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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

INITIAL-BOUNDARY VALUE PROBLEM FOR THE DISCRETE BOLTZMANN EQUATION

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I Introduction

In this note we study a typical initial-boundary value problem for the discrete Boltzmann equation in one dimensional space.

We consider a discrete velocity gas which consists of particles with a finite number of velocities $\underline{v}_i, i \in I$, where $\underline{v}_i$ are constant vectors in $\mathbf{R}^3$. Let $F_i(x,y,z,t)$ be the mass density of gas particles with the velocity $\underline{v}_i$ at time t and position $(x,y,z) \in \mathbf{R}^3$. In the discrete kinetic theory, the state of the gas in motion is determined by the densities F_i whose time-evolution is described by the so-called discrete Boltzmann equation [3].

We suppose that the gas occupies a strip $\Omega = \{(x,y,z) \in \mathbf{R}^3; 0 < x < 1\}$ between two rigid walls located at $x = 0$ and $x = 1$, and that the motion of the gas is homogeneous in both of the y and z-directions. In this case the densities F_i are depending only on (x,t) and satisfy the following one-dimensional discrete Boltzmann equation in the region $0 < x < 1$.

$$(1) \qquad \frac{\partial F_i}{\partial t} + v_i \frac{\partial F_i}{\partial x} = \sum_{jk\ell} (A^{ij}_{k\ell} F_k F_\ell - A^{k\ell}_{ij} F_i F_j), \quad i \in I,$$

where the summation is taken over all $j, k, \ell \in I$. Here we denoted by v_i the x-component of the velocity $\underline{v}_i$. The coefficients $A^{ij}_{k\ell}$ in (1) are non-negative constants satisfying the usual conditions :

$$(A1) \qquad A^{ji}_{k\ell} = A^{ij}_{k\ell} = A^{ij}_{\ell k} \ ,$$

$$(A2) \qquad A^{ij}_{k\ell}(v_i + v_j - v_k - v_\ell) = 0 \ ,$$

$$(A3) \qquad A^{ij}_{k\ell} = A^{k\ell}_{ij} \ ,$$

for $i, j, k, \ell \in I$. Note that (A2) implies the conservation of momentum (in the x-direction) in the microscopic collision process.

Concerning the interaction between particles and rigid walls, we assume, for simplicity, that the particles impinging on the wall $x = 0$ or $x = 1$ rebound according to the so-called diffuse reflection law [4] :

$$(2) \qquad \begin{aligned} F_i &= \Sigma_j^- B_{ij} F_j, \quad v_i > 0, \quad \text{on} \quad x = 0, \\ F_i &= \Sigma_j^+ \tilde{B}_{ij} F_j, \quad v_i < 0, \quad \text{on} \quad x = 1, \end{aligned}$$

where and in what follows Σ_j^- (resp.Σ_j^+) means the summation taken over all $j \in I$ with $v_j < 0$ (resp.$v_j > 0$). The coefficients B_{ij} and $\tilde{B}_{ij}$ in (2) are non-negative constants satisfying

$$(B1) \qquad \begin{aligned} \Sigma_i^+ v_i B_{ij} + v_j = 0, \quad v_j < 0, \\ \Sigma_i^- v_i \tilde{B}_{ij} + v_j = 0, \quad v_j > 0. \end{aligned}$$

This condition implies that there is no momentum flow (in the x-direction) on the walls $x = 0$ and $x = 1$. We assume in addition that there exists a constant Maxwellian $\{M_i; i \in I\}$ satisfying both of the conditions in (2), namely,

$$(B2) \qquad \begin{aligned} M_i &= \Sigma_j^- B_{ij} M_j, \quad v_i > 0, \\ M_i &= \Sigma_j^+ \tilde{B}_{ij} M_j, \quad v_i < 0. \end{aligned}$$

Finally we prescribe the initial densities :

$$(3) \qquad F_i(x,0) = F_{i0}(x), \quad i \in I,$$

where $0 \leq x \leq 1$. Thus our problem has been formulated as an initial-boundary value problem (1), (2), (3).

Recently, for the pure initial value problem (1), (3) on the whole space $-\infty < x < \infty$, the global existence of solution was proved when the initial densities are bounded and non-negative [1], [2]. On the other hand, nothing is known about initial-boundary value problems. The aim of this note is to show a global existence result for the initial-boundary value problem (1), (2), (3).

This work was done while the author was visiting the Laboratoire de Modélisation en Mécanique, Université de Paris VI. He is grateful to Professor H. Cabannes, Professor R. Gatignol, and to the other members of the laboratory for their hospitality.

2. Main result

Let us denote by C_+^1 be the space of functions which are continuously differentiable and are positive. Our global existence result is then stated as follows.

Theorem.— *(global existence) Suppose that the initial data $F_{i0}, i \in I$, belong to $C_+^1([0,1])$ and satisfy the compatibility conditions up to order 1 on the boundaries $x = 0$ and $x = 1$. Then the problem (1), (2), (3) has a unique global solution $\{F_i, i \in I\}$ satisfying $F_i \in C_+^1([0,1] \times [0,\infty)), i \in I$.*

For the proof of this theorem, it suffices to show a local existence result and a suitable a priori estimate. In order to state these two results shortly, we introduce the following notations.

$$(4) \qquad E_0 = \max_{i,x} F_{i0}(x),$$

$$(5) \qquad E(t) = \sup_{0 \leq \tau \leq t} \max_{i,x} F_i(x,\tau),$$

where the maximum is taken over all $i \in I$ and $x \in [0,1]$.

Proposition 1.— *(local existence) Suppose that the initial data satisfy the same conditions as in Theorem. Then there exists a positive constant T_0 depending only on E_0 such that the problem (1), (2), (3) admits a unique solution satisfying $F_i \in C_+^1([0,1] \times [0,T_0]), i \in I$. Moreover, we have*

$$(6) \qquad E(t) \leq 2E_0, \quad t \in [0,T_0].$$

The life span T_0 of the local solution depends only on E_0, and therefore we can extend the solution globally in time, if we have the following a priori estimate for $E(t)$.

Proposition 2.— *(a priori estimate) Consider a solution to the problem (1), (2), (3), which satisfies* $F_i \in C^1_+([0,1] \times [0,T]), i \in I$, *for* $T > 0$. *Then we have*

$$E(t) \leq C E_0 e^{Kt}, \quad t \in [0,T], \tag{7}$$

where $C > 1$ *is a number and* $K > 0$ *is a constant depending only on* E_0.

The local existence result stated in Proposition 1 can be obtained without great difficulty, because the standard iteration method is applicable to the problem (1), (2), (3). So we omit its proof. In order to prove the a priori estimate (7), we derive two kind of inequalities. The first one, given in Lemma 1 below, is regarded as a difference inequality for $E(t)$ but it involves the following quantity :

$$M(t,r) = \sup_{|J| \leq r} \int_J \sum_i F_i(x,t) dx, \tag{8}$$

where Σ_i is the summation taken over all $i \in I$, and where J is an interval in $[0,1]$ and $|J|$ denotes its length. The second one, in Lemma 2, gives a sharp upper-bound of the quantity $M(t,r)$.

Lemma 1.— *Let us fix a constant* V *such that* $V > \max |v_i|$. *Then there are two numbers* $C_1 > 1$ *and* $C_2 > 0$ *such that for any* $h > 0$ *with* $2Vh \leq 1$, *we have*

$$E(t+h) \leq C_1 E(t) + C_2 E(t+h) M(t, 2Vh), \quad t \geq 0. \tag{9}$$

Lemma 2.— *There is a number* $C_3 > 0$ *such that for any* r *with* $0 < r \leq 1$, *we have*

$$M(t,r) \leq C_3 E_0 \delta(r), \quad t \geq 0, \tag{10}$$

where $\delta(r)$ *is a continuous function of* r *such that* $\delta(r) > 0$ *for* $0 < r \leq 1$ *and* $\delta(r) \to 0$ *as* $r \to 0$.

Remark. Among conditions for the coefficients $A^{ij}_{k\ell}$, B_{ij} and $\tilde{B}_{ij}$, stated in the first paragraph, we use $(A1), (A2)$ and $(B1)$ in the proof of (9), whereas we need $(A1), (A3), (B1)$ and $(B2)$ to prove (10).

Once the above two lemmas are established, it is straightforward to derive the a priori estimate (7). In fact, substituting (10) into (9), we have

$$E(t+h) \leq C_1 E(t) + C_2 C_3 E_0 \delta(2Vh) E(t+h).$$

We choose a positive constant h so small that $C_2 C_3 E_0 \delta(2Vh) \leq 1/2$; h depends only on E_0. By this choice of h, we obtain

$$E(t+h) \leq 2 C_1 E(t),$$

which shows that the estimate (7) holds true for $C = 2C_1$ and $K = h^{-1} \log(2C_1 E_0)$. This completes the proof of Proposition 2.

3. Proof of the lemmas.

We show Lemma 1 by a method similar to the one employed in [2]. Consider a solution in the rectangle $R = [0,1] \times [t_0, t_0 + h]$, where $t_0 \geq 0$ and $h > 0$ with $2Vh \leq 1$. Let $\alpha \in I$. We put $I_\alpha = \{i \in I; v_i = v_\alpha\}$ and

$$\tilde{F}_\alpha(x,t) = \sum_{i \in I_\alpha} F_i(x,t). \tag{11}$$

Then, summing up the equations in (1) for $i \in I_\alpha$, we obtain

$$\frac{\partial \tilde{F}_\alpha}{\partial t} + v_\alpha \frac{\partial \tilde{F}_\alpha}{\partial x} = \sum_{i \in I_\alpha} \sum_{k \notin I_\alpha} \sum_{j\ell} (A_{k\ell}^{ij} F_k F_\ell - A_{ij}^{k\ell} F_i F_j). \tag{12}$$

Note that the right-hand side of (12) does not contain the summation with respect to $k \in I_\alpha$. Let $(x_1, t_1) \in R$ and let us denote by ℓ_α the backward v_α-characteristic through the point (x_1, t_1). We integrate (12) along the characteristic ℓ_α in order to obtain an expression of $\tilde{F}_\alpha(x_1, t_1)$. Here we only treat a typical case where $v_\alpha > 0$ and the characteristic ℓ_α meets the boundary $x = 0$ at a time $t = t_*$ with $t_0 < t_* < t_1$, and therefore the resulting expression of $\tilde{F}_\alpha(x_1, t_1)$ contains the boundary value $\tilde{F}_\alpha(0, t_*)$. We see from (2) that this boundary value is estimated in terms of $\Sigma_\beta^- \tilde{F}_\beta(0, t_*)$. Furthermore, we can derive an expression of $\tilde{F}_\beta(0, t_*), v_\beta < 0$, by integrating (12) (with α replaced by β) along the backward v_β-characteristic through the point $(0, t_*)$; note that this v_β-characteristic meets the base $t = t_0$ of the rectangle R. Thus we can reach an estimate of the form

$$F_\alpha(x_1, t_1) \leq CE(t_0) + CE(t_0 + h) \left\{ \int_{t_*}^{t_1} \sum_{k \notin I_\alpha} F_k(x_1 - v_\alpha(t_1 - \tau), \tau) d\tau \right. \tag{13}$$

$$\left. + \sum\nolimits_\beta^- \int_{t_0}^{t_*} \sum_{k \notin I_\beta} F_k(-v_\beta(t_* - \tau), \tau) d\tau \right\},$$

where C is a constant.

In order to estimate the integrals on the right-hand side of (13), we make use of the following conservation equations of mass and momentum :

$$\frac{\partial}{\partial t}(\sum_i F_i) + \frac{\partial}{\partial x}(\sum_i v_i F_i) = 0, \tag{14}$$

$$\frac{\partial}{\partial t}(\sum_i v_i F_i) + \frac{\partial}{\partial x}(\sum_i v_i^2 F_i) = 0. \tag{15}$$

We have used (A1) and (A2) to derive (15). Besides the v_α-characteristic ℓ_α defined previously, we consider the following space-like lines ℓ_+ and ℓ_- in the rectangle R : $\ell_\pm$

are the straight line with slope $\pm V$, which pass through the point (x_1, t_1). Now we make a combination (14) $\times v_\alpha -$ (15) and integrate the resulting conservation equation over the triangle defined by $\ell_\alpha, t = t_*$ and ℓ_-. Next, we integrate (14) over the trapezium defined by $\ell_+, x = 0, t = t_*$ and ℓ_-, and also over another trapezium defined by $t = t_*, x = 0, t = t_0$ and ℓ_-, and then use the fact that $\Sigma_i v_i F_i = 0$ on $x = 0$, which is a consequence of (B1). Finally, combining these three integrals, we find that

$$\int_{t_*}^{t_1} \sum_{k \notin I_\alpha} F_k(x_1 - v_\alpha(t_1 - \tau), \tau) d\tau \leq CM(t_0, 2Vh) \tag{16}$$

for some constant C. The second integral in (13) can be estimated similarly. Substituting these bounds into (13) gives the desired inequality (9), and therefore the proof of Lemma 1 is complete.

In order to prove Lemma 2, we use the following equality which is a modified version of the H-theorem.

$$\frac{\partial}{\partial t}\left\{\sum_i F_i \log(F_i/M_i)\right\} + \frac{\partial}{\partial x}\left\{\sum_i v_i F_i \log(F_i/M_i)\right\} = -\frac{1}{4}\sum_{ijk\ell} A_{k\ell}^{ij}(F_iF_j - F_kF_\ell)\log(F_iF_j/F_kF_\ell) \leq 0. \tag{17}$$

Here $\{M_i; i \in I\}$ is the constant Maxwellian in (B2) ; we have used (A1) and (A3) to derive (17). Making use of (B1) and (B2) together with the fact that $\eta \log \eta$ is a convex function of $\eta > 0$, we see that the flux $\sum_i v_i F_i \log(F_i/M_i)$ in (17) is non-positive on $x = 0$ and non-negative on $x = 1$. Therefore, integrating (17) over $x \in [0, 1]$ yields

$$\frac{d}{dt}\int_0^1 \sum_i F_i \log(F_i/M_i) dx \leq 0. \tag{18}$$

This inequality was first obtained by Gatignol [4]. On the other hand, integration of (14) over $x \in [0, 1]$ yields

$$\frac{d}{dt}\int_0^1 \sum_i F_i dx = 0, \tag{19}$$

since we have $\Sigma_i v_i F_i = 0$ on $x = 0$ and $x = 1$ as a consequence of (B1). We make a combination (18) $-$ (19) $\times \log E_0$ and integrate the resulting inequality over the time interval $[0, t]$. Then, taking account of the inequality $\eta|\log \eta| < \eta \log \eta + 1$ (for $\eta > 0$), and using the integral form of (19) over $[0, t]$, we arrive at the estimate

$$\int_0^1 \sum_i F_i |\log(F_i/E_0)| dx \leq CE_0, \tag{20}$$

where C is a constant. This estimate combined with the argument of Grandall and Tartar [5] yields the desired inequality (10). We omit the details and refer the reader to [5].

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