} = \Omega^2\mathbf{g}$ for some scalar Ω, and W is a solution then so is $\Omega^{-1}\phi_{*}W$.

It is well known that the causal structure of an arbitrary Einstein space time can have undesirable pathologies. All these can be avoided by postulating the existence of a Cauchy hypersurface in M, i.e. a hypersurface Σ with the property that any causal curve intersects it at precisely one point.[2] Einstein space-times with this property are called *globally hyperbolic.* Such space-times are in particular stable causal, i.e. they allow the existence of a globally defined differentiable function t whose gradient, $\mathbf{D}t$ is everywhere time-like. We call t a *time function* and the foliation given by its level surfaces a *t- foliation.* We denote by T the future directed unit normal to the foliation.

Topologically, a space-time foliated by the level surfaces of a time function is diffeomorphic to a product manifold $\Re \times \Sigma$ where Σ is a three

[2]In particular Σ is a space-like hypersurface

dimensional manifold. Indeed the space-time can be parametrized by points on the slice t=0 by following the integral curves of $\mathbf{D}t$. Moreover, relative to this parametrization the space-time metric takes the form,

$$ds^2 = -\phi^2(t,x)dt^2 + \sum_{i,j=1}^{3} g_{ij}(t,x)dx^i dx^j \tag{1.0.1}$$

where $x = (x^1, x^2, x^3)$ are arbitrary coordinates on the the slice t=0. The function $\phi(t,x) = \frac{1}{<\mathbf{D}t,\mathbf{D}t>^{1/2}}$ is called the *lapse function* of the foliation, g_{ij} its first fundamental form. We refer to 1.0.1 as the canonical form of the space-time metric with respect to the foliation.

The foliation is said to be normalized at infinity if

Normal Foliation Condition

$$\phi \longrightarrow 1 \quad \text{as} \quad x \longrightarrow \infty \quad \text{on each leaf } \Sigma_t$$

The second fundamental form of the foliation, i.e. the extrinsic curvature of the leaves Σ_t, is given by,

$$k_{ij} = -(2\phi)^{-1}\partial_t g_{ij} \tag{1.0.2}$$

We denote by ∇ the induced covariant derivative on the leaves Σ_t and by R_{ij} the corresponding Ricci curvature tensor. Relative to an orthonormal frame e_1, e_2, e_3 tangent to the leaves of the foliation we have the following formulas:

$$\begin{aligned}
\mathbf{D}_i e_j &= \nabla_i e_j - k_{ij}T \\
\mathbf{D}_i T &= -k_{ij}e_j \\
\mathbf{D}_T e_i &= \bar{\mathbf{D}}_T e_i + (\phi^{-1}\nabla_i\phi)T \\
\mathbf{D}_T T &= (\phi^{-1}\nabla_i\phi)e_i
\end{aligned}$$

where $\bar{\mathbf{D}}_T e_i$ denotes the projection of $\mathbf{D}_T e_i$ to the tangent space of the foliation. It is convenient to calculate relative to a frame for which $\bar{\mathbf{D}}_T e_i = 0$[3].

[3] called Fermi propagated

Since Σ_t is three dimensional, we recall that the Ricci curvature R_{ij} completely determines the induced Riemann curvature tensor R_{ijkl} according to the formula,

$$R_{ijkl} = g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk} - 1/2(g_{ik}g_{jl} - g_{jk}g_{il})R$$

where R is the scalar curvature $g^{ij}R_{ij}$. The second fundamental form k, the lapse function ϕ and the Ricci curvature tensor R_{ij} of the foliation are connected to the space-time curvature tensor $\mathbf{R}_{\alpha\beta\gamma\delta}$ according to the following,

The Structure Eqts. Of The Foliation

$$\begin{aligned} \partial_t k_{ij} &= -\nabla_i\nabla_j\phi + \phi(\mathbf{R}_{iTjT} - k_{ia}k^a{}_j) && (1.0.3a) \\ \nabla_i k_{jm} - \nabla_j k_{im} &= \mathbf{R}_{mTij} && (1.0.3b) \\ R_{ij} - k_{ia}k^a{}_j + k_{ij}trk &= \mathbf{R}_{iTjT} + \mathbf{R}_{ij} && (1.0.3c) \end{aligned}$$

where ∂_t denotes the partial derivative with respect to t and $\mathbf{R}_{iTjT}$, $\mathbf{R}_{mTij}$ are the components $\mathbf{R}(\partial_i, \mathbf{T}, \partial_j, \mathbf{T})$ and respectively, $\mathbf{R}(\partial_m, \mathbf{T}, \partial_i, \partial_j)$ of the space-time curvature relative to arbitrary coordinates on Σ. The equations 1.0.3a are the second variation formulas, while 1.0.3b and 1.0.3c are the classical Gauss-Codazzi and, respectively, Gauss equations of the foliation.

In view of 1.0.3c, the equation 1.0.3a becomes,

$$\partial_t k_{ij} = -\nabla_i\nabla_j\phi + \phi(-\mathbf{R}_{ij} + R_{ij} + trkk_{ij} - 2k_{ia}k^a{}_j) \qquad (1.0.3d)$$

Taking the trace of the equations 1.0.3c, 1.0.3b and 1.0.3a respectively, we derive,

$$\begin{aligned} R - |k|^2 + (trk)^2 &= 2\mathbf{R}_{TT} + \mathbf{R} && (1.0.4a) \\ \nabla^j k_{ji} - \nabla_i trk &= \mathbf{R}_{Ti} && (1.0.4b) \\ \partial_t trk &= -\triangle\phi + \phi(\mathbf{R}_{TT} + |k|^2) && (1.0.4c) \end{aligned}$$

where $|k|^2 = k_{ij}k^{ij}$

By contrast to Riemannian geometry where the basic covariant equations one encounters are of elliptic type, in Einstein geometry the basic equations are hyperbolic. The causal structure of the space-time is tied to the evolutions feature of the corresponding equations. This is particularly true for the Einstein field equations where the space-time itself is the dynamic variable.

The Einstein field equations were proposed by Einstein as a unified theory of space-time and gravitation. As mentioned above the space-time (**M** ,**g**) is the unknown; one has to find an Einstein metric **g** such that,

Einstein Field Equations

$$\mathbf{G}_{\mu\nu} = 8\pi \mathbf{T}_{\mu\nu}$$

where $\mathbf{G}_{\mu\nu}$ is the tensor $\mathbf{R}_{\mu\nu} - 1/2\mathbf{g}_{\mu\nu}\mathbf{R}$, with $\mathbf{R}_{\mu\nu}$ the Ricci curvature of the metric, $\mathbf{R}$ its scalar curvature and $\mathbf{T}_{\mu\nu}$ the energy momentum tensor of a matter field (e.g. the Maxwell equations). Contracting twice the Bianchi identities $\mathbf{D}_{[\epsilon}\mathbf{R}_{\alpha\beta]\gamma\beta} = 0$ we derive

Contracted Bianchi Identities

$$\mathbf{D}^{\nu}\mathbf{G}_{\mu\nu} = 0$$

which are equivalent to the divergence equations of the matterfield,

$$\mathbf{D}^{\nu}\mathbf{T}_{\mu\nu} = 0$$

In the simplest situation of the physical vacuum, i.e. **T**=0, the Einstein equations take the form,

Einstein-Vacuum Equations

$$\mathbf{R}_{\mu\nu} = 0$$

In view of the four contracted Bianchi identities mentioned above, the Einstein-Vacuum equations, or shortly E-V, can be viewed as a system of 10-4=6 equations for the 10 components of the metric tensor **g**. The remaining 4 degrees of freedom correspond to the general covariance of the equations. Indeed if $\Phi : M \longrightarrow M$ is a difeomorphism then the pairs (M,g) and $(M, \Phi^{\star}\mathrm{g})$ represent the same solution of the field equations.

Written explicitely in an arbitrary system of coordinates the E-V equations lead to a degenerate system of equations. The well posedness of the Cauchy problem , which we discuss below, was proved however by Y.Choquet-Bruhat in harmonic coordinates(see [Br]), yet as she has pointed out later these are unstable in the large. This problem of finding globally stable, well posed coordinate conditions is the first major dificulty one has to overcome in the construction of global solutions to the Einstein equations.

To emphasise the dynamic character of the E-V equations it is helpful to express them in terms of the parameters ϕ, g, k of an arbitrary t- foliation. Thus, assuming that the space-time $(\mathbf{M},\mathbf{g})$ can be foliated by the level surfaces of a time function t, and writing $\mathbf{g}$ in its canonical form 1.0.1, the E-V equations are equivalent to the following,

Constraint Equations for E-V

$$\nabla^j k_{ji} - \nabla_i trk = 0 \tag{1.0.5a}$$

$$R - |k|^2 + (trk)^2 = 0 \tag{1.0.5b}$$

Evolution Equations for E-V

$$\partial_t g_{ij} = -2\phi k_{ij} \tag{1.0.6a}$$

$$\partial_t k_{ij} = -\nabla_i \nabla_j \phi + \phi(R_{ij} + trk k_{ij} - 2k_{ia}k^a{}_j) \tag{1.0.6b}$$

Indeed the equivalence of the equations 1.0.5a, 1.0.5b, 1.0.6a, 1.0.6b with the E-V is an immediate consequence of 1.0.4a , 1.0.4b and 1.0.5a.

Also, 1.0.4c becomes;

$$\partial_t trk = - \Delta \phi + \phi(R + (trk)^2) \tag{1.0.7}$$

Given a t-foliation we denote by E,H the electric-magnetic decomposition of the curvature tensor $\mathbf{R}$ of an E-V manifold with respect to T, the future oriented unit normal to the time foliation. Clearly E,H are symmetric traceless 2 tensors tangent to the foliation. In view of these definitions the equations 1.0.3b and 1.0.3c become

$$\nabla_i k_{jm} - \nabla_j k_{im} = \in_{ij}{}^l H_{lm} \tag{1.0.8a}$$

$$R_{ij} - k_{ia}k^a{}_j + k_{ij} trk = E_{ij} \tag{1.0.8b}$$

Remark that the total number of unknowns in the Evolution Equations 1.0.6a and 1.0.6b is 13 while the total number of equations is only 12. This discrepancy correponds to the remaining freedom of choosing the time function t which defines the foliation. To emphasize the crucial importance of making an appropriate choice of time function we note that the natural choice $\phi = 1$, corresponding to the temporal distance function from an initial hypersurface, leads to finite time break-down. This can be seen from the equation 1.0.7 which becomes in this case, in view of 1.0.5b,

$$\partial_t trk = |k|^2 \geq (\mathrm{tr}k)^2$$

We also remark that, in view of the twice contracted Bianchi identities, if g, k satisfy the Evolution Equations, then the Constraint Equations 1.0.5a and 1.0.5b are automatically satisfied on any Σ_t provided they are satisfied on a given initial slice Σ_{t_0}. Therefore they can be regarded as constraints on given initial conditions for g and k. According to this an *initial data set* for E-V is defined to be a triplet (Σ, g, k) consisting of a three dimensional manifold Σ together with a Riemannian metric g and covariant symmetric 2-tensor k which satisfy the constraint equations 1.0.5a, 1.0.5b on Σ.

A development of an initial data set consists of an Einstein-Vacuum space-time (**M**,g) together with an embedding $i : \Sigma \longrightarrow \mathbf{M}$ such that g and k are the induced first and second fundamental forms of Σ in **M**. The central problem in the mathematical theory of E-V equations is the study of the evolution of general initial data sets.

The simplest solution of E-V equations is the Minkowski space-time $\mathbf{R}^{3+1}$, the i.e the space $\mathbf{R}^4$ together with a given Einstein metric $<,>$ and a canonical coordinate system (x^0, x^1, x^2, x^3) such that

$$< \partial_\alpha, \partial_\beta >= \eta_{\alpha\beta}; \ \ \alpha, \beta = 0, 1, 2, 3$$

The issue we want to adress in our work is that of the global nonlinear stability of the Minkowski space-time. More precisely we want to investigate whether Cauchy developments of initial data sets which are close, in an appropriate sense, to the trivial data set lead to global, smooth, geodesically complete solutions of the Einstein-Vacuum equations which remain close, in an appropriate, global sense, to the Minkowski space-time. We like to stress the fact that at the present time it is not even known whether there are, apart from the Minkowski space-time, *any smooth, geodesically complete solution which becomes flat at infinity on any given space-like direction.* Any attempt to significantly simplify the problem by looking for solutions with additional symmetries fails as a consequence of the well known results of Lichnerowitz for static solutions [4], and Birkhoff for spherically symmetric solutions. According to the first, a static solution which is geodesically complete and flat at infinity on any space-like hypersurface must be flat. The Birkhoff theorem asserts that all spherically symmetric solutions of the E-V equations are static. Thus, disregarding the Schwartzschild solution which is not geodesically complete, the only such solution which becomes flat at space-like infinity, is the Minkowski space-time.

The problem of stability of the Minkowski space-time is closely related to that of characterizing the space-time solutions of the Einstein-Vacuum equations which are *globally asymptotically flat* i.e., as defined in the physics literature, space-times which become flat as we approach infinity in any direction. Despite the central importance that such space-times have in General Relativity, as corresponding to isolated physical systems, it is not at all setled how to define them correctly, consistent with the field equations. Attempts to develop such a notion have been made however in the last 25 years (see [Ne-To] for a survey) beginning with the work of Bondi([Bo-Bu-Me], [Bo])(see also Sacks [Sa]) who introduced the idea to analyze solutions of the field equations along null hypersurfaces. The present state of understanding was set by Penrose([Pe2] , [Pe1]) who formalized the idea of asymptotic flatness by adding a boundary at infinity attached through a smooth *conformal compactification.* However it remains questionable whether there exists any nontrivial [5] solution of the field equations which satisfy the Penrose requirements. Indeed his regularity assumptions translate into fall-off conditions of the curvature which may be too stringent and thus may fail to be satisfied by any solution which would allow gravitational waves. Moreover, the picture given by the conformal compactification

[4] A space-time is said to be stationary if there exists a one parameter group of isometries whose orbits are time-like curves. It is said to be static if, in addition, the orbits of the group are orthogonal to a space-like hypersurface

[5] Namely a nonstationary solution.

fails to adress the crucial issue of the relationship between conditions in the past and behavior in the future.

We believe that a real understanding of asymptotically-flat spaces can only be accomplished by constructing them from initial data, and studying their asymptotic behaviour. This is precisely the objective we set up to achieve.

To make our discussion more precise we have to introduce the notion of an asymptotically-flat initial data set. By this we understand an initial data set (Σ, g, k) with the property that the complement of a finite set in Σ is diffeomorphic to the complement of a ball in R^3 (i.e. Σ is diffeomorphic to R^3 at *infinity*) and the notion of energy, linear and angular momentum are well defined and finite. These can be unambiguously defined for the following class of initial data sets which we will refer to as *strongly asymptotically flat.*

We say that an initial data set (Σ, g, k) satisfies the S.A.F. condition if g,k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighborhood of infinity such that, as $r = [\sum_{i=1}^3 (x^i)^2]^{1/2} \to \infty$

S.A.F. Initial Data Sets[6]

$$g_{ij} = (1 + 2M/r)\delta_{ij} + o_4(r^{-3/2}) \qquad (1.0.9a)$$

$$k_{ij} = o_3(r^{-5/2}) \qquad (1.0.9b)$$

We shall call the leading term, $(1 + 2M/r)\delta_{ij}$, in the the expansion 1.0.9a the *Schwartzschild Part* of the metric g. [7]

Given such a data set the ADM (Arnowitt, Deser and Misner) definitions of energy E, linear momentum P and angular momentum J are given by,

$$E = 1/16\pi \lim_{r\to\infty} \int_{S_r} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii}) N^j dA$$

$$P_i = 1/8\pi \lim_{r\to\infty} \int_{S_r} (k_{ij} - trk g_{ij}) N^j dA \quad ; \ i = 1, 2, 3$$

[6] A function f is said to be $o_m(r^{-k})$, resp. $O_m(r^{-k})$, as $r \to \infty$ if $\partial^l f(x) = o(r^{-k-l})$, resp. $O(r^{-l-k})$, for any $l = 0, 1 \ldots m$, where ∂^l denote all the partial derivatives of order l relative to the coordinates x^1, x^2, x^3

[7] It is the same as that of a space-like hypersurface, ortogonal to the Killing vectorfield of a Schwartzschild space-time

$$J_i = 1/8\pi \lim_{r\to\infty} \int_{S_r} \in_{iab} x^a(k^{bj} - g^{bj} trk)N_j dA \quad ; i = 1,2,3$$

where S_r is the coordinate sphere of radius r, N is the exterior unit normal to it and dA its area element. Clearly the limits on the right hand side of the formulas defining E and P exist and are finite.To check that J is well defined one has to remark that the difference between the integrals on two different spheres r_1, r_2, can be written as a volume integral of an expression which involves, as higher order term, $\partial_j k_b{}^j - \partial_b(trk) = \nabla_j k_b{}^j - \nabla_b(trk) + o(r^{-9/2})$. The assertion follows then with the help of the constraint equations 1.0.5a.

Moreover, due to our conditions 1.0.9a and 1.0.9b we have

$$E = M \quad , \quad P = 0$$

Thus, the S.A.F. condition implies that the initial data set is in a center of mass frame. In view of the positive mass theorem M must be a positive number vanishing only if the initial data set is flat.

The definition of the energy - momentum (E, P_1, P_2, P_3) and the angular momentum J_1, J_2, J_3 are independent of the particular choice of the coordinates x^1, x^2, x^3 in the definition of S.A.F initial data sets.[8] Moreover, they are preserved by the evolution equations 1.0.6a and 1.0.6b of a normally foliated(see definition on page 6) E-V space-time. This can be easily checked by taking the time derivatives of the expressions defining E,P,J.

We believe that the question we are investigating here, namely the stability of the Minkowski space-time, requires initial data sets with finite energy, linear and angular mometum.

In its least precise version our main result asserts the following,

First Version of the Main Theorem

Any Strongly Assymptotically Flat initial data set which satisfies, in addition, a Global Smallness Assumption, leads to a unique, globally hyperblic, smooth and geodesically complete development, solution of

[8]Indeed, first remark that the definitions are invariant under rigid transformations of the coordinates x^1, x^2, x^3. It thus sufices to show that the variations of the integrals defining E,P,J, with respect to one parameter groups of diffeomorphisms generated by vectofields $\xi = O_3(1)$ as $r \to \infty$, vanish in the limit.

the Einstein-Vacuum Equations. Moreover, this development is globally asymptotically flat, by which we mean that its Riemann curvature tensor approaches zero on any causal or space-like geodesic, as the corresponding affine parameter tends to infinity.

The main difficulties one encounters in the proof of our result are the following:

1. The problem of coordinates.
2. The strongly nonlinear hyperbolic features of the Einstein equations.
3. The logarithmic divergence of the light cones.

1. The problem of coordinates is, as we have mentioned above the first major difficulty one has to overcome when trying to solve the Cauchy problem for the Einstein equations. Our strategy is based on two ideas. First, we describe our space-time by specifying, instead of full coordinate conditions, only a time function whose level hypersurfaces are *maximal.* [9] More precisely we impose, in addition to the equations 1.0.5a, 1.0.5b, the constraint

$$trk = 0 \tag{1.0.10}$$

With this choice we remove the indeterminacy of the evolution equations 1.0.6a, 1.0.6b and obtain the following *determined* system of equations for the maximal foliation of an E-V space-time:

Constraint Equations of a Maximal foliation

$$\begin{aligned} trk &= 0 && (1.0.11a) \\ \nabla^j k_{ji} &= 0 && (1.0.11b) \\ R &= |k|^2 && (1.0.11c) \end{aligned}$$

[9] In Einstein geometry a maximal hypersurface is one which is space-like and maximizes the volume among all possible compact perturbations of it.

Evolution Equations of a Maximal foliation

$$\partial_t g_{ij} = -2\phi k_{ij} \quad (1.0.12a)$$
$$\partial_t k_{ij} = -\nabla_i \nabla_j \phi + \phi(R_{ij} - 2k_{ia}k^a{}_j) \quad (1.0.12b)$$

Lapse Equation of a Maximal foliation

$$\triangle \phi = |k|^2 \phi \quad (1.0.13)$$

Remark that the time function t is defined by specifying the level sets only up to a transformation of the form $t \longrightarrow f \circ t$ with f any, orientation preserving, diffeomorphism of the real line. However we can specify a unique t by requiring that, regarded as a parameter on an integral curve Γ_x of T which passes through a point x of Σ_0, it converges to the arclength on Γ_x as x tends to infinity on Σ_0. This is equivalent to the condition that ϕ tends to 1 at infinity on each Σ_t, which is precisely the normal foliation condition introduced above. Indeed, with the exception of the Minkowski space-time, the above definition specifies a unique time function. This is due to the fact that, when the A.D.M. energy E is non-zero, there is a unique maximal foliation with respect to which the linear momentum P vanishes. In physical terms, this foliation constitutes the center of mass frame of the corresponding isolated system.

The second idea is to make use in a fundamental way of the Bianchi identities of the space-time and the Bell-Robinson tensor introduced below. The basic observation is that, once we have good estimates for the curvature tensor **R**, all the parameters of the foliation, i.e. g, k, ϕ, are determined purely by solving the elliptic system,

$$R_{ij} - k_{ia}k^a{}_j = E_{ij} \quad (1.0.14a)$$
$$\operatorname{curl} k_{ij} = H_{ij} \quad (1.0.14b)$$
$$\nabla^j k_{ji} = 0 \quad (1.0.14c)$$
$$trk = 0 \quad (1.0.14d)$$

together with the lapse equation 1.0.13. The equations 1.0.14a, 1.0.14b are immediate consequences of, respectively, 1.0.8b, 1.0.8a with $curl\, k_{ij} = \in_i{}^{ab} k_{jab}$ Thus, all the evolution features of the Einstein equations are contained in the Bianchi identities, which have the great advantage of being covariant.

2. The other major obstacle in the study of the Einstein equations consists in their hyperbolic and strongly nonlinear character. The only powerful analytic tool we have in the study of nonlinear hyperblic equations, in the physical space-time dimension, are the energy estimates. Yet the classical energy estimates are limited to proving estimates which are local in time. The difficulty has to do with the fact that, in order to control the higher energy norms of the solutions, one has to control the integral in time of their bounds in uniform norm. In recent years however, new techniques were developed, based on modified energy estimates and the invariance property of the corresponding linear equations, which were applied to prove global or long time existence results for nonlinear wave equations (see [Kl3], [Kl1]). More precisely, one uses the Killing and conformal Killing vector fields generated by the conformal group of the Minkowski space-time to define a global energy norm which is invariant relative to the linear evolution. The precise asymptotic behaviour, including the uniform bounds mentioned above, are then an immediate consequence of a global version of the Sobolev inequalities (see [Kl3], [Kl2], [Ho]).

The relevant linearized equations for the E-V field equations are the Bianchi equations (see page 4) in Minkowski space-time. As a first preliminary step in our program, we have analyzed the complete asymptotic properties of the Bianchi equations[10] in Minkowski space-time by using only energy estimates and the conformal invariance properties of the equations in the spirit of the ideas outlined above(see [Ch-Kl]).

However to derive a global existence result one also needs to investigate the structure of the nonlinear terms [11]. It is well known that arbitrary quadratic nonlinear perturbations of the scalar wave equation, even when derivable from a Lagrangean, could lead to formation of singularities unless a certain structural condition, which we have called the *Null condition*, is satisfied(see [Ch], [Kl1]). It turns out that

[10] In [Ch-Kl] they were called Spin-2 equations

[11] generated each time we commute the Bianchi Identities with a one of the vectorfields used in the definition of the global energy norm.

the apropriate, tensorial version of this structural condition is satisfied by the Einstein equations. One could say that the troublesome nonlinear terms, which could have led to formation of singularities are in fact excluded due to to the covariance and algebraic properties of the Einstein equations. This is in sharp contrast with the basic nonlinear hyperbolic equations of classical mechanics. Indeed the equation of Nonlinear Elasticity [John] or Compressible fluids [Si], in four space and time dimensions, form singularities even for arbitrary small initial conditions.

3. In implementing the strategy outlined in (1) and (2) one encounters a very serious technical difficulty. The *mass term* which appears in the Schwartzschild part of an (S.A.F.) initial data set, 1.0.9a, has the long range effect of distorting the asymptotic position of the null geodesic cones. They are expected to diverge logarithmically from their corresponding position in flat space-time. In addition to this their asymptotic shear[12] differs drastically from that in the Minkowski space-time. This difference reflects the presence of gravitational radiation in any nontrivial perturbation of the Minkowski space-time. To take this effect into account one has to appropriately modify the Killing and conformal Killing vectorfields used in the definition of the basic energy norm. We achieve this by an elaborate construction of an *optical function* whose level surfaces are outgoing null hypersurfaces related by a translation at infinity. By an optical function we mean a solution u of

Eikonal Equation

$$g^{\alpha\beta}\frac{\partial u}{\partial x^\alpha}\frac{\partial u}{\partial x^\beta}=0$$

The construction of the optical function and the approximate Killing and conformal Killing vectorfields related to it requires more than half of our work. The most demanding part in the construction is taken by the angular momentum vectorfields [13]. These are particularly important to our construction as they are crucial in circumventing the

[12] the traceless part of their null second fundamental form

[13] i.e the vectorfields which can be viewed as deformation of $\Omega_{ij}=x_i\partial_j-x_j\partial_i$, for i,j=1,2,3, of Minkowski space-time.

problem of slow decay at infinity of the initial data set. Thus, we do not estimate directly **R** from the Bianchi identities but only its Lie derivatives with respect to these vectorfields. This allows us to consider higher weighted norms than will be possible for **R**. Yet, as it turns out, the latter can be easilly estimated in terms of the former. [14] Similarly, we use the approximate Killing vectorfield T, the unit normal to the foliation, to allow higher weighted norms for the Lie derivatives of the curvature tensor with respect to T [15].

As outlined above, our construction requires initial data sets which satisfy, in addition to the constraint equations, the maximal condition $trk = 0$. We will refer to them as maximal in what follows.

To make the statemant of our main theorem precise we need also to define what we mean by the global smallness assumption. Before stating this condition, we assume the metric g to be complete and introduce the following quantity:

$$\begin{aligned} Q(x_{(0)}, b) &= \sup_{\Sigma}\{b^{-2}(d_0^2 + b^2)^3|Ric|^2\} + \\ & b^{-3}\{\int_\Sigma \sum_{l=0}^{3}(d_0^2 + b^2)^{l+1}|\nabla^l k|^2 + \int_\Sigma \sum_{l=0}^{1}(d_0^2 + b^2)^{l+3}|\nabla^l B|^2\} \end{aligned}$$

where $d_0(x) = d(x_{(0)}, x)$ is the Riemannian geodesic distance between the point x and a given point $x_{(0)}$ on Σ, b a positive constant, $|Ric|^2 = R^{ij}R_{ij}$, ∇^l denote the l covariant dervatives and B is the symmetric, traceless 2-tensor tensor [16],

$$B_{ij} = \in_j{}^{ab} \nabla_a(R_{ib} - 1/4 g_{ib}R)$$

[14] this fact seems entirely plausible in view of the Birkhoff theorem.

[15] in view of the Lichnerowitz theorem, this procedure allows us to obtain information about **R** itself.

[16] Remark that B, called the Bach tensor, is dual to the tensor $R_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + 1/4(g_{ik}\nabla_j R - g_{ij}\nabla_k R)$ whose vanishing characterizes locally conformally flat three dimensional manifolds (see [Eisen]). Thus, up to lower order terms, the Schwartzschild part of g does not affect it.

The symmetry and tracelessness of B follow immediately from the twice contracted Bianchi identities $\nabla^j R_{ij} - \frac{1}{2}\nabla_i R = 0$. In fact we can write;

$$B_{ij} = \frac{1}{2}(\in_i{}^{ab}\nabla_a \hat{R}_{jb} + \in_j{}^{ab}\nabla_a \hat{R}_{ib})$$

where $\hat{R}_{ij}$ is the traceless part of R_{ij}, $R_{ij} = \hat{R}_{ij} + \frac{1}{3}Rg_{ij}$.

We say that an S.A.F. initial data set, (Σ, g, k), satisfies the global smallnes assumption if

The Global Smallnes Assumption

The metric g is complete and there exists a sufficiently small positive ϵ s.t.

$$\inf_{x_{(0)}\in\Sigma, b\geq 0} Q(x_{(0)}, b) < \varepsilon \tag{1.0.15}$$

Second Version of the Main Theorem [17]

Any Strongly Assymptotically Flat, Maximal, initial data set which satisfies the Global Smallness Assumption 1.0.15, leads to a unique, globally hyperblic, smooth and geodesically complete solution of the Einstein-Vacuum Equations foliated by a normal, maximal time foliation. Moreover, this development is globally asymptotically flat. [18]

Remark: In view of the scale invariance property of the Einstein-Vacuum equations any initial data set Σ, g, k for which $Q(x_0, b) < \varepsilon$ can be rescaled to the new initial data set Σ, g', k' with $g' = b^{-2}g$, $k' = b^{-1}k$ for which $Q(x_0, 1) < \varepsilon$. The global existence for the new set is equivalent to global existence for the original set. This is due to the fact that the develpments $\mathbf{g}, \mathbf{g}'$ of the two sets are related by $\mathbf{g}' = b^{-2}\mathbf{g}$. It thus suffices to prove the theorem under the global smallness assumption

$$\inf_{x_{(0)}\in\Sigma Q(x_{(0)},1)} < \varepsilon$$

[17] The first version of the Theorem is not an immediate consequence of the second. It can be proved however by, first, devloping the initial data set locally in time and then, imbedding in it a maximal hypersurface. Imbedding results of the type one needs were obtained by Bartnik (see [Ba1]).

[18] A precise statement of the asymptotic behaviour for the curvature tensor $\mathbf{R}$ and also for the lapse function ϕ and second fundamental form k of the foliation will be given in the third version of the theorem.

We next indicate how to construct maximal initial data sets which are asymptotically flat and satisfy 1.0.15. This is based on the observation that the constraint equations 1.0.11a and 1.0.11b are conformal invariant. More precisely they are invariant with respect to the transformation, $g_{ij} \longrightarrow \Phi^4 g_{ij}$ and $k_{ij} \longrightarrow \Phi^{-2} k_{ij}$. Thus, given arbitrary solutions $\tilde{g}, \tilde{k}$ to the equations,

$$\begin{aligned} tr_{\tilde{g}} \tilde{k} &= 0 \qquad (1.0.16a) \\ \tilde{\nabla}^j \tilde{k}_{ji} &= 0 \qquad (1.0.16b) \end{aligned}$$

where $\tilde{\nabla}$ denotes the covariant differentiation with respect to the metric $\tilde{g}$, we infer that $g_{ij} = \Phi^4 \tilde{g}_{ij}$ and $k_{ij} = \Phi^{-2} \tilde{k}_{ij}$ are solutions to the same equations for arbitrary function Φ. To satisfy also the equation 1.0.11c we have to subject Φ to the Lichnerowitz equation

$$\tilde{\Delta}\Phi - \frac{1}{8}\tilde{R}\Phi + |\tilde{k}|^2_{\tilde{g}}\Phi^{-7} = 0 \qquad (1.0.17a)$$

In practice one does not solve directly the Lichnerowitz equation. The standard approach is to look for Φ of the form $\Phi = \Omega\Psi$ where Ω and Ψ are the conformal factors corresponding to transformations which take, first, an arbitrary solution of the equations 1.0.16a, 1.0.16b to a solution $\bar{g}, \bar{k}$ of the same equations and then, take $\bar{g}, \bar{k}$ to the desired solution g,k. The first conformal factor Ω is chosen so that the Ricci curvature $\bar{R}$ of $\bar{g}$ vanishes identically. Thus, Ω has to be a solution of the linear equation equation,

$$\tilde{\Delta}\Omega - \frac{1}{8}\tilde{R}\Omega = 0 \qquad (1.0.17b)$$

The second conformal factor Ψ is chosen such that the transformed variables g,k satisfy $R = |k|^2$. For this to happen Ψ has to be a solution to the equation,

$$\bar{\Delta}\Psi + \frac{1}{8}|\bar{k}|^2_{\bar{g}}\Psi^{-7} = 0 \qquad (1.0.17c)$$

Remark that, by virtue of the maximal principle, the equation 1.0.17c has allways a smooth solution, $\Psi \geq 1$, with $\Psi \longrightarrow 1$ as $x \longrightarrow \infty$ on Σ.

On the other hand a sufficient condition so that the equation 1.0.17b have a positive solution with the same property is that the $L^{3/2}$ norm of the negative part of $\bar{R}$ be sufficiently small. Therefore (Σ, g, k) is an initial data set satisfying the S.A.F. conditions 1.0.9a, 1.0.9b provided that the corresponding solutions $\tilde{g}, \tilde{k}$ of 1.0.16a, 1.0.16b verify,

$$\begin{aligned} \tilde{g}_{ij} &= \delta_{ij} + o_4(r^{-3/2}) \\ \tilde{k}_{ij} &= o_3(r^{-5/2}) \end{aligned}$$

and the negative part of $\bar{R}$ satisfy the smallness condition mentioned above. Moreover, g,k satisfy the Global Smallnes Assumption of the Theorem provided that the metric $\tilde{g}$ is complete and there exists a small positive ε such that,

$$\begin{aligned} \inf_{x_{(0)}\in\Sigma, a\geq 0} \{\sup_{\Sigma}(a^2 + \tilde{d}_0^2)^3 |\widetilde{Ricc}|^2 &+ \int_\Sigma \sum_{l=0}^{2} (a^2 + \tilde{d}_0^2)^{l+2} |\tilde{\nabla}\widetilde{Ricc}|^2 \\ &+ \int_\Sigma \sum_{l=0}^{3} (a^2 + \tilde{d}_0^2)^{l+1} |\tilde{\nabla}^l \tilde{k}|^2\} < \varepsilon \end{aligned}$$

where $\tilde{d}_0(x)$ denotes the Riemannian geodesic distance relative to $\tilde{g}$ between the point x and a given point $x_{(0)}$ on Σ.

It remains to discuss whether the equations 1.0.16a, 1.0.16b have solutions verifying the above properties. This can be done using the orthogonal, York, decomposition, of any symmetric, traceless 2-covariant tensor h, on a three dimensional Riemannian manifold $(\Sigma, \tilde{g})$, into a divergence free part $\tilde{k}$ and the traceless part of the deformation tensor of a vectorfield X,

$$h_{ij} = \tilde{k}_{ij} + \widehat{\mathcal{L}_X \tilde{g}_{ij}}$$

The vectorfield X has to be a solution of the York equation,

$$L_{\tilde{g}} X = \tilde{\text{div}}\, h \qquad (1.0.18)$$

where,

$$L_{\tilde{g}} X_j = \tilde{\nabla}^i (\tilde{\nabla}_i X_j + \tilde{\nabla}_j X_i - \frac{2}{3} \tilde{g}_{ij} \tilde{\nabla}^l X_l)$$

One can show that $L_{\tilde{g}}$ is injective on spaces of vectorfields X with appropriate decay at infinity. It is also onto for appropriate spaces of vectorfields. Thus, for given $\tilde{g} = \delta_{ij} + o_4(r^{-3/2})$, we select an appropriate $\tilde{k}$ by decomposing any symmetric traceless tensor $h = o_3(r^{-5/2})$ according to the definition above, where X is a solution to the York equation. For details of how to achieve this we refer to [Ch-Mu]. We also remark that the corresponding tensor $\tilde{k}$ is of order $o_3(r^{-5/2})$ if the principal term in the expansion of solutions X at infinity, namely the term of order r^{-1}, vanishes. This is so if and only if the linear momentum P_i of the corresponding initial data set is zero.

The proof of the Main Theorem hinges on an elaborate comparison argument with the Minkowski space-time at the level of the three geometric structures with which this is equiped.

- *The canonical space-like foliation* of Minkowski space-time is given by any choice of a one parameter family of parallel space-like hyperplanes, the level sets of the time function $t = x^0 = const.$

- *The null structure* of the Minkowski space-time is specified by one family of future null cones and another of past null cones with vertices on a time-like geodesic orthogonal to to the canonical space-like foliation. These families are the level sets of the *optical* functions $u = r - t$ and respectively, $v = r + t$, where $r = (\sum_{i=1}^{3} |x^i|^2)^{1/2}$ The null vectors $e_+ = \partial_t + \partial_r$ and $e_- = \partial_t - \partial_r$ are parallel to their respective gradients and span all the asymptotic null directions.

- *The conformal group structure* is given by the 15 parameter group of translations, Lorentz rotations, scaling and inverted translations. The corresponding infinitesimal generators of the group are,

1. The 4 generators of translations,

$$T_\mu = \partial_\mu \quad ; \mu = 0, 1, 2, 3$$

2. The 6 generators of the Lorentz group,

$$\Omega_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \quad ; \mu, \nu = 0, 1, 2, 3$$

where, $x_\mu = \eta_{\mu\nu} x^\nu$

3. The scaling vectorfield,

$$S = x^\mu \partial_\mu$$

4. The 4 inverted translation vectorfields,

$$K_\mu = -2x_\mu S + < x, x > \partial_\mu \quad ; \mu = 0, 1, 2, 3$$

We recall that the vactorfields in the first two groups are Killing while all the others are conformal Killing. [19] In particular the deformation tensors of S and K_0 are given by,

$$^{(S)}\pi = 2\eta \quad ; \quad ^{(K_0)}\pi = 4t\eta$$

As small perturbations of the Minkowski space-time, the solutions of the E-V which we want to construct will mirror the structures outlined above. In other words we construct them together with the following:

- A maximal space-like foliation, of the type described above, given by the level surfaces of a time function t.
- An appropriately defined optical function u whose level surfaces describe the structure of future null infinity.
- A familly of almost Killing and conformal Killing [20] vectorfields tied to the definition of t and u.

The intersection between a t-slice Σ_t and a u- null hypersurface C_u is a 2-surface $S_{t,u}$. Thus, the (t,u) foliations of the space-time define a codimension 2 foliation by 2-surfaces. This foliation is crucial in our work, the asymptotic behaviour of the curvature tensor $\mathbf{R}$ and the hessians of t and u can only be fully described by decomposing them to components tangent to $S_{t,u}$. We achieve this by introducing null pairs consisting of two future directed null vectors e_4 and e_3 orthogonal to $S_{t,u}$, with e_4 tangent to C_u and

$$< e_4, e_3 >= -2$$

[19] A vectorfield S in a space-time (M,g) is called Killing, resp. conformal Killing, if its deformation tensor $^{(X)}\pi = \mathcal{L}_X \mathbf{g}$ is zero, resp. proportional to g

[20] By almost conformal Killing we mean vectorfields whose trace-less part of their deformation tensors are small in an appropriate fashion.

The null frame can be standardized by picking e_4 such that $< e_4, T >= -1$. Onother possible choice is to take $e_4 = l, e_3 = \underline{l}$ with

$$l^\mu = -\mathrm{g}^{\mu\nu}\frac{\partial u}{\partial x^\nu}$$

We call this the l-null pair of t,u. A null pair together with an orthonormal frame e_1, e_2 on $S_{t,u}$ forms a null frame. The null- decomposition of a tensor relative to a null frame e_4, e_3, e_2, e_1 is obtained by taking contractions with the vectorfields e_4, e_3. For example, the null decomposition of the Riemann curvature tensor of an Einstein-Vacuum space-time consists of two S-tangent[21] symmetric traceless[22] tensors $\underline{\alpha}, \alpha$, two S-tangent 1-forms $\underline{\beta}, \beta$ and two scalars ρ, σ. They are defined by;

$$\begin{array}{llll} \mathbf{R}_{A3B3} = & \underline{\alpha}_{AB} & \mathbf{R}_{A4B4} = & \underline{\alpha}_{AB} \\ \mathbf{R}_{A334} = & 2\underline{\beta}_A & \mathbf{R}_{A434} = & 2\beta_A \\ \mathbf{R}_{3434} = & 4\rho & {}^*\mathbf{R}_{3434} = & 4\sigma \end{array} \quad (1.0.19)$$

As part of our Main Theorem we deduce the following asymptotic properties for the null components of the curvature tensor,

$$\sup_{\Sigma_t} \tau_+ \tau_-^{5/2} |\underline{\alpha}| \leq c \quad (1.0.20a)$$

$$\sup_{\Sigma_t} \tau_+^2 \tau_-^{3/2} |\underline{\beta}| \leq c \quad (1.0.20b)$$

$$\sup_{\Sigma_t} \tau_+^3 |\rho| \leq c \quad (1.0.20c)$$

$$\sup_{\Sigma_t} \tau_+^3 \tau_-^{1/2} |\sigma| \leq c \quad (1.0.20d)$$

$$\sup_{\Sigma_t} \tau_+^{7/2} |\beta| \leq c \quad (1.0.20e)$$

$$\sup_{\Sigma_t} \tau_+^{7/2} |\alpha| \leq c \quad (1.0.20f)$$

$$(1.0.20g)$$

where $\tau_- = (1 + u^2)^{1/2}$, $\tau_+ = (1 + v^2)^{1/2}$ with $v = 2r - u$ and $r = r(t, u)$ defined by requiring that the area of a surface $S_{t,u}$ is equal to $4\pi r^2$.

[21] A space-time tensorfield is called S-tangent if, at every point is tangent to the $S_{t,u}$ 2-surface passing through that point.

[22] This is an immediate consequence of the trace zero condition of the Einstein equations

Note that the "peeling property", connected to the smoothness of the conformal compactification, fails to be satisfied for the component α. Nevertheless our results confirm the picture given by the conformal compactification in its main outlines.

The proof of the Main Theorem relies on a continuity type argument. Using an adequate version of the local existence theorem we assume our space-time to be maximally extended up to a value t_* of the t-function. Starting with the "last slice" Σ_{t_*} we then define an appropriate optical function u. We use u and t to define a time-translation vectorfield T, which is simply the future directed unit normal to the t-foliation, an inverted time-translation[23] vectorfield K_0, a scaling vectorfield S and angular momentum operators ${}^{(a)}\Omega$. These vectorfields are used, in conjunction with the Bell-Robinson tensor associated to $\mathbf{R}$, to define the basic quantities $\mathcal{Q}_1(t), \mathcal{Q}_2(t)$;

$$\mathcal{Q}_1 = \int_\Sigma Q(\hat{\mathcal{L}}_{\mathbf{O}}\mathbf{R})(\bar{K},\bar{K},T,T) + \int_\Sigma Q(\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},T) \qquad (1.0.21)$$

$$\begin{aligned}\mathcal{Q}_2 &= \int_\Sigma Q(\hat{\mathcal{L}}_{\mathbf{O}}^2\mathbf{R})(\bar{K},\bar{K},T,T) + \int_\Sigma Q(\hat{\mathcal{L}}_{\mathbf{O}}\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},T)\\ &+ \int_\Sigma Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R})(\bar{K},\bar{K},\bar{K},T) + \int_\Sigma Q(\hat{\mathcal{L}}_T^2\mathbf{R})(\bar{K},\bar{K},\bar{K},T)\end{aligned} \qquad (1.0.22)$$

where $Q(\hat{\mathcal{L}}_{\mathbf{O}}\mathbf{R}) = \sum_{a=1}^3 Q(\hat{\mathcal{L}}_{{}^{(a)}\Omega}\mathbf{R})$.

In linear theory[24] the time derivatives of the corresponding quantities would be zero in flat space-time, in our case they give rise to cubic error terms which depend linearly on the deformation tensors of $K_0, T, S, {}^{(a)}\Omega$, and quadratic with respect to $\mathbf{R}$ and its covariant and Lie derivatives in the direction of $T, S, {}^{(a)}\Omega$. The crucial point of our overall

[23] This is the analogue of the vectorfield $K_0 = (t^2 + |x|^2)\partial_t + 2tx^i\partial_i$ in Minkowski space-time

[24] By this we mean the replacement of the curvature tensor of the space-time with an arbitrary Weyl tensor verifying the Bianchi equations

strategy is to control the time integral of these error terms. This depends on the one hand on the asymptotic behaviour of all components of $\mathbf{R}$ and its covariant derivatives, on the other hand on the asymptotic behavior of the deformation tensors of the vectorfields, and finally, due to the general covariance of the equations, on the cancellations of the "worst possible " cubic terms. The asymptotic behaviour of $\mathbf{R}$ and its covariant derivatives can be traced, due to global Sobolev and Poincaré inequalities and the Bianchi identities, back to the basic quantities $\mathcal{Q}_1$, $\mathcal{Q}_2$[25]. The same is true for the deformation tensors of $K_0, T, S, {}^{(a)}\Omega$; we show this by elaborate estimates for the lapse function ϕ, the second fundamental form k_{ij} of the t-foliation and the components of the Hessian of the optical function u. We are thus able to control the time integral of the error terms and show that $\mathcal{Q}_1, \mathcal{Q}_2$ cannot exceed a constant multiple of their values at $t = 0$. This fact allows us to continue our space-time beyond t_*.

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