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SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES 1985 - 1986

# COVARIANT FUNCTIONAL QUANTIZATIONS OF SUPERSTRINGS.

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1. INTRODUCTION.

String theories have recently emerged as compelling candidates for a unified theory of all fundamental interactions. Perhaps one way of conveying the excitement over the subject is to retrace briefly some stages in its development.

The original dual resonance models were actually proposed as phenomenological theories of hadrons. That the masses squared of the states rose linearly with the mode number suggested shortly afterwards their interpretation as theories of one-dimensional extended objects or "strings", which interacted by joining and splitting. The Virasoro constraints on physical states could then be explained as gauge constraints arising from reparametrization invariance of the action, which was the area of the world-sheet swept out by the string in space-time. Although the theory had strong resemblances with the modern theory QCD of strong interactions (in particular color flux tubes behave like strings, and QCD is believed to reduce to a string theory in the infinite color limit) there were always difficulties in interpreting strings as a model of just hadrons. Besides the presence of tachyons and the neccesity of working in critical dimensions to maintain Lorentz invariance and decoupling of ghosts (the critical dimension is $d = 26$ for the original bosonic model, and $d = 10$ for the fermionic model of Neveu-Schwarz-Ramond), the spectrum of the theory contained massless particles, including particles of spin 1 coupling like vector bosons, and particles of spin 2 coupling like gravitons. A drastic change of viewpoint was suggested in 1975 by Scherk and Schwarz, who proposed instead to interpret strings as models of unified interactions including gravity, and to view the extra dimensions as physical. Another key discovery was that the fermionic string projected onto the even G parity sector was free of tachyons and supersymmetric at each mass level. This space time supersymmetry was rather obscure in the original formalism, and Green and Schwarz pioneered a new approach which put it better in evidence. They classified fermionic strings into

Type I superstrings, consisting of open and closed strings, could carry a classical group as Yang-Mills gauge group. This was done by attaching group factors to the ends of open strings by the Chan-Paton method. One-loop amplitudes indicated that the infinities of the theory could be absorbed into a renormalization of the string tension. It was feared however that potential anomalies of both Yang-Mills and gravitational nature could render the theory inconsistent ;

Type II superstrings consisted only of closed strings and was expected to be finite. This theory contained no elementary Yang-Mills gauge group, and hence certainly no gauge anomaly. Furthermore in the chiral $N = 2$ ten dimensional supergravity theory which is its low energy limit, gravitational anomalies had been found by Alvarez-Gaumé and Witten to cancel. The problem here was that it seemed difficult to obtain the chiral fermions observed in nature by compactifying à la Kaluza-Klein a theory without elementary Yang-Mills group.

A breakthrough came in 1984 when Green and Schwarz discovered the anomaly cancellation mechanism which singled out SO(32) as the anomaly free gauge group for Type I superstrings. At the effective field theory level the same cancellation worked for $E_8 \times E_8$ , which started a search for an $E_8 \times E_8$ superstring. Such a theory - the "heterotic string" - was found by Gross, Harvey, Martinec, and Rohm, as a hybrid mixture of closed strings, with the right sector being the corresponding truncated fermionic string and the left sector a compactification from $d = 26$ to $d = 10$ of the bosonic string producing exactly the desired gauge group. The phenomenological prospects of the $E_8 \times E_8$ theory in particular are especially bright [1] .

It is thus urgent to improve our understanding of the principles and techniques of string theory, in particular to formulate Feynman rules for loop amplitudes. For scattering of pure particle states, the Feynman diagram for closed oriented strings at the $h$-loop level is a compact surface $M$ with $h$ handles, with insertion of the corresponding vertex operators. In the Polyakov formulation of string theory [2] , the amplitudes are given by functional integrals over all (super) geometries of $M$ . Conformal and reparametrization invariances in the critical dimensions suggest that the amplitudes should reduce to integrals over the (finite-dimensional) moduli space of $M$ , and Feynman rules in the string case should correspond to an explicit identification of the integrand as well as of the measure on moduli space occuring in the amplitude. We shall show that the Polyakov quantization of strings actually leads to the Weil-Petersson measure on moduli space, and that the integrand is a combination of determinants and propagators for the Laplacian and Dirac operators with respect to constant curvature metrics. In the superstring case, zero modes of the gravitino field also appear, which can be viewed as the super analogues of quadratic differentials. These ingredients in turn can be evaluated in terms of fundamental notions in the mathematics of Riemann surfaces and number theory, such as Poincaré series, prime forms, theta functions, and special values of Selberg zeta functions.

We would also like to call attention to other related recent developments, especially those described in [3] [4] (which are also based on the Polyakov approach), [5] (based on a light cone gauge approach), and [6] (based on methods of two dimensional conformal field theory).

## 2. THE BOSONIC STRING.

The simplest string theory is the purely bosonic one, whose treatment can serve as an introduction to the more complicated superstrings. The original action proposed by Nambu and Goto is the area of the world-sheet, which however is difficult to quantize while preserving manifest covariance. In the Polyakov approach we start instead from the action

$$I(X,g) = \frac{1}{2} \int_M d^2\xi \sqrt{g}\, g^{mn}\, \partial_m X^\mu\, \partial_n X_\mu \tag{1}$$

which reduces to the Nambu-Goto action when $\delta I/\delta g = 0$ . Here $g_{mn}$ is a metric on $M$ , and $X^\mu$ $\mu = 1,\ldots,d = 26$ describe the coordinates of the string in flat Euclidian space-time. (Strictly speaking, renormalization effects will require a cosmological term whose coefficient will be fixed later by requiring Weyl invariance, and a coupling constant of the form $\exp(\lambda\chi(M))$. We shall ignore them in this simplified discussion. Also the string tension has been set to $T = 1$). Besides translational invariance in the $X_\mu$'s , the action possesses two fundamental symmetries :

(i) Invariance under any reparametrization $\eta \in \mathrm{Diff}(M)$. The $\eta$'s which are deformable to the identity can be represented by vector fields $\delta V^m$ . The space of such $\eta$'s will be denoted by $\mathrm{Diff}_o(M)$, and the corresponding infinitesimal changes in the metric are given by

$$\delta g_{mn} = \nabla_m(\delta V_n) + \nabla_n(\delta V_m) \tag{2}$$

There are however global diffeomorphisms which cannot be so deformed, and the mapping class group $\mathrm{Diff}(M)/\mathrm{Diff}_o(M)$ will be of importance in the sequel ;

(ii) Invariance under Weyl scalings : $\delta g_{mn} = \delta\sigma g_{mn}$ (3)

The partition function is now given by

$$Z = \int Dg\, DX \exp(-I(X,g))$$

and the key task is to factor out the volumes of the gauge groups. The result is [7]

$$Z = \int_{\text{Moduli}} d(\text{Weil-Pet.})(\det P_1^{\dagger} P_1)^{1/2} (2\pi \det{}' \Delta_{\hat{g}} / \int d^2\xi \sqrt{\hat{g}})^{-13} \qquad (4)$$

Here $\Delta_{\hat{g}}$ is the scalar Laplacian, and $P_1$ is the operator sending vectors into traceless symmetric 2-tensors given by

$$P_1(\delta V) = \nabla_m \delta V_n + \nabla_n \delta V_m - g_{mn} g^{pq} \nabla_p (\delta V_q) \qquad (5)$$

Note that $P_1(\delta V)$ is precisely the traceless part of the infinitesimal changes in the metric induced by $\delta V$, and that $\operatorname{Ker} P_1^{\dagger}$ parametrizes infinitesimal deformations of holomorphic structures, i.e., quadratic differentials. Both $\Delta_{\hat{g}}$ and $P_1$ in (4) are taken with respect to constant curvature metrics $\hat{g}$.

To understand formula (4), we observe that the integral DX is Gaussian and produces $(2\pi \det{}' \Delta_g / \int d^2\xi \sqrt{g})^{-d/2}$, so the issue is to carry out the Dg integral over metrics. At this point conformal invariance is broken, and according to [2]

$$\frac{2\pi \det{}' \Delta g}{\int d^2\xi \sqrt{g}} = \frac{2\pi \det{}' \Delta \hat{g}}{\int d^2\xi \sqrt{\hat{g}}} \exp(-S(\sigma)) \qquad (6)$$

$$S(\sigma) = \frac{1}{24\pi} \int_M d^2\xi \sqrt{\hat{g}} \left[\frac{1}{2} \hat{g}^{mn} \partial_m \sigma \partial_n \sigma + \mu^2 (e^{\sigma} - 1) + \frac{1}{2} R_{\hat{g}} \sigma\right]$$

for $g = e^{\sigma} \hat{g}$. Thus Dg should be viewed as an integral along a slice for Weyl scalings, as well as along its orbits. A natural slice is the space $M_{-1}$ of metrics of constant negative curvature -1 (for one loop choose instead the metrics of zero curvature and area 1), from which Teichmüller space is easily obtained, since it is just $M_{-1}/\text{Diff}_0(M)$. Now $M_{-1}$ is naturally a Riemannian manifold on which $\text{Diff}_0(M)$ acts by isometries, so this provides Teichmüller space with a metric, which is the Weil-Petersson metric. Explicitly, it is obtained by representing tangent vectors to Teichmüller space by quadratic differentials, and their inner products by pairing them and integrating over M using constant curvature metrics. Next we parametrize metrics by a conformal factor $\delta\sigma$, a Teichmüller parameter, and an infinitesimal reparametrization $\delta V^m \in \text{Diff}_0(M)$. The Jacobian of this change of variables can be computed in terms of the natural decomposition of infinitesimal changes of metrics (which are just the symmetric 2 tensors) :

$$T_g(\text{Metrics}) = \{\delta\sigma g_{mn}\} \oplus \text{Range } P_1 \oplus \text{Ker}P_1^\dagger \tag{7}$$

A careful analysis taking sito account the zero modes of $P_1^\dagger$ gives a surprisingly simple answer :

$$Dg = \text{Vol}(\delta V)\ \text{Vol}(\delta\sigma)[d(\text{Weil-Pet.})(\det P_1^\dagger P_1(\hat{g}))\ \exp(-13\ S(\sigma))] \tag{8}$$

Thus in the critical dimension d = 26 the conformal anomaly cancels and formula (4) for the partition function follows, after factoring out $\text{Diff}(M)/\text{Diff}_o(M)$ as well, which means restricting the integration from Teichmüller space to moduli space. With this it is rather easy to compute scattering amplitudes. For example the vertex for the tachyon is

$$V_o(k) = \int d^2\xi\sqrt{g}\ e^{ik_\mu X^\mu(\xi)} \qquad k^2 = 8\pi$$

the amplitudes for p tachyons are

$$< V_o(k_1)\ldots V_o(k_p) > = \int Dg\ DX\ V_o(k_1)\ldots V_o(k_p)\ \exp(-I(X,g))$$

and the above arguments yield easily

$$< V_o(k_1)\ldots V_o(k_p) = (2\pi)^{13}\ \delta(k_1+\ldots+k_p) \times$$

$$\int_{\text{Moduli}} d(\text{Weil-Pet.})(\det P_1^\dagger P_1)^{1/2} \left(\frac{2\pi}{\int d^2\xi\sqrt{\hat{g}}} \det{}'\Delta_{\hat{g}}\right)^{-13} \times$$

$$\prod_{i=1}^{P} \int d^2\xi_i\sqrt{\hat{g}}(\xi_i)\exp\left(\frac{1}{2}\sum_{ij} k_{i\mu}\ k_j^\mu\ G_{\hat{g}}(\xi_i,\xi_j)\right). \tag{9}$$

In generalvertices for on shell particles (see e.g. [8] for the bosonic case, and [9] for fermionic ones) consistent with the symmetries of the action can be treated in the same way as partition functions, so we shall henceforth restrict our discussion to this latter case.

## 3. THE NEVEU-SCHWARZ-RAMOND MODEL.

The action of the NSR model is that of a matter multiplet $(X^\mu,\psi^\mu)$ coupled to two dimensional supergravity. In component language it can be written as

$$I = \int d^2\xi\sqrt{g}\left[\frac{1}{2}g^{mn}\partial_m X^\mu\partial_n X_\mu - \frac{i}{2}\psi^\mu\gamma^m\partial_m\psi_\mu - \frac{1}{2}\,\psi^\mu\gamma^a\gamma^m\chi_a\partial_m X_\mu + \frac{1}{8}\psi^\mu\gamma^a\gamma^b\chi_a(\chi_b\psi_\mu)\right]$$

with $X^\mu$ , $\mu = 1,\ldots,d = 10$ coordinates of the string in Euclidean space time $\psi^\mu_\alpha$ are Majorana spinors $g_{mn} = \delta_{ab} e^a_m e^b_n$ a metric on $M$ , and $\chi_{m\alpha}$ a spin 3/2 gravitino field. (For simplicity we omit scalar auxiliary fields as well as some explicit formulas for symmetry transformations below). Besides Lorentz invariance of the vielbeins $e^a_m$ and the reparametrization and Weyl invariances described in (i) and (ii) of Section 2, the action is also invariant under the superpartners of (i) and (ii) :

(iii) Supersymmetry :

$$\delta e^a_m = i\zeta\gamma^a\chi_m$$

$$\delta\chi_m = 2D_m\zeta$$

Here $D_m = \partial_m - \omega_m \gamma 5/2$ with $\omega_m$ the connection given by

$$\omega_m = e^a_m \epsilon^{pq}\partial_p e^b_q \delta_{ab} - \frac{i}{2} \chi_m\gamma_5\gamma^n\chi_n$$

(iv) Super Weyl transformation

$$\delta\chi_m = \gamma_m\lambda$$

The partition function is now

$$Z = \sum_{\nu,\bar{\nu}} \int DgD\chi DXD\psi \quad \exp(-I) \tag{10}$$

where $\nu,\bar{\nu}$ represent summation over spin structures. In Euclidean signature, this is done by splitting real spinors into sums of Weyl spinors and their complex conjugates, using the holomorphic structure to separate their contributions in the derivation of the measure, and assigning independent spin structures to Weyl spinors and their conjugates. Now in addition to the metric $g_{mn}$ , the gravitino field $\chi_m$ is non-dynamical and should be gauged away. In analogy with $P_1$ , the $\gamma$-traceless piece of supersymmetry transformations are $P_{1/2}\zeta = 2 D_m\zeta - \gamma_m\gamma^n D_n\zeta$ , and the gravitino modes decompose as

$$\{\delta\chi_m\} = \{\gamma_m\lambda\} \oplus \text{Range } P_{1/2} \oplus \text{Ker } P^\dagger_{1/2} \tag{11}$$

The modes $\text{Ker}P^\dagger_{1/2}$ are the ones that cannot be gauged away by supersymmetry and super-Weyl transformations. They are usually called supermoduli parameters. Their real dimension is 4h-4 when h is $\geqslant 2$ , and 0 or 2 when h = 1 depending on the spin structure.

Our facturing out of the gauge groups will be based on the key assumption of no local supersymmetry anomaly. In particular there will be no super-Weyl anomaly if there is no Weyl anomaly. With this we may determine the quantum measure by computing Faddeev-Popov determinants along any slice for combined supersymmetry and super-Weyl transformations. We construct such a slice $(g_{mn}, \chi_m)$ by writing $g = e^{\sigma}\hat{g}$ with $\hat{g}$ of constant curvature, and setting $\chi_m^{(j)} = e^{\sigma/4}\,\hat{\chi}_m$ with $\hat{\chi}_m^{(j)}$, $j = 1,\ldots,4h-4$ an orthonormal basis for $\mathrm{Ker}P_{1/2}^{\dagger}(\hat{g})$. Observe that such a slice is Weyl covariant, but <u>not</u> orthogonal to $\{\gamma_m\lambda\} \oplus (\mathrm{Range}\ P_{1/2})$, so that Jacobians have to be computed carefully. The result is

$$D\chi_m = \mathrm{Vol}(\gamma_m\delta\lambda)\ (\det P_{1/2}(\hat{g}))^{-1}\ \exp(-\frac{11}{2}S(\sigma))\ D\zeta \prod_{j=1}^{4h-4} da_j \qquad (12)$$

where $\zeta$ is the supersymmetry transformation parameter and $a_j$, $j = 1,\ldots,4h-4$ are real anticommuting parameters for the space $\{\chi_m^{(j)}\}$. Next the integrals over $DX$, $D\psi$, and $\Pi\, da_j$ can be carried out explicitly since the first two are Gaussian while the third is a Berezin integral. We check that left and right Weyl spinors remain separate, and that the conformal anomalies arising from the $DX$ and $D\psi$ integrals cancel exactly the ones from (8) and (12) in the critical dimension $d = 10$ to produce [10]

$$Z = \int_{\mathrm{Moduli}} d(\mathrm{Weil\text{-}Pet.}) \sum_{\substack{\text{spin}\\ \text{structures}}} (\det P_1^{\dagger}P_1)^{1/2} \left(\frac{2\pi}{\int d^2\xi\sqrt{\hat{g}}}\det{}'\Delta_{\hat{g}}\right)^{-5} (\det{}'\not{D}_{\hat{g}})^{5} \times (\det P_{1/2})^{-1} \langle \mathrm{Pfaff}\ K(\hat{g}) \rangle \qquad (13)$$

The last factor is a contribution of the supermoduli parameters, and can be expressed in terms of the $\hat{\chi}_m^{(j)}$'s and propagators for the Laplacian and Dirac operator $\not{D}_{\hat{g}}$ .

Measures for fermionic strings such as (13) usually involve combinations of chiral Dirac determinants which should be holomorphic on Teichmüller space. Using the recent constructions of connections on determinant bundles of [11] , characteristic classes computations for the Teichmüller curve [12] and the pseudo convexity of Teichmüller space, it is possible to define holomorphic determinants which however are not gauge invariant. This anomaly will cancel for the above combination of determinants in the critical dimension $d = 10$ [13] . For a discussion of these aspects in terms of the curvature and torsion of determinants bundles, we refer to [14] .

We also remark that it would be valuable to have a manifestly supersymmetric derivation of the measure. In superspace the multiplet $(g_{mn},\chi_m)$ corresponds to a superzweibein. In the Wess-Zumino gauge all components of the supercurvature and torsion can be expressed in terms of a single superfield [15] . Setting this superfield to be constant provides a slice for Weyl and super-Weyl scalings which is supersymmetric. It is however difficult to identify a slice for supersymmetry within this slice.

## 4. THE HETEROTIC STRING.

A measure for the heterotic string in the Polyakov formulation has been recently derived by Moore, Nelson, and Polchinski [3] . Here we brietly indicate how the above methods for the NSR model apply to the heterotic case to yield a formula analogous to (13). Covariant quantization is most easily performed in the fermionic representation where the action is

$$I = \int d^2\xi\sqrt{g}[\frac{1}{2}\, g^{mn}\partial_m X^\mu \partial_n X_\mu - \frac{i}{2}\psi^\mu\gamma^-\, e_-^{\ m}\partial_m\psi_\mu - \psi^\mu\chi_- e_+^m\, \partial_m X_\mu]$$

$$+ \int d^2\xi\sqrt{g}\psi^I\gamma^+ e_+^m\partial_m\psi^I$$

Here we are again in space-time dimension d = 10, $\gamma^\pm = (\gamma^1 \pm i\gamma^2)/\sqrt{2}$ , etc..., $\gamma^+\psi^\mu = 0$ , $\gamma^-\chi_a = 0$ , and the $\psi^I$'s, I = 1,...,32 are internal fermions. The gauge symmetries of the heterotic string are simple modifications of those of the NSR string, with however the following important restrictions.

(iii)' N = 1/2 supersymmetry

$$\delta e_m^+ = i\zeta\gamma^+\chi_m \quad \delta e_m^- = 0$$

$$\delta\chi_m = 2D_m\zeta \qquad \text{with} \quad \gamma^-\zeta = 0$$

(iv)' Super-Weyl scalings

$$\delta\chi_m = \gamma_m\lambda \qquad \text{with} \quad \gamma^+\lambda = 0 .$$

The partition function is

$$Z = \sum_{\nu,\bar{\nu}} \int Dg\ DX\ D\chi_m^-\ D\psi_\mu^+ \exp(-I)$$

where again spin structures $\nu$ and $\bar{\nu}$ for left and right movers are assigned independently. Concerning the left movers we still have the choice of assigning the same spin structure to all the $\psi^I$'s , which corresponds to the

Spin $(32)/\mathbb{Z}_2$ string, or to split the $\psi^I$'s into two groups of 16, and assign independent spin structures to each group. This would give the $E_8 \times E_8$ string. (For other choices and their string theories, see the recent developments in [16]).

In the heterotic case the analogue of $P_{1/2}$ is just $P_{1/2}$ restricted to spinors $\zeta$ satisfying $\gamma^- \zeta = 0$ , and the supermoduli modes are the (2h-2) complex basis vectors of negative chirality in $\mathrm{Ker}P^\dagger_{1/2}$ . We can now proceed as before by choosing a slice for supersymmetry and super-Weyl transformations which is Weyl covariant, compute Faddeev-Popov determinants along the slice, then the integrals $DX\, D\psi^+\, D\psi^I$ which are all Gaussian, and the integral over supermoduli which is just a finite dimensional fermionic integral. Combining with (8) for the Dg integral we easily see that all conformal anomalies cancel in the crtitical dimension d = 10 , and the final formula is

$$Z = \int_{\text{Moduli}} d(\text{Weil-Pet.}) \sum_{\substack{\text{spin}\\ \text{structures } \nu,\bar\nu}} (\det P_1^\dagger P_1)^{1/2} \left(\frac{2\pi}{\int d^2\xi\sqrt{\hat g}} \det{}' \Delta_{\hat g}\right)^{-5} (\det_{\bar\nu} \bar P_{1/2})^{-1} \times$$

$$(\det{}'_{\bar\nu} \bar{\not D})^5\, (\det_\nu {}^+\!\not D)^{16} < \mathrm{Pfaff}_{\bar\nu}\, K^-(\hat g) > \qquad (14)$$

Here all determinants are taken with respect to constant curvature metrics, and $K^-(\hat g)$ is a $(2h-2) \times (2h-2)$ matrix arising from the contributions of the supermoduli parameters. (It corresponds to the upper block of the matrix $K(\hat g)$ encountered in the NSR model (13), if we write $K(\hat g)$ in terms of spinors of definite chirality) :

$$K^-_{jk}(\hat g) = - \int d^2\xi \sqrt{\hat g}(\xi) \int d^2\eta \sqrt{\hat g}(\eta) \psi^\mu \hat\chi_-^{(j)}(\xi) \psi_\mu \hat\chi_-^{(k)}(\eta)\; \Sigma_{++}(\xi,\eta)$$

$$\Sigma_{++}(\xi,\eta) = e_+^m(\xi) e_+^n(\eta)\; (\partial_{\xi_m} \partial_{\eta_n} \Delta_{\hat g}^{-1}(\xi,\eta) - \tfrac{1}{2}\, \hat g_{mn}\, \delta_{\hat g}(\xi,\eta)) \qquad (15)$$

Finally we would like to mention the problem of consistency of global phases of the chiral Dirac determinants. Invariance under the mapping class group will provide powerful constraints on the phases of the spin structures within each orbit, but the relative phases between different orbits will require more subtle considerations.

## 5. EVALUATION OF DETERMINANTS.

We conclude by describing some of the recent progresses in the explicit evaluation of determinants. In the one-loop case all the above formulas can be written explicitly in terms of theta and Dedekind eta functions, so we restrict our discussion to the case of higher genus. For $\Delta$, $P_{1/2}$, $P_1$, their absolute values can be expressed in terms of special values of the Selberg zeta function [17]. The Selberg zeta function is defined in terms of lengths of closed geodesics on the surface M, which are more tractable geometric invariants than the spectrum of Laplacians. In particular approaching the boundary of moduli space corresponds to pinching the lengths of closed geodesics, and in the limit one obtains the divisor of Riemann surfaces with nodes which is the Deligne-Mumford way of compactifying moduli. A second advantage is that Selberg zeta functions admit functional equations and Dirichlet series expansions which can be exploited to estimate their values outside the region of their absolute convergence. Rigorous studies of these questions have been undertaken in [18]. The determinant of the Dirac operator has not been derived by these methods up to this point. A very interesting different type of formula has been proposed in [19]. We should also point out that the parametrization of moduli by Schottky groups by Mandelstam also leads to remarkable zeta functions [5]. In all the above it will be important to understand better the dependence of (combinations of) special values of zeta functions on the complex structure of moduli space.

In a different direction it would be natural to try to build amplitudes for any number of loops out of a basic ingredient, in a tighter analogy with the Feynman rules of field theory. Such an ingredient is the "pair of pants", which corresponds to the three string vertex. In [20] it is shown that the determinant of the Laplacian can be written in terms of Fenchel-Nielsen coordinates and Green's functions on pants. The Green's fucntions can be expressed in terms of prime forms on pants. We still need to understand better the changes in the mapping class group as we sew on diagrams. Recent results [21] on the homology of the mapping class group may help shed light on this issue. We shall report on such investigations elsewhere.

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