Séminaire Équations aux dérivées partielles – École Polytechnique

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Critical points at infinity in the variational calculus

Séminaire Équations aux dérivées partielles (Polytechnique) (1985-1986), exp. nº 21, p. 1-31

<http://www.numdam.org/item?id=SEDP_1985-1986____A21_0>

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SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES 1985 - 1986

CRITICAL POINTS AT INFINITY IN THE VARIATIONAL CALCULUS

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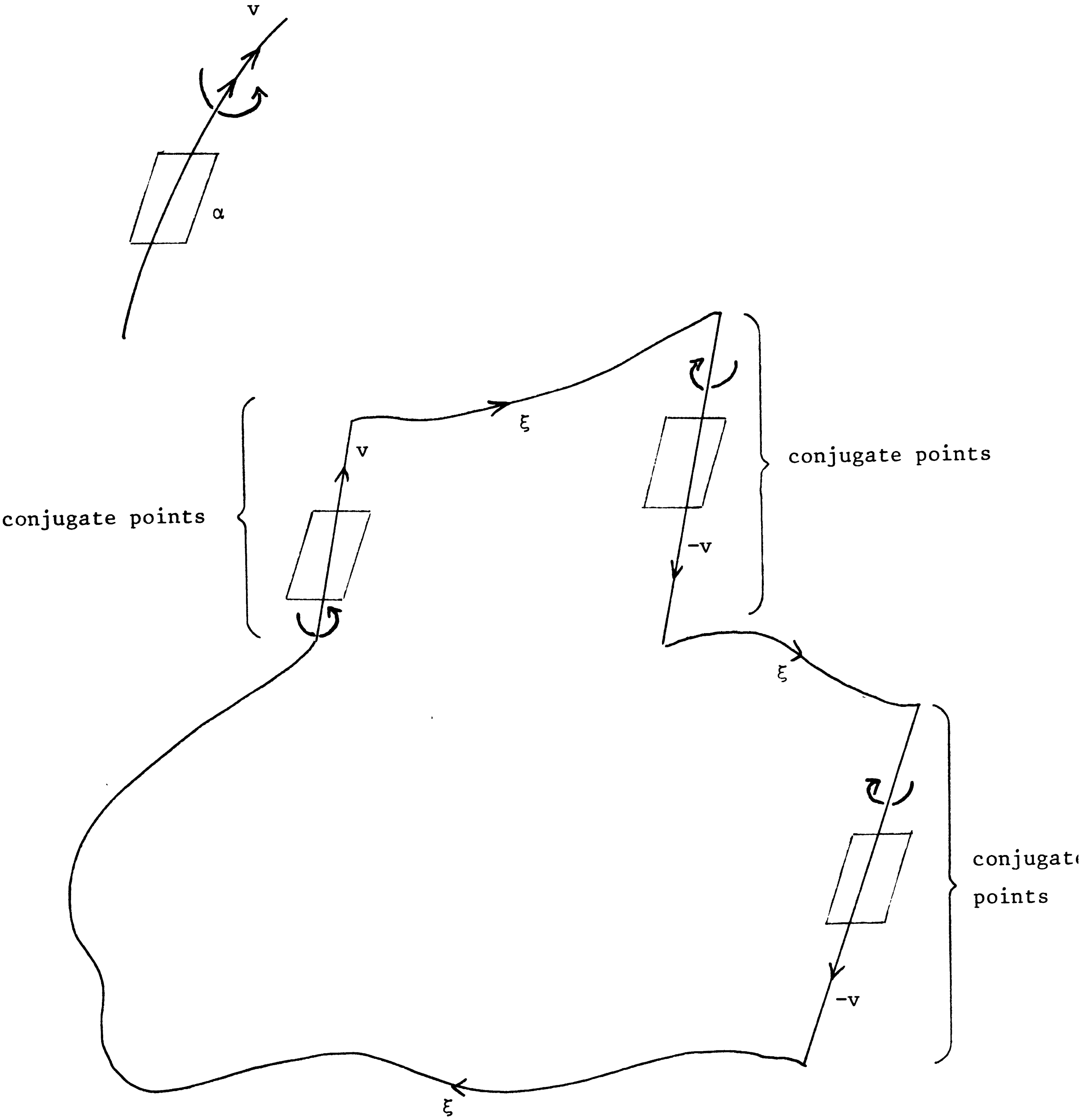
in the honour of H. Lewy

Exposé n° XXI 17 Décembre 1985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1. ORBITS OF THE FLOW.

Let E be a space of variations (either a Hilbert space or a manifold modelled on a Hilbert space for sake of simplicity) and let :

(1) $$f : E \longrightarrow \mathbb{R}$$

be a functional which we will assume to be C^∞, for sake of simplicity also. Let :

(2) $(\ ,\ )$ be the scalar product on E

and

(3) $\partial f(x)$ be the gradient of f at x in E.

We are concerned with finding a solution to the equation :

(4) $$\partial f(x) = 0 \qquad x \in E \quad .$$

For $a \in \mathbb{R}$, we introduce the level sets of the functional f at a :

(5) $$\begin{cases} f^a = \{x \in E \mid f(x) \leqslant a\} & \text{(sub-level set)} \\ f_a = \{x \in E \mid f(x) \geqslant a\} & \text{(upper-level set)} \\ {}^a f = \{x \in E \mid f(x) = a\} & \text{(level surface)} \end{cases} \quad .$$

We also consider the differential equation :

(6) $$\frac{\partial x}{\partial s} = -\,\partial f(x) \quad ; \quad x(0) = x_o \quad .$$

Let :

(7) $x(s,x_o)$ be the solution of (6).

We then have the following very simple principle to solve (4) :

Proposition 1 : Let $b < a$. Assume f^b is not retract by deformation of f^a (e.g. $H_*(f^a, f^b; G) \neq 0$ for a certain coefficient group G ; and H_* is a homology theory ; or $\pi_*(f^a, f^b; G) \neq 0$; π_* is homotopy ...) Then :

either (4) has a solution $\bar{x}$ with $b \leqslant f(\bar{x}) \leqslant a$

or there exists an x_o such that $b < f(x_o) \leqslant a$ such that

(a) $\lim\limits_{s \to +\infty} f(x(s, x_o)) \geqslant b$

(b) the closure of the set $\{x(s, x_o) \; ; \; s \geqslant 0\}$ is non compact .

There are several directions where one can make the content of Proposition 1 more precise and in the same time more general :

1st precision : In case $\bar{x}$ exists, the basic assumption which is used to study the situation near $\bar{x}$ is that ∂f is Fredholm at $\bar{x}$. Then, if $\bar{x}$ is a non degenerate critical point of f, one computes the Morse index of f at $\bar{x}$.

We then have :

Assume the injection $f^b \to f^a$ is not a homotopy equivalence, then $\bar{x}$ cannot contribute to the difference of topology between f^b and f^a if the Morse index of f at $\bar{x}$ is infinite.

Thus a critical point $\bar{x}$ is relevant in the calculus of variations in the large if it comes with some properness assumption (Fredholm structure) and if it has a finite Morse index.

In case the Fredholm structure is available, there is a way to drop the assumption of non degeneracy (see Marino and Prodi [1]). Otherwise, very few is known.

2nd precision : In the second case of the alternative provided by Proposition 1, we have :

$$\int_0^{+\infty} |\partial f(x(s, x_o))|^2 \, ds < f(x_o) - b \tag{8}$$

(9) $\int_0^\tau \partial f(x, (s, x_o)) ds$ cannot be included in a compact set for $\tau \in [0, +\infty[$.

We then have a sequence (s_n) or either $(x(s_n, x_o))$ such that :

$$b \leqslant f(x(s_n, x_o)) \leqslant a \quad ; \quad \partial f(x(s_n, x_o)) \longrightarrow 0 \quad . \tag{10}$$

Classically, in the calculus of variations, there is an assumption, called the (C) condition, introduced by Palais and Smale forbidding (10). The content of this condition is the following :

(11) for any sequence (x_n) such that $b \leqslant f(x_n) \leqslant a$ and $\partial f(x_n) \to 0$, there is a convergent subsequence.

This condition forbids the second case in Proposition 1 and thus allows to find solutions to (4) via the study of the difference of topology in the level sets.
However the condition (C) forbids more tant (8)-(9). Indeed, in (8)-(9), we deal with sequences which lie on the same orbit of the flow ; while with the (C) condition, we deal with arbitrary sequences (both of them satisfying $b \leqslant f(x_n) \leqslant a$; $\partial f(x_n) \to 0$).

Thus, to do variational calculus, we need :

1. either a weaker condition than (C) or (11) : namely, we can impose on the x_n's to lie on the same deformation line,

2. or to study the difference of topology induced in the level sets by theses orbits of the flow which satisfy (8)-(9).

The first case 1 is just an improvement of condition (C).
The second case 2 is concerned with getting rid of any condition of this type, by considering the flow as a dynamical system, having possibly singularities "at infinity".
This goal of getting rid of the condition (C) has been set by S. Smale in his book : "Mathematics of the time" [2].

It is far from being concretly achieved in all variational problems of interest.

3rd precision : The invariants.

The orbits of the flow satisfying (8)-(9) are of course relevant to the flow itself , i.e. to the differential equation (6).

On the other hand, the variational calculus is concerned with the function f ; in fact, equation (4) means that it is impossible to decrease strictly with respect to f a whole neighbourhood in E of the critical point x.

From this correct interpretation of the calculus of variations, we may replace (6) by any other differential equation corresponding to a (pseudo)-gradient for f (in particular consider another scalar product on E) or even more consider any decreasing (with respect to f), globally defined deformation of the level sets.

Thus we see immediately that there is some ambiguity, if we do not make some further specification in case 2 .

We discuss here this ambiguity :

1 - First of all, there is an intrinsic notion relevant to f and not ∂f (or either deformation). This notion is the difference of topology of the level sets when crossing the critical level, say c.
i.e. if this critical level c is isolated, which we will assume for sake of simplicity , then for $\varepsilon > 0$ small enough, the homotopy type of the space $f^{c+\varepsilon}/f^{c-\varepsilon}$ is an invariant.

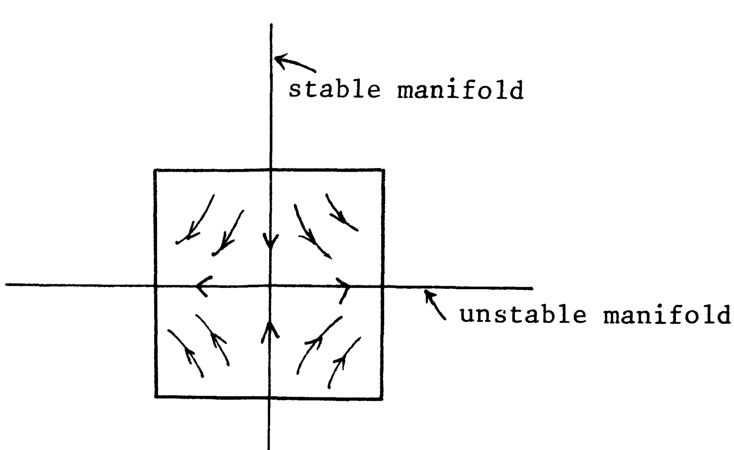
2 - There is a second invariant which is of dynamical type.

In dynamical systems, an invariant set for a flow comes with a stable and unstable manifold, at least when it is non degenerate (or either hyperbolic), which holds generically (otherwise some perturbative argument is necessary using some kind of Fredholm structure ; these days, even degenerate cases are studied, see for instance the work of Yomdin in algebraic geometry and Cappell-Weinberger in algebraic topology).

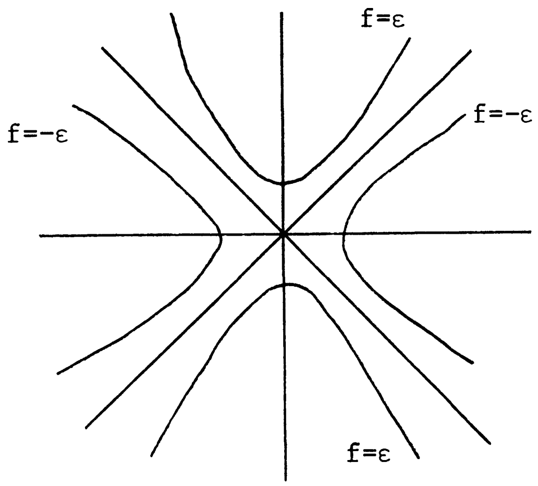
What is invariant in a hyperbolic situation is not the stable and unstable manifolds but rather their dimension and the qualitative behaviour of the flow on the boundary of an isolating block in the sense of C.C. Conley [3].

To give the simplest picture of this invariant, the best example is the situation nearby a non degenerate critical point of a function f on a finite dimensional manifold. We then have, by Morse lemma, the following local situation (see M. Hirsch [4], for instance).

(12)



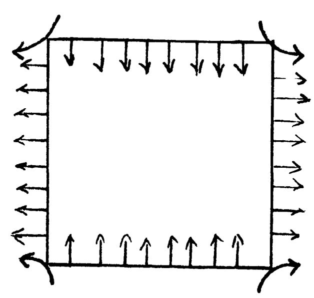
dynamics of the flow and of any pseudo-gradient hyperbolic flow

(13) 

0 is the critical level.

behaviour of the level sets.

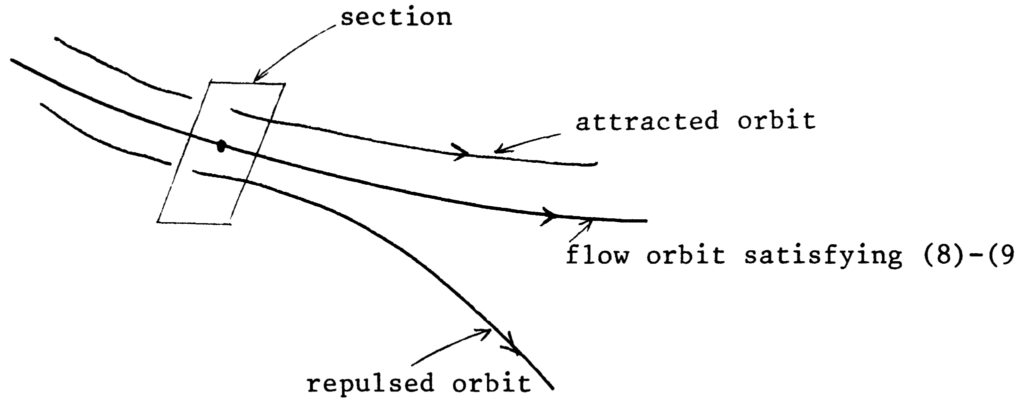
In (12), we can retain the behaviour of the flow on the boundary of the box :

(14) 

C.C Conley [3] has introduced invariants related to this Morse decomposition, namely the homology of what comes in and what comes out.
C.C. Conley introduced these invariants for a general hyperbolic flow. In such a genral situation, these invariants do not determine the invariant set and its hyperbolic structure (i.e. stable and unstable manifolds) inside the isolating blowk. However, when the flow is pseudo-gradient and the critical (or rest) point is isolated (without assumption of degeneracy), these invariants completely determine the nature of the critical point inside.

This notion of Morse decomposition and isolating block can be extended to the situation of (8)-(9), giving rise to Conley invariants related to this situation, which provide with a second set of invariants, more precise than the difference of topology in the level sets.
Qualitatively, we draw the following picture of the flow, under (8)-(9).

(15)



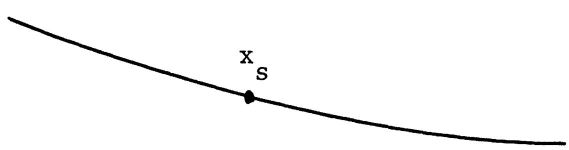
The analysis of the flow on sections provides with this second set of invariants.

4th precision : The global nature of the deformation.

With this phenomenon of the failure of the (C)-condition, there is something important which enters into account :

Consider the case of a usual non degenerate critical point for a functional f, then there is no way one can decrease with respect to the functional a whole neighbourhood of this critical point.

With a flow line going to $+\infty$, the situation is different :



Indeed, consider a point x_s on this flow line : then there is no problem to decrease with respect to the functional a whole neighbourhood to x_s.
But, in fact, there is more : namely, one can decrease in most cases where we studied critical points at infinity a whole neighbourhood of the flow line ; in particular, the flow line itself.
What really matters is how "large" this neighbourhood can be taken, with respect to the structure of the flow in section and also (it is somewhat the same thing) how "large" it is with respect to the "end of the orbit". There is a real ambiguity here, of the same kind than one encounters with the ends of analytic functions. What is defined is a way to arrive to them (when they are "accessible") ; not really themsleves ; i.e. the way to arrive to them defines them. This is why the global nature of the

deformation, in particular the flow when necessary, is an important tool.

(For a discussion on the ends and the accessibility, see Moïse, "Geometric topology in dimension 2 and 3".)

5th precision : The representation device.

Assume there are natural ends for the flow lying in some manifold V. This manifold might be given by the sequences (x_n) violating the (C)-condition. This manifold depends then only on the notion of pseudo-gradient, not on the precise pseudo-gradient chosen.
Consider then a bundle F over V with fiber a space of parameters $\Lambda = \Lambda_1 \times H$ where Λ_1 is, for sake of simplicity, finite dimensional and H is a neighbourhood of zero in a Hilbert space (or so).
Let $(\Lambda_1 \times H)_x$ be the fiber at x.
Assume now there is a way to represent the functions x of E such that $|\partial f(x)| < \varepsilon$; $|f(x) - c| < \varepsilon$ in F. Calling this representation R, we have a functional defined on F ; namely f(Rx).
Assume also that over any point u in V and for any $\lambda_1 \in \Lambda_1$, we can minimize this functional for the variations in H.
We then have a functional on Γ, a bundle over V, with fiber Λ_1 ; and we are reduced to a variational problem of finite dimensional type, which depends on the representation R.
In the simplest case, $\Lambda_1 = (\mathbb{R}^+)^p$, $p \in \mathbb{N}$ and when $\varepsilon \to 0$, $(\lambda_1,\dots,\lambda_p) \to (0,\dots,0)$ in $(\mathbb{R}^+)^p$.
It is then natural to look at the functional f(Rx) over V in a neighbourhood of (0,...,0) in the fiber. Then V might be thought as the space of variations at infinity (not the critical set at infinity) ; and the behaviour of the functional in a neighbourhood of Γ will select, under good conditions, the part of V which is critical "at infinity".
This relates this kind of variational problems to the study of the singularities. This is the way C.H. Taubes proceeds [13] ; and he proves that the singularities do not interfere with certain homology or homotopy classes.

After these five precisions, we introduce the notion of critical point at infinity (see Abbas Bahri [5]).

Definition 1 : A critical point at infinity is an orbit of a (pseudo)-gradient flow for the functional f, starting at a point x_o for $s \geqslant 0$, such that $f(x(s,x_o)) \to c \in \mathbb{R}$, whose closure in E is non compact.
Thus a critical point at infinity is related to a (pseudo)-gradient.
If it is of hyperbolic type, or if we are dealing with a hyperbolic set of

critical points at infinity, there are invariants (in particular Conley invariants) related to such a critical point at infinity.
Finally, in case there is an appropriate extension of the variational problem nearby infinity (i.e. the bundle F) with a normal form of f, then there might exist a space of representation for these critical points at infinity.
In any case, the notion of critical point at infinity is <u>not</u> intrinsic to the variational problem.

2. AN ABSTRACT DEFORMATION LEMMA.

Assume we know that, with $b < a$,

(16) $\quad f^b$ is not retract by deformation of f^a ,

but we do not know the condition (C) to hold on $[b,a]$.
There is then a way to analyze the possible defect of compactness, which amounts to a deformation lemma we present now.
Assume also we have a function $g : E \to \mathbb{R}$ such that if

(17) $\quad (x_n)$ is a sequence such that $b \leqslant f(x_n) \leqslant a$; $\partial f(x_n) \to 0$ and $(g(x_n))$ is bounded, there is a convergent subsequence.

Thus, if a sequence (x_n) violates the (C)-condition, $g(x_n)$ goes to $+\infty$.
In general, there are many possible choices of g and the best one is in some sense the function g (if it exists) which measures how a sequence violating the (C)-condition leaves the compact sets of E (examples will be provided later on).

Let :

(18) $$\varphi(x) = \frac{|\partial f|}{|\partial g|}(x)(f(x)+g(x)) \quad \text{if} \quad \partial g(x) \neq 0 \quad ; \quad \varphi(x) = +\infty \quad \text{if} \quad \partial g(x) = 0 \quad .$$

We assume

(19) $$\partial g(x) = 0 \Longrightarrow \partial f(x) \neq 0 \quad \text{if} \quad b \leqslant f(x) \leqslant a \quad .$$

Let

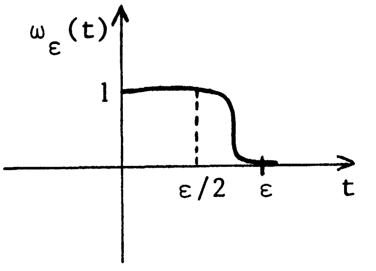
$$Z(x) = (|\partial f|\ \partial g + |\partial g|\ \partial f)(x) \tag{20}$$

Z has the property that $(Z,\partial f) \geqslant 0$; $(Z,\partial g) \geqslant 0$; and

$$(Z,\partial f)(x) = 0 \quad \text{if and only if} \quad Z(x) = 0 \quad . \tag{21}$$

Let $\varepsilon>0$ be given and let :

(22) $\qquad \omega_\varepsilon : \mathbb{R}^+ \longrightarrow \mathbb{R}$:



and let now :

$$Z_\varepsilon(x) = (1 - \omega_\varepsilon(\varphi(x)))\quad \partial f(x) + Z(x) \quad . \tag{23}$$

Z_ε is locally Lipschitz.
Qualitatively, $-Z_\varepsilon$ is obtained from $-\partial f$ by adding $-Z$ which decreases f <u>and</u> g whenever φ is small.

We point out here that, in general, we may choose g such that :

$$\frac{(f+g)(x)}{|\partial g(x)|} \geqslant C > 0 \quad ; \tag{24}$$

C uniform for any x such that $b \leqslant f(x) \leqslant a$ or even, in some situations (see section 3 below)

$$\frac{(f+g)(x)}{|\partial g(x)|} \xrightarrow[g(x)\to+\infty]{} +\infty \quad \left(\text{in section 3} \quad \frac{(f+g)(x)}{|\partial g(x)|} \geqslant C\sqrt{g(x)} + C' \quad ; \quad C>0\right) \tag{25}$$

Thus $\varphi(x) \geqslant C\ |\partial f(x)|$ and $\varphi(x)$ goes to zero <u>implies</u> that $\partial f(x)$ goes at zero.
In this way, $-Z_\varepsilon(x)$ is a vector-field which controls the sequences violating the condition (C).
We then consider the differential equation :

$$\frac{\partial x}{\partial s} = -Z_\varepsilon(x) \tag{26}$$

and we have :

Proposition 2 : If for $b<a$, f^b is not retract by deformation of f^a, then, for each $\varepsilon>0$, there exists x_ε such that :

(27) $$b \leqslant f(x_\varepsilon) \leqslant a$$

and

either $\partial f(x_\varepsilon)=0$; if $f^b \to f^a$ is not a homotopy equivalence, x_ε has a finite Morse index if ∂f is Fredholm at x_ε and x_ε is note degenerate ;

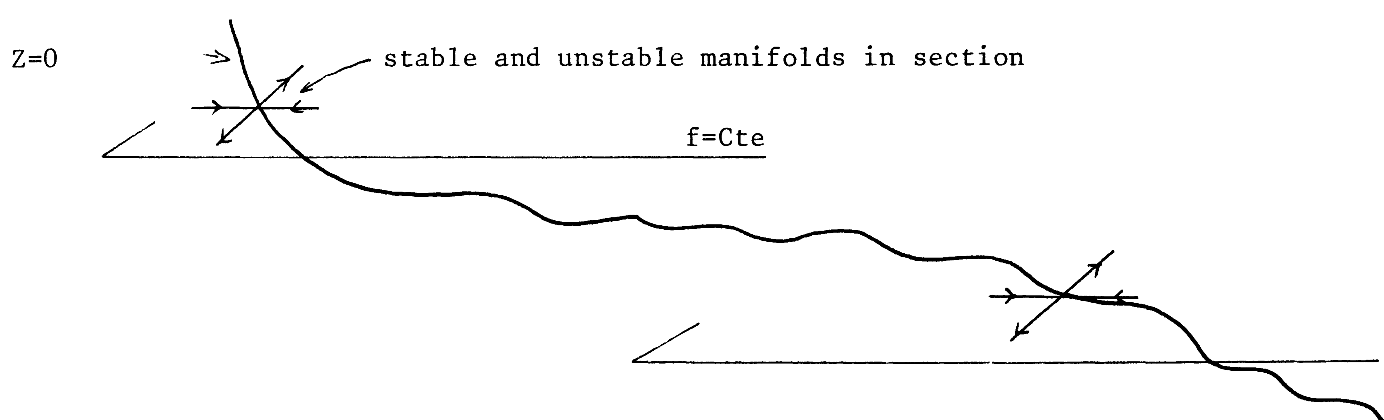
or $Z(x_\varepsilon) = 0$; $\varphi(x_\varepsilon) \leqslant \varepsilon$. In this case, there are three relevant notions which are :

a - the set $Z(x)=0$ around x . Generically, this is a line transverse to f ;

b - if it is a line transverse to f, the index of Z at x_ε in section to the level surface at x_ε ;

c - in the simple case when this line goes to $+\infty$ (i.e. $g(x) \to +\infty$ on this line) the unstable manifold of this line of zeros of Z.

This is summed up in the following drawing :



Is there a convergence process ?

In case such a line has a natural end, we end up with a critical point at infinity as defined in the previous section. We illustrate this by the following example (section 3).

3. PSEUDO-ORBITS OF CONTACT FORMS.

The framework is the following :

We consider a contact form α on a three dimensional compact, orientable manifold M. The assumption $n = 3$ is not essential here. There is a version of what we present in dimension 2p+1. For sake of simplicity, we restrict ourselves to this case.

Let ξ be the Reeb vector-field of α, i.e. ξ is defined by the equations

$$\alpha(\xi) \equiv 1 \quad ; \quad d\alpha(\xi,.) \equiv 0 \quad . \tag{28}$$

Let v be a vector-field in the kernel of α which we assume to be nowhere vanishing. The existence of such a v means that the bundle in planes $\alpha = 0$ over M is trivial ; which we will assume.

Let θ_s be the one parameter group generated by v, $D\theta_s$ is differential and let θ_s be the associated map in the differential forms of M.

Let x_o be a point of M and $x_s = \theta_s(x_o)$ be the generating point of the v-orbit through x_o.

Let :

(29) $e_1(o)$ and $e_2(o)$ be two vectors tangent to M at x_o such that :

$$\alpha \wedge d\alpha(v_{x_o}, e_1(o), e_2(o)) < 0 \quad . \tag{30}$$

Let

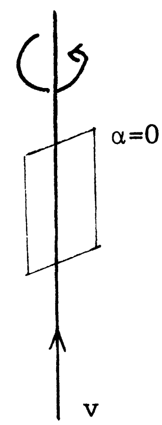
$$\begin{aligned} e_1(s) &= D\theta_s(e_1(o)) \quad ; \quad e_2(s) = D\theta_s(e_2(o)) \quad ; \\ w(s) &= \alpha(e_1(s))e_2(s) - \alpha(e_2(s))e_1(s) \quad . \end{aligned} \tag{31}$$

3.1 Some geometrical facts ; a notion of conjugacy.

The following proposition expresses along the v-orbits the fact that α is a contact form :

Proposition 3 : w(s) rotates in the direct sense of the frame $(e_1(s),e_2(s))$ when s increases.

This is expressed in the following picture :



Let then :

(32) $\psi(s,x_o)$ be the angle in the moving frame $(e_1(s),e_2(s))$ which measures the rotation of $w(s)$ from 0 to s.

We define :

Definition 1 : We call coincidence points of x_o (relatively to α and v) along the v-orbit through x_o those points x_s such that $\psi(s,x_o) = 2k\pi$; $k \in \mathbb{Z}$.
At these points x_s, we have :

(33) $$(\theta_s^*\alpha)x_o = \lambda(s,x_o)\alpha_{x_o} \quad : \quad \lambda(s,x_o) > 0 \quad .$$

Definition 2 : We call conjugate point of x_o (relatively to α and v) along the v-orbit through x_o a coindence point such that :

(34) $$\lambda(s,x_o) = 1 \quad .$$

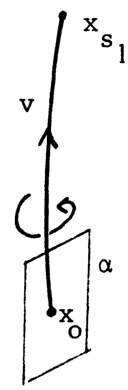
Definition 3 : We say that α turns well along v if any point x_o of M has a coincidence point distinct from itself. Let then $\gamma^i : M \to \mathbb{R}$ be the function which associates at a point x_o of M the i-th time $s = \gamma^i(x_o)$ such that x_s is a coincidence point of x_o $(i \in \mathbb{Z})$. Let $f^i : M \to M$ be the diffeomorphism of M which sends x_o on $x_{\gamma^i(x_o)}$. We denote $\mu_i(x_o)$ the colinearity coefficient of $(\theta^*_{\gamma^i(x_o)}\alpha)_{x_o}$ and α_{x_o} :

(35) $$(\theta^*_{\gamma^i(x_o)}\alpha)_{x_o} = \mu_i(x_o)\alpha_{x_o} \quad .$$

As can be noticed, the notion of conjugate point is a delicate notion :

not all points of M have conjugate points distinct from themselves. At the contrary, these points live on a hypersurface of M.

If we draw the segment of v-orbit between x_o and x_{s_1}, a coincidence point :



we have a natural differential equation which comes with this piece : namely the one satisfied by α from x_o into x_{s_1} ; or either the one satisfied by w(s) in a transported frame $(e_1(s), e_2(s))$.

It has the intrinsic form :

$$(36)\qquad \begin{aligned} \frac{\partial}{\partial s}\alpha_{x_s} &= d\alpha_{x_s}(v,.) \\ \frac{\partial}{\partial s}(d\alpha_{x_s}(v,.) &= a(x_s)\alpha_{x_s} + b(x_s)d\alpha_{x_s}(v,.) \end{aligned}$$

thus

$$(37)\qquad \frac{\partial^2}{\partial s^2}(\alpha_{x_s}) = a(x_s)\alpha_{x_s} + b(x_s)\frac{\partial}{\partial s}\alpha(x_s)$$

with $a(x_s) < 0$.

Hence the pendulum equation, with a periodic solution if and only of we have a conjugate point.

3.b A variational problem on a submanifold of the loop space on M.

Let

$$(38)\qquad \beta = d\alpha(v,.)\quad .$$

We leave aside here the question of β not being a contact form (which is treated in [5]) and we study the case :

(39) $\beta \wedge d\beta > 0$ with respect to $\alpha \wedge d\alpha$.

We then normalize v by multiplication by a factor λ so that :

(40) $$\beta \wedge d\beta = \alpha \wedge d\alpha \quad .$$

Let then :

(41) $$\mathcal{L}_\beta = \{x \in H^1(S^1,M)/\beta(\dot{x}) \equiv 0\}$$

(42) $$C_\beta = \{x \in H^1(S^1,M)/\beta(\dot{x}) \equiv 0 \quad \text{and} \quad \alpha(\dot{x}) = Cte > 0\} \ .$$

We have :

<u>Proposition</u> 4 : C_β is a submanifold of $H^1(S^1,M)$. If α (hence β) turns well along v, then the injection of C_β in $H^1(S^1,M)$ is a weak homotopy equivalence.

Consider now a curve of C_β, x. Its tangent vector $\dot{x}$ can be split on (ξ,v) and we have :

(43) $$\dot{x} = a\xi + bv \qquad a = \text{positive constant} \ ; \quad b \in L^2(S^1;\mathbb{R}) \quad .$$

We then consider the functional :

(44) $$f(x) = a \text{ on } C_\beta \quad .$$

As one might expect, from the first glance, the functional f does not control at all the v-component of $\dot{x}$.
We introduce :

(45) $$g(x) = \int_0^1 b^2 \, dt \quad .$$

Some further analysis (see [5]) shows that f does not satisfy the (C)-condition and in fact, on a sequence (x_n), with $\dot{x}_n = a_n\xi + b_n v$, the boundedness of the functional and the fact that $\partial f(x_n) \to 0$ just tells us that $\int_o^t b_n/(\int_o^1 b_n^2) \to 0$, at best.

So :

1 - f doest not satisfy (C) ;

2 - ∂f has no Fredholm structure (see [5]) ;

3 - there is a difference of topology in the level sets ; but it is by far too heavy to be due only to the critical points of f which are the periodic orbits of ξ (see also [5]).

Yet, such an ill posed variational problem has a precise meaning, of interest. For this, consider for instance the case :

(46) $M = P\mathbb{R}^3$; $\alpha = \lambda\alpha_o$ where α_o is the standard contact form on $P\mathbb{R}^3$ considered as the cotangent sphere bundle over S^2 .

Let

(47) $p : P\mathbb{R}^3 \to S^2$ be the fibration ; and v be the vector-field of the S^1-fibers.

The variational problem (44) on C_β corresponds then to the geodesic problem on S^2 (to any metric on S^2, there is a corresponding λ in (46)), in the space of immersed curves.

Solving this problem is important for the issue of counting the minimal number of geodesics on a surface Σ (then M is the S^1-fiber bundle over Σ provided by the sphere cotangent bundle).

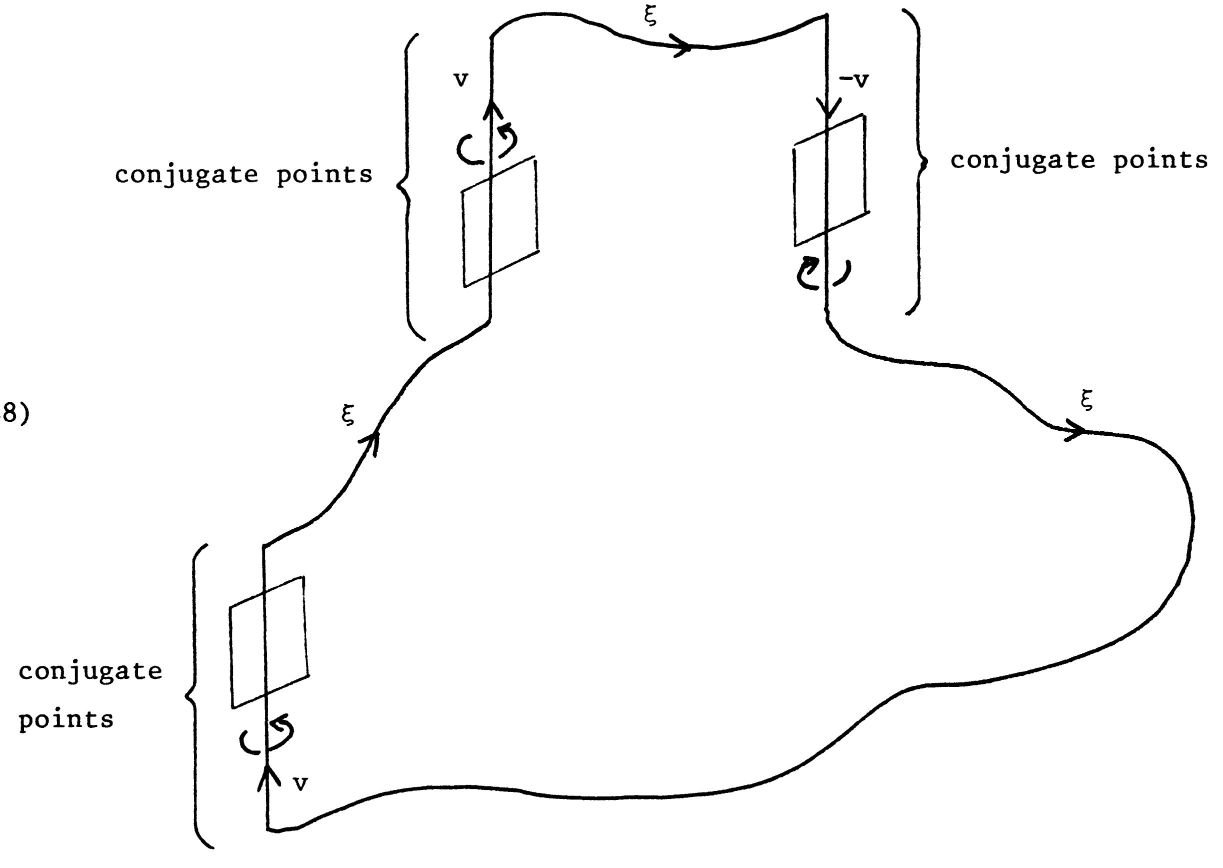
R. Hamilton pointed out that the normal curvature flow for the projected curve from C_β along p is a pseudo-gradient for such a variational problem ; it is as well a pseudo-gradient for the area enclosed by such curves.

We thus see that this variational problem has a common pseudo-gradient with other important variational problems.

One should note here that this implies that all these problems have related critical points at infinity.

3.c The critical points at infinity of the variational problem [Note aux Comptes-Rendus, Abbas Bahri, July 1984].

(48)



(48) is a geometrical description of the critical points at infinity.

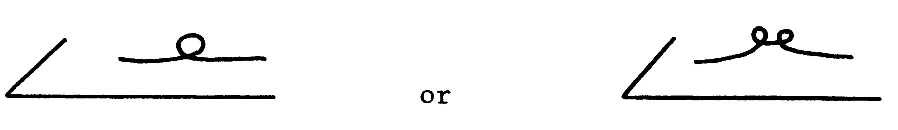
To understand qualitatively what is going on, we have to apply a convergence process to these curves :

1 - the conjugate points

For $\varepsilon \to 0$, the curve x approximating this object (on the deformation line) forms a small bubble in section to v.

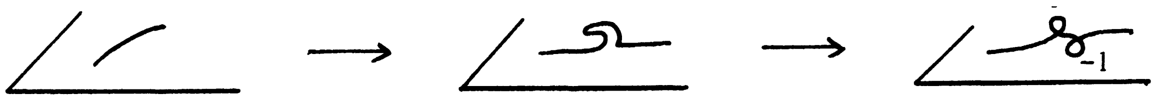
i.e. if one projects a neighbourhood of the piece rather tangent to v on x on a section to v, one finds :

(49)



or more bubbles

These are thus points where the tangent vector to x_ε makes very rapidly an integer number of rotations, possibly growing when $\varepsilon \to 0$ by the following process :

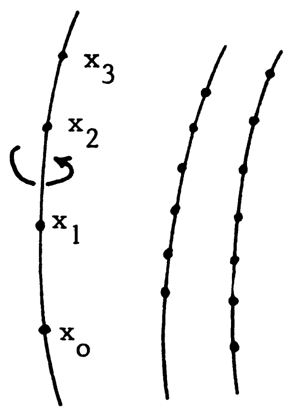
(50) 

no bubble

+1 bubble
-1 bubble

However the resulting movement is very particular, i.e. the bubble as depolyed along v will go from one point to a conjugate of this point. Thus, generically, these bubble build up at precise locations in M. At these precise locations, the singularity has a very precise and restricted normal form given by a periodic solution to (37).

To see the phenomenon, we could draw v-orbits passing through each point x of M and distinguish on these orbits the coincidence points to x :

(51) 

We thus have a $\mathbb{Z}$-structure along M related to α along v. For some distinguished points, we have conjugate points.

(52) 

$x_o \in$ hypersurface of M

Assume x_p corresponds to time s_1 on the v-orbit starting at x_o. We have :

(53) $$\theta^*_{s_1} \alpha = \alpha .$$

and we can compute in this situation the second variation of α i.e.

(54) $$\delta(\theta_s^*\alpha - \alpha)(s_1) \quad .$$

This gives rise to a quadratic form on tangent vectors to M at x_o, q_o ; and in the same time to a quadratic form on tangent vectors to M at $x_{s_1} = x_p$, q_1.

Thus, these conjugate points come out with :

1 - a precise location ;
2 - a precise normal form to the singularity ;
3 - an integer (the rotation of α from x_o to $x_{s_1} = x_p$) ;
4 - two quadratic forms q_o and q_1 ;
5 - a way to approach them by curves which project on local sections on bubbles.

Remark : Each time integers are encountered in "geometry", the word "quantization" commonly appears, with an evident abuse of this concept, which is due to Kostant and Souriau (see Guillemin-Sternberg : "Geometric Asymptotics" for a scientific presentation of the concept of quantization).

2 - the ξ-pieces

These are pieces where the curve is tangent to the Reeb vector-field ; thus the curve is tangent to ξ (in this $\mathbb{Z}$-structure we introduced) until it hits a point admitting a conjugate point.
Then, under certain conditions stated in [5], it jumps to the conjugate point.

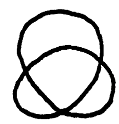
(55) The ξ-pieces come also with a quadratic form q_3 defined by the second variation of f along them with fixed ends. This quadratic form is related to a rotation of v along the ξ-piece (see [5]).

The Reeb vector-field ξ is, in the case of the cotangent unit sphere bundle of a Riemannian manifold Σ, such that its periodic orbits project on geodesics of Σ. In that case, there is no other conjugate point for a point x_o than itself and the Dirac masses describe a complete circle S^1 over a given point in Σ in $ST^*\Sigma$.
In other simple, but more complicated cases, this is what happens :

Take the case of S^3 fibering over S^2 with the Hopf fibration

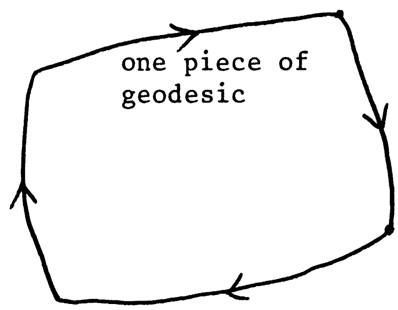
(56) $$p : S^3 \longrightarrow S^2 \quad .$$

Consider $\alpha=\lambda\alpha_o$, λ a positive function on S^3 and α_o the standard contact form of S^3. Let v be the vector-field of the fibers over S^2. In this case, the Reeb vector-field ξ, when describing a fiber S^1 over a point x_o of S^2, describes in the tangent plane to S^2 at x_o the following :

(57)  i.e. two circles.

We thus have two choices of length on S^2, hence two notions of geodesics. Then, (48) projects as :

(58)



The location of the corners is very precise and there is a Morse index related to q_o, q_1, q_3.

This is a general picture of what happens.

[A clean statement of these results has been made in a Note aux Comptes Rendus de l'Académie des Sciences de Paris, July 1984.
A manuscript was available ; sent to my friends Paul H. Rabinowitz, H. Brézis, J.M. Coron, A. Chenciner, D. Bennequin, R. Narasimhan.
A typed version of this manuscript is available now : "Pseudo-orbits of contact forms".]

We reproduce here the theorem we announced in [6] :

Assume :

(H_1) α turns well along v ;

(H_2) v has a periodic orbit ,

(H_3) for one vector-field v_1, non singular and colinear to v, we have : $\exists\, k_1 > 0$ such that $\|D\theta^1_s\| \leq k_1$ $\forall\, s \in \mathbb{R}$, where θ^1_s is the one-parameter group of v_1 ;

(H_4) $\exists\, k_2$ and $k_3 > 0$ such that $\forall\, i \in \mathbb{Z}$, we have :

$$k_2 d(x,y) \leq d(f^i(x),f^i(y)) \leq k_3 d(x,y) \qquad \forall\, x, y \in M \ ;$$

(H_5) $\exists\, k_4 > 0$ such that $|\mu_i(x) - \mu_i(y)| \leqslant k_4\, d(x,y)$ $\forall\, x,y \in M$;

(H_6) $\exists\, \rho > 0$ such that for any $x \in M$, the set $C_\rho(x) = \{f^i(x) / |\mu_i(x) - 1| < \rho ; i \in \mathbb{Z}\}$ is finite.

Then, under these hypotheses which can be considerably weakened (see [5]), we have :

Theorem 1 : The critical points at infinity of the variational problem are continuous and closed curves made up with pieces $[x_{2i}, x_{2i+1}]$ tangent to ξ and pieces $[x_{2i+1}, x_{2i+2}]$ tangent to v. x_{2i+2} is conjugate to x_{2i+1}. If the Betti numbers of the loop space are unbounded, there are unfinitely many of these curves.
Furthermore, if n is the number of v-pieces of one of these curves, we have :

$$n \leqslant Ca \tag{59}$$

where a is the length of the curve along ξ and C is a universal constant.

As pointed out in [5] (see also [17] for application of the method developped in [5] to Yamabe type equations), the defect of compactness reveals, when analyzed along deformation lines, new geometrical structures which govern in fact the variational problem.
In turn, this variational problem somewhat expresses them through the dynamics of the flow and the difference of topology in the level sets.
This link is the important point which defines a critical point at infinity.
Out of it, one either looses the variations, hence the reason why the phenomenon exists ; or the ends of the orbits, hence the way these variations interact at infinity and are governed by these interactions.

4. THE NIRENBERG PROBLEM ; THE EQUATION $\begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \\ u = 0_{|\partial\Omega} \quad u > 0 \end{cases}$; $\Omega \subset \mathbb{R}^n$ an open bounded set.

4.a The problems.

We are concerned with studying the equations :

$$\begin{cases} \Delta u - \dfrac{n-1}{4(n-2)}\, u + K(x)\, u^{\frac{n+2}{n-2}} = 0 \\ \Omega \subset S^n \;\; ; \;\; u = 0_{|\partial\Omega} \;\; \text{if} \;\; \Omega \subset\subset S^n \end{cases} \quad u > 0 \; . \tag{59}$$

In case $\Omega = S^n$, problem (59) is the Nirenberg's problem : namely finding the functions K such that they are scalar curvature of a metric conformal to the standard one on S^n.
In case $\Omega \subset\subset S^n$, we will here mainly look at the case K(x) is a positive constant.

We wish to point out here that Lee and Jerison [7] and Chern and Hamilton [8] showed that there was a Yamabe type equation on the contact form situation. The paper by Chern and Hamilton appeared in the publication of the Berkeley MSRI in October 1984, shortly after the proof the Yamabe conjecture by R. Schoën (summer 1984). This shows some connection, in time and in content between the two fields.

4.b The functional and the flow.

Let

$$(60) \qquad \Sigma^+ = \{u > 0 \ ; \int_\Omega |\nabla u|^2 \, dx + \frac{n-1}{4(n-2)} \int_\Omega u^2 dx = 1 \ ; \ u = 0_{|\partial\Omega} \ \text{ if } \Omega \subset\subset S^n\} \ .$$

Let

$$(61) \qquad J(u) = \frac{1}{\int_\Omega K(x) u^{2n/n-2}} \quad ; \quad u \in \Sigma^+ \ .$$

Let

$$(62) \qquad \lambda(u) = J(u)^{\frac{n-2}{4}} \ .$$

The $H^1_{(o)}$-flow of J over Σ^+ is (the parenthesis around o is meant to cover the case $\Omega = S^n$) :

$$(63) \qquad -\partial J(u) = -\lambda(u) u - \lambda(u)^{\frac{n+2}{n-2}} \, L^{-1} \, u^{\frac{n+2}{n-2}}$$

where

$$(64) \qquad L = \Delta - \frac{n-1}{4(n-2)} \quad \text{under Dirichlet boundary conditions if } \ \Omega \subset\subset S^n \ .$$

We consider the differential equation :

$$(65) \qquad \frac{\partial u}{\partial s} = -\partial J(u) \ .$$

To simplify the notations, we will rather work in $\mathbb{R}^n$, transforming the problem by stereographic projection.

Then :

$$(66) \qquad \Sigma^{+} = \{u > 0 \ ; \ |u|^2 = \int_\Omega |\nabla u|^2 dx = 1 \quad ; \quad u = 0_{|\partial\Omega} \quad \text{if } \bar{\Omega} \text{ compact} \subset \mathbb{R}^n\} \ .$$

Let

$$(67) \qquad \delta_i(x) = \delta(x_i,\lambda_i)(x) = c_o \frac{\lambda_i^{\frac{n-2}{2}}}{(1+\lambda_i^2 \ |x-x_i|^2)^{\frac{n-2}{2}}}$$

where c_o is such that :

$$(68) \qquad \Delta\delta_i + \delta_i^{\frac{n+2}{n-2}} = 0 \quad .$$

The variational problem does not satisfy the condition (C).
The sequences which violate this condition have been analyzed by J. Sacks and K. Uhlenbeck [9] for the harmonic map problem in dimension 2.
Their intuition has been extended to the Yamabe (type) equations throughout the work of Y.T. Siu and S.T. Yau [11], P.L. Lions [10], C.H. Taubes [13], M. Struwe [18] , H. Brézis and J.M. Coron [12].

We have the following proposition :

<u>Proposition</u> 3 : Let (u_n) be a sequence in Σ^+ such that $(J(u_n))$ is bounded and $\partial J(u_n) \to 0$. There exists $\bar{u} \geqslant 0$, $\partial J(\frac{\bar{u}}{\bar{u}}) = 0$ if $\bar{u} \neq 0$, $p \in \mathbb{N}$, such that for an extracted subsequence again denoted (u_n), we have :

$$(69) \qquad \frac{u_n - \bar{u}}{|u_n - \bar{u}|} \in V(p,\varepsilon_n) \quad ; \quad \varepsilon_n \to 0$$

where

$$(70) \qquad V(p,\varepsilon) = \Big\{ u \in \Sigma \mid \exists \ (a_1,\dots,a_p) \in \Omega^p \ ; \ \exists \ (\lambda_1,\dots,\lambda_p) \in [0,+\infty[^p$$

$$\text{s.t. } |u = \frac{1}{\sqrt{p}} \sum_{i=1}^{p} \frac{1}{K(a_i)^{\frac{n-2}{4}}} \delta(a_i,\lambda_i)| < \varepsilon \ ; \ \lambda_i d(a_i,\partial\Omega) > \frac{1}{\varepsilon} \quad \forall\, i \ ;$$

$$\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j \ |a_i - a_j|^2 > \frac{1}{\varepsilon} \quad \forall\, i \neq j \Big\}$$

$$(71) \qquad V(o,\varepsilon) = \{u \in \Sigma \mid |u| < \varepsilon\} \quad .$$

From the point of view of critical points at infinity, the sets $V(p,\varepsilon)$ when $\varepsilon \to 0$ are the <u>candidates</u> (or potential) <u>critical points at infinity</u>. These are <u>not critical</u> points at infinity.

In fact the critical points at infinity are analyzed in the following. Consider the problems :

(72) $\quad$ Minimize $|u - \sum_{i=1}^{p} \alpha_i \, \delta(x_i,\lambda_i)|$; $\alpha_i > 0$; $x_i \in \Omega$; $\lambda_i > 0$

(73) $\quad$ Minimize $|u - \sum_{i=1}^{p} \alpha_i \, P\delta(x_i,\lambda_i)|$; $\alpha_i > 0$; $x_i \in \Omega$; $\lambda_i > 0$

where

(74) $\quad$ $P\delta(x_i,\lambda_i)$ is the orthogonal projection of δ_i onto $H_0^1(\Omega)$
$(P\delta_i = \delta_i$ if $\Omega = \mathbb{R}^n)$.

<u>Proposition</u> 4 : For any p, there exists $\varepsilon_o > 0$ such that for any u in $V(p,\varepsilon_o)$, problems (72)-(73) have a unique solution up to permutation.

We consider now the differential equation (65) with a starting point

(75) $$u_o \in V(p,\varepsilon) \quad ; \quad \varepsilon < \varepsilon_o \quad .$$

We want to analyze the behaviour of the solution $u(s,u_o)$.
As long as the solution remains in $V(p,\varepsilon_o)$, we have well defined quantities :

(76) $$\alpha_i(s),\ x_i(s),\ \lambda_i(s),\ v(s) = u(s,u_o) - \sum_{i=1}^{p} \alpha_i(s)\ \delta(x_i(s),\lambda_i(s)) \quad .$$

In order for a gradient line to build up a critical point at infinity, we must have $\lambda_i(s) \to +\infty \ \forall\, i$.
In fact, the crucial behaviour is the one of the $\lambda_i(s)$.
It turns out that such a behaviour has a normal form, which we give now in the case K(x) is constant, $\overline{\Omega}$ compact.
For the general case, see A. Bahri [14].

We first introduce the regular part of the Green's function of the Laplacian on Ω.

$$(76) \qquad \begin{cases} y \longrightarrow H(x,y) \\ \Delta_y H(x,y) = 0 \quad \text{in } \Omega \\ H(x,y) = \dfrac{1}{|x-y|^{n-2}} \quad \text{on } \partial\Omega \quad . \end{cases}$$

We then define the matrix :

$$(77) \qquad M(x_1,\dots,x_p) = (M_{ij}) \quad \text{on } \Omega^p$$

where

$$(78) \qquad M_{ij}(x_1,\dots,x_p) = H(x_i,x_j) - \frac{1}{|x_i-x_j|^{n-2}} = G(x_i,x_j) \qquad i \neq j$$

$$(79) \qquad M_{ii}(x_1,\dots,x_p) = H(x_i,x_i) \quad .$$

We have :

$$(80) \qquad u(s) = \sum_{i=1}^{p} \alpha_i(s)\delta(x_i(s),\lambda_i(s)) + v(s) \quad .$$

<u>Theorem</u> 2 : For any $\delta > 0$, there exists $\varepsilon_o > 0$ and $s_o > 0$ such that if $u(s) \in V(p,\varepsilon_o)$ for $0 \leqslant s \leqslant s_o$ and $d(x_i(s),\partial\Omega) \leqslant \delta$ for $0 \leqslant s \leqslant s_o$, then for all $\bar{s} \geqslant s_o$ such that $u(s)$remains in $V(p,\varepsilon_o)$ for $s \in [o,\bar{s}]$, we have :

$$\begin{cases} \dfrac{\dot{\lambda}_i}{\lambda_i}(\bar{s}) = \dfrac{C_1}{\alpha_i\lambda_i^{\frac{n-2}{2}}}\left[\alpha_i^{\frac{n+2}{n-2}} \dfrac{H(x_i,x_i)}{\lambda_i^{\frac{n-2}{2}}} - \Big(\displaystyle\sum_{i\neq j} \frac{1}{\lambda_j^{n-2}} \frac{\alpha_i^{4/n-2}\alpha_j + \alpha_j^{\frac{n+2}{n-2}}}{|x_i-x_j|^{n-2}} - \alpha_i^{\frac{n+2}{n-2}} H(x_i,x_j)\Big) \right. \\ \qquad\qquad \left. + \displaystyle\sum_{j\neq i} \frac{\alpha_j}{\lambda_j^{\frac{n-2}{2}} |x_i-x_j|^{n+2}}\right] + o\left(\displaystyle\sum \frac{1}{\lambda_k^{n-2}}\right) \\ |\dot{x}_i|(\bar{s}) \leqslant \dfrac{C}{\lambda_i^{n-2}}\left(\displaystyle\sum \frac{1}{\lambda_k^{n-2}}\right) \\ \dot{\alpha}_i(\bar{s}) = C\alpha_i(1 - \alpha_i^{4/n-2} C_1 \displaystyle\int_{\mathbb{R}^n} \delta^{2n/n-2}) + 0\left(\frac{1}{\lambda_k^{n-2}}\right) \\ |v|^2(\bar{s}) \leqslant K\left(\displaystyle\sum \frac{1}{{}_{k}^{n-2}}\right) \quad . \end{cases}$$

This is the dynamical behaviour of the flow nearby the singularities (i.e. in the $V(p,\varepsilon)$'s ; $\varepsilon \to 0$).

Let then :

(81) $\rho(x_1,\ldots,x_p)$ be the least eigenvalue of $M(x_1,\ldots,x_p)$.

Observe that the matrix $M(x_1,\ldots,x_p)$ is related to the equation

$$\begin{cases} \Delta G(x,x_i) = \delta_{x_i} \\ G(x,x_i) = o|_{\delta\Omega} \end{cases}$$

(δ_{x_i} = Dirac mass at x_i. Not be confounded with δ_i of (67).)
The energy interactions, as we prove it, are governed by the matrix M and ρ. In partiuclar, for two points x_i and x_j, this interaction increases along grad ρ. This gradient is related to the vector fields $D_i = \mathrm{grad}(x,x_i)$ and $D_j = \mathrm{grad}(x,x_j)$ which satisfy :

$$\begin{cases} \mathrm{div}\, D_i = \delta_{x_i} \\ \mathrm{div}\, D_j = \delta_{x_j} \end{cases} \qquad \delta_x = \text{Dirac mass at } x.$$

If there is no boundary, then D_i and D_j are $\nabla\left(\frac{1}{x-x_i{}^{n-2}}\right)$ and $\nabla\left(\frac{1}{|x-x_j|^{n-2}}\right)$.
Increasing amounts, by symmetry arguments, to move the points along $D_i(x_j) - D_j(x_i)$ which is directed by $x_i - x_j$. Hence the interaction lies along the geodesic from x_i to x_j.
Otherwise, one has to take the boundary into account ; hence the distance of the points x_i and x_j to the boundary (see [14]).
We have :

Theorem 3 : If an orbit of the flow defines a critical point at infinity then, $\lim\limits_{s\to+\infty} \rho(x_1(s),\dots,x_p(s)) \geqslant 0$. Conversely, if $\rho(x_1(s),\dots,x_p(s))$ remains larger than $\delta > 0$ for a certain time interval, then the orbit will define a critical point at infinity and $\lim\limits_{s\to+\infty} \rho(x_1(s),\dots,x_p(s))$ will be strictly positive on such an orbit.

Then the points $x_i(s)$ converge in Ω and $\lambda_i(s) \underset{s\to+\infty}{\sim} C_i\, s^{1/n-2}$.

Note here that ρ is a "natural" extension of J at infinity (see [14]) ; its critical points in $\{x\in\Omega^p \mid \rho(x) > 0\}$ providing somewhat "more critical" points at infinity than the orbits of the flow. This is the full variational problem at infinity ; or either the variational problem with singularities.

On the other hand, we derive local expansions of J which we will give in the general case.

Using (73), we write a function u of $V(p,\varepsilon)$ in the form :

$$u = \sum_{i=1}^{p} \alpha_i\, P\delta_i + v \tag{82}$$

and we have setting

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\, |x_i-x_j|^2\right)^{-\frac{n-2}{2}} \quad . \tag{83}$$

Theorem 4 :

$$J\Big(\sum_{i=1}^{p} \alpha_i P\delta_i + v\Big) = \frac{(\Sigma\,\alpha_i^2)^{n/n-2}}{\Sigma\alpha_i^{2n/n-2}K(x_i)}\left(\int \delta^{2n/n-2}\right)\Bigg\{1 - (\Sigma\alpha_j^{2n/n-2}K(x_j))^{-1} c_7 \sum_i \alpha_i^{2n/n-2}\frac{\Delta K(x_i)}{\lambda_i^2}$$

$$+ o_K\Big(\Sigma \frac{1}{\lambda_i^2}\Big) - c_8\Bigg[\Sigma\,\frac{H(x_i,x_i)}{\lambda_i^{n-2}}\Big(\frac{1}{2}\frac{\alpha_i^2}{\Sigma\,\alpha_j^2} - \alpha_i^2\,\frac{\alpha_i^{4/n-2}K(x_i)}{\Sigma\,\alpha_j^2(\alpha_j^{4/n-2}K(x_j))}$$

$$+ \sum_{i\neq j}\Big(\frac{H(x_i,x_j)}{(\lambda_i\lambda_j)^{n-2/2}} - \varepsilon_{ij}\Big)\Big(\frac{1}{2}\frac{\alpha_i\alpha_j}{\Sigma\,\alpha_k^2} - \alpha_i\alpha_j\,\frac{\alpha_i^{4/n-2}K(x_i)}{\Sigma\,\alpha_k^2(\alpha_k^{4/n-2}K(x_k))}\Bigg]\Bigg\}$$

$$+ \frac{c_9}{\Sigma\alpha_i^2}\Big(\int |\nabla v|^2 dx - \frac{n+2}{n-2}\,\Sigma\,\alpha_j^2 \cdot \Sigma\,\frac{\alpha_i^{4/n-2}K(x_i)}{\Sigma\,\alpha_j^2(\alpha_j^{4/n-2}K(x_i))}\int \delta_i^{4/n-2}\, v^2\Big)$$

$$+ (f,v) + \begin{cases} + O((\int |\nabla v|^2)^{n/n-2} & \text{if } n \geq 6 \\ + O((\int |\nabla v|^2)^{3/2} & \text{if } n < 6 \end{cases} + O((\int |\nabla v|^2 dx)(\sum_{i \neq j} \varepsilon_{ij}^{2/n-2} (\log \varepsilon_{ij}^{-1})^{2/n}$$

$$+ O_K \left(\Sigma \frac{1}{\lambda_i}\right) + \Sigma \frac{1}{\lambda_i^2 \lambda_i^4} + \begin{cases} \Sigma \dfrac{1}{\lambda_j^2 d_j^{n-2}} & n \geq 6 \\ \Sigma \dfrac{1}{\lambda_i^{6-n/2} \lambda_j^{n-2/2}} \dfrac{1}{d_j^{n-2}} & n < 6 \end{cases} + O\left(\sum_{i \neq j} \varepsilon_{ij}^{n/n-2}\right.$$

$$\left. + \Sigma \left(\frac{1}{\lambda_i^{n-2/2} \lambda_j^{n-2/2}} \frac{\log \lambda_i}{d_j^n} + \frac{1}{\lambda_i^{n+2/2}} \frac{1}{\lambda_j^{n-2/2}} \frac{1}{d_i^n} + \frac{1}{(\lambda_i d_i)^n} + \frac{1}{\lambda_i^2} \frac{1}{\lambda_j^{n-2}} \frac{1}{d_j^{2(n-2)}}\right)\right)$$

where

$$(f,v) = \int_{\mathbb{R}^n} K(x) (\Sigma \alpha_i P\delta_i)^{\frac{n+2}{n-2}} v = O((\int |\nabla v|^2)^{1/2} (\sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2(n-2)}} (\log \varepsilon_{ij}^{-1})^{\frac{n+2}{2n}}$$

$$+ \Sigma \frac{1}{\lambda_i^{n+2/2} d_i^{n+2}} + \frac{1}{\lambda_i^2} \frac{1}{\lambda_j^{n-2/2}} \frac{1}{d_j^{n-2}} + O_K\left(\Sigma \frac{1}{\lambda_i}\right) + (\text{if } n < 6) \sum_{i \neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{n-2}{n}}$$

and

$$o_K(\varepsilon_{ij}) \leq \frac{C}{\lambda_i} O(\varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{n-2}{2}} ((\frac{\log \lambda_i}{\lambda_i})^{2/n} + 1)$$

and $d_i = d(x_i, \partial\Omega)$.

The indexed (by K) quantities appear when K is non constant.

The proof of Theorem 2 is dynamical (see [14]).

The proof of Theorem 4 requires some computations (see [16]).

Theorem 4 is a kind of Morse lemma nearby infinity or more precisely the "versal deployment" of the singularities in the sense of René Thom.

4.c <u>The defect of compactness</u> : <u>topological invariants</u>.

We illustrate this problem with $K = 1$ and $\overline{\Omega}$ compact.

Setting :

$$b_p = (pS)^{2/n-2} \tag{83}$$

we give a theorem which provides the difference of topology when crossing the energy level b_p ; i.e. for $\varepsilon > 0$ small enough a description of :

(84) $$(J^{b_p+\varepsilon}/J_p^{b_p-\varepsilon}) \cap V(p,\varepsilon_o)$$

i.e. the contribution of the critical points at infinity in the difference of topology at level b_p. For this, we need to introduce :

(85) $$I_p = \{(x_1,\ldots,x_p) \in \Omega^p / \rho(x_1,\ldots,x_p) < 0\}$$

(86) $$\Delta_{p-1} = \{(t_1,\ldots,t_p) \ ; \ t_i > 0 \quad , \quad \Sigma\, t_i = 1\}$$

(87) σ_p the symmetric group.

We then have :

<u>Theorem</u> 5 : Assume that ρ has no critical point in $\{x/\rho(x) = 0\}$. Then

$$(J^{b_p+\varepsilon}/J^{b_p-\varepsilon}) \cap V(p,\varepsilon_o) \simeq \Omega^p \times_{\sigma_p} \Delta_{p-1} / (\Omega^p \times \partial\Delta_{p-1} \times_{\sigma_p} I_p \times \Delta_{p-1} .$$

Finally, we state an existence theorem in case $K = 1$, $\overline{\Omega}$ is compact. Let Ω be connected, regular.

<u>Theorem</u> 6 : Assume the reduced $\mathbb{Z}_2$-homology of Ω is non zero, the equation

$$\begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \\ \\ u = 0_{|_{\partial\Omega}} \quad u > 0 \end{cases}$$

has a solution.

The proof of theorem 6 is <u>not</u> difficult, <u>neither</u> technical. It has been outlined in [16]. The complete proof will appear in [15].

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