Séminaire Équations aux dérivées partielles - École Polytechnique

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Séminaire Équations aux dérivées partielles (Polytechnique) (1983-1984), exp. nº 21, p. 1-12

<http://www.numdam.org/item?id=SEDP_1983-1984____A21_0>

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SEMINAIRE GOULAOUIC - MEYER - SCHWARTZ 1983 - 1984

BOUNDARY VALUE PROBLEMS OF LINEAR ELASTOSTATICS AND HYDROSTATICS ON LIPSCHITZ DOMAINS

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Exposé n°XXI

15 Mai 1984

Section 1 Introduction

In this note I will report on some recent progress in the study of boundary value problems for systems of equations on Lipschitz domains D in $\mathbb{R}^n$, with boundary data in $L^2(\partial D, d\sigma)$. The specific problems I will discuss here arise from elastostatics and hydrostatics.

The Dirichlet problem for a single equation (the Laplacian) on a Lipschitz domain D with $L^2(\partial D, d\sigma)$ data and optimal estimates was first treated by B.E.J. Dahlberg (see [3], [4] and [5]). His approach relied on positivily, Harnack's inequality and the maximum principle, and thus, it could not be used to study for example the Neumann problem, or systems of equations. Shortly afterwards, E. Fabes, M. Jodeit Jr and N. Rivière [6] were able to utilize A.P. Calderon's ([1]) theorem on the boundedness of the Cauchy integral on C^1 curves, to extend the classical method of layer potentials to the case of C^1 domains. In this work they were able to resolve the Dirichlet and Neumann problem with $L^2(\partial D, d\sigma)$ data, and optimal estimates, for C^1 domains. They relied on the Fredholm theory, exploiting the compactness of the layer potentials in the C^1 case. In 1979, D. Jerison and C. Kenig [9] were able to give a simplified proof of Dalhberg's results, using an integral identity that goes back to Rellich ([15]). However, our method still relied on positivity. Shortly afterwards, we were also able to treat the Neumann problem on Lipschitz domains, with $L^2(\partial D, d\sigma)$ data and optimal estimates [10]. To do so we combined the Rellich type formulas with Dahlberg's results. This still restricted the applicability of the method to a single equation.

In 1981, R. Coifman, A. Mc Intosh and Y. Meyer [2] established the boundedness of the Cauchy integral on any Lipschitz curve, opening the door to the applicability of the layer potential method to Lipschitz domains. This method is very flexible, and does not in principle differentiate between a single equation or a system of equations. The difficulty becomes the solvability of the integral equations, some unlike in the C^1 case, the Fredholm theory is not applicable, because on a Lipschitz domain operators like the double layer potential are not compact.

For the case of a single equation (the Laplacian) this difficulty was overcome by G. Verchota ([16]) in his doctoral dissertation. He made the key observation that the Rellich identities mentioned before are the appropriate substitute to compactness, in the case of Lipschitz domains.

Thus, he was able to recover the results of Dahlberg [4], and of Jerison and Kenig [10], for Laplace's equation on a Lipschitz domain, but using the method of layer potentials.

This note sketches the extension of the ideas of G. Verchota to the case of systems of equations. The results thus obtained had not been previously available for general Lipschitz domains, although a lot of work had been devoted to the case of piecewise linear domains. For the case of the systems of elastostatics, the result that we are about to state had been previously obtained for C^1 domains by A. Gutierriez [7], using the Fredholm theory as in [6]. Once again, compactness was a crucial element in his analysis. This is of course, not available for Lipschitz domains.

The organization of the paper is as follows. In section 2 we treat the systems of elastotastics. This is joint work in progress with G. Verchota. In section 3 we treat the Stokes problem of hydrostatics. This is joint work in progress with E. Fabes. Full proofs of the results stated here will appear in future publications.

It is a pleasure to express my deep gratitude to my collaborators, E. Fabes and G. Verchota, for their contributions to our joint work, and for allowing me to announce here our unplublished results.

Section 2 Linear elastostatics on a Lipschitz domain.

For simplicity, in the rest of this note we will treat domains D above the graph of a Lipschitz function φ, i.e. $D = \{(x,y): y > \varphi(x)\}$, where $\varphi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ is a Lipschitz function and $n = 3$. Points $(x,\varphi(x))$ or $(y,\varphi(y))$ on ∂D will usually be denoted by P or Q. Points (x,y) in D or $\complement\overline{D}$ will be denoted by X. The surface measure on ∂D will be denoted by $d\sigma$, and the inward unit normal will be n. By $\Gamma^+(Q)$, $Q \in \partial D$ we will denote a vertical circular cone completely contained in D. Note that the opening of $\Gamma^+(Q)$ can (and will) be taken to depend only on the Lipschitz constant of φ. By $\Gamma^-(Q)$ we will denote the reflection of $\Gamma^+(Q)$, this time it is contained in $\complement\overline{D} = D-$. For a function $u(X)$ defined on D, $(u)^*(P) = \sup_{X\in\Gamma^+(P)} |u(X)|$, $P \subset \partial D$. We will say that $u(X)$ converges non tangentially at P to a limit ℓ if $\lim_{X \in \Gamma^+(P)} u(X) = \ell$. If u is defined in $\mathbb{R}^n \setminus \partial D$ and converges non-tangentiably at $P \in \partial D$ from D and $D-$, we will denote the respective limits by $u^+(P)$ and $u^-(P)$.

Let $\lambda \geq 0$, $\mu > 0$ be constants (Lamé moduli). We will seek to solve the following boundary value problems, where $\vec{u} = (u_1, u_2, u_3)$

$$(1) \qquad \begin{cases} \mu \Delta \vec{u} + (\lambda+\mu) \nabla \operatorname{div} \vec{u} = 0 & \text{in } D \\ \vec{u}\big|_{\partial D} = \vec{f} \in L^2(\partial D, d\sigma) \end{cases}$$

$$(2) \qquad \begin{cases} \mu \Delta \vec{u} + (\lambda+\mu) \nabla \operatorname{div} \vec{u} = 0 \qquad \text{in } D \\ (\lambda+\mu)\, n_i(X) \dfrac{\partial u_j}{\partial x_j}(X) + k\left(n_j(X) \dfrac{\partial u_j}{\partial x_i}(X) - n_i(X) \dfrac{\partial u_j}{\partial x_i}(X)\right) + \mu\, \delta_{ij} \dfrac{\partial u_j}{\partial n}(X) \Big|_{\partial D} = \\ = f_i \in L^2(\partial D, d\sigma) \end{cases}$$

Here and in the sequel we will use the summation convention. Problem (1) is a Dirichlet problem, while Problem (2) is a Neumann type problem in which k is an arbitrary, (but fixed) positive number. To ease the notation we introduce the operator $T^k \vec{u} =$

$$(\lambda+\mu) n_i(X) \frac{\partial u_j}{\partial X_j}(X) + k\left(n_j(X) \frac{\partial u_j}{\partial X_i}(X) - n_i(X) \frac{\partial u_j}{\partial X_i}(X)\right) + \mu \delta_{ij} \frac{\partial u_j}{\partial n}(X).$$ The operator T^k is called the generalized stress.

Problem (3) is the particular case $k = \mu$ of problem (2). The operator $T = T^\mu$ is called the stress.

<u>Theorem 2.1</u> : a) There exists a unique solution of problem (1) in D, with $(\vec{u})^* \in L^2(\partial D, d\sigma)$ and $\vec{u}$ having non-tangential limit $\vec{f}(P)$ for almost every $P \in \partial D$. The solution $\vec{u}$ belongs to the Sobolev space $H^{1/2}(D)$.

b) For every $k > 0$, $k \neq \mu$ there exists a unique solution of problem (2) in D, which is 0 at infinity, with $(\nabla \vec{u})^* \in L^2(\partial D, d\sigma)$, and with $T^k \vec{u}$ having non-tangential limit $\vec{f}(P)$ for almost every $P \in \partial D$. The solution $\vec{u}$ belongs to the Sobolev space $H^{3/2}(D)$.

<u>Theorem 2.2</u> : If $\|\nabla \varphi\|_\infty \leq 1$, the same conclusion as in Theorem 2.1 b) holds for problem (3).

At thus time we do not know whetter Theorem 2.2 holds without the restriction $\|\nabla\varphi\|_\infty \leq 1$. The proof of Theorem 2.2 is some what complicated, and will not be presented here.

In order to sketch the proof of Theorem 2.1 , we first introduce the Kelvin matrix of fundamental solutions (see [11] for example), $\Gamma(X) = (\Gamma_{ij}(X))$, where $\Gamma_{ij}(X) = \frac{A}{4\pi}\frac{\delta_{ij}}{|X|} + \frac{C}{4\pi}\frac{X_i X_j}{|X|^3}$, and $A = \frac{1}{2}\left[\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right]$, $C = \frac{1}{2}\left[\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right]$. Our solution of (1) will be given in the form of a double layer potential

$$\vec{u}(X) = \mathcal{D}\vec{g}(X) = \int_{\partial D} \{T^k(Q)\Gamma(X-Q)\}^t \,\vec{g}(Q)\, d\sigma(Q),$$

where the operator T^k is applied to each column of the matrix Γ .

Our solution of (2) will be given in the form of a single layer potential

$$\vec{u}(X) = S\vec{g}(X) = \int_{\partial D} \Gamma(X-Q)\vec{g}(Q)\, d\sigma(Q) .$$

<u>Lemma 2.3</u> : Let $\mathcal{D}\vec{g}(X)$, $S(\vec{g})(X)$ be defined as above, with $\vec{g} \in L^2(\partial D, d\sigma)$. Then, they both solve the system $\mu\Delta\vec{u} + (\lambda+\mu)\nabla \operatorname{div}\vec{u} = 0$ in D and D- . Moreover ,

(a) $\|(\mathcal{D}\vec{g})^*_+\|_{L^2(\partial D,d\sigma)} + \|(\mathcal{D}\vec{g})^*_-\|_{L^2(\partial D,d\sigma)} + \|\mathcal{D}\vec{g}\|_{H^{1/2}(D)} \leq C\|\vec{g}\|_{L^2(\partial D,d\sigma)}$

(b) $(\mathcal{D}\vec{g})^{\pm}(P) = \pm\frac{1}{2}\vec{g}(P) + \text{p.v.}\int_{\partial D} \{T^k(Q)\Gamma(P-Q)\}^t\vec{g}(Q)\,d\sigma(Q)$

(c) $\|(\nabla S\vec{g})^*_+\|_{L^2(\partial D,d\sigma)} + \|(\nabla S\vec{g})^*_-\|_{L^2(\partial D,d\sigma)} + \|S\vec{g}\|_{H^{3/2}(D)} \leq C\|\vec{g}\|_{L^2(\partial D,d\sigma)}$

(d) $\left(\frac{\partial}{\partial X_i}(S\vec{g})_j\right)^{\pm}(P) = \mp\left\{\frac{(A+C)}{2}\; n_i(P)\, G_j(P) - \left(n_i(P)\, n_j(P) < n(P), \vec{g}(P) >\right\}\right)$

$$+ \left(\text{p.v.}\int_{\partial D} \frac{\partial}{\partial P_i}\Gamma(P-Q)\,\vec{g}(Q)\, d\sigma(Q)\right)_j .$$

Therefore,

$$(T^k S\vec{g})^{\pm}(P) = \mp\frac{1}{2}\vec{g}(P) + \text{p.v.}\int_{\partial D} T^k(P)\Gamma(P-Q)\vec{g}(Q)\,d\sigma(Q) .$$

The proof of Lemma 2.3 follows from the theorem of R. Coifman, A. Mc Intosh and Y. Meyer ([2]). See [16] for the details in a similar situation.

Thus, the proof of Theorem 2.1 reduces to the invertibility on $L^2(\partial D, d\sigma)$ of the operators

$$\pm \frac{1}{2} I + K^k$$

$$\pm \frac{1}{2} I + (K^k)^* \quad , \text{ where}$$

$$K^k \vec{g}(P) = \text{p.v.} \int_{\partial D} \{T^k(Q)\Gamma(P-Q)\}^t \, \vec{g}(Q) \, d\sigma(Q) \ .$$

This is accomplished by means of the following lemma :

Lemma 2.4 : There exists a constant C , which depends only on the Lipschitz constant of ∂D , and on the number k , such that, if $k \neq \mu$ we have, for all $\vec{g} \in L^2(\partial D, d\sigma)$

$$\left\| \left(\frac{1}{2} I - (K^k)^*\right) \vec{g} \right\|_{L^2(\partial D, d\sigma)} \leq C \left\| \left(\frac{1}{2} I + (K^k)^*\right) \vec{g} \right\|_{L^2(\partial D, d\sigma)}$$

and,

$$\left\| \left(\frac{1}{2} I + (K^k)^*\right) \vec{g} \right\|_{L^2(\partial D, d\sigma)} \leq C \left\| \left(\frac{1}{2} I - (K^k)^*\right) \vec{g} \right\|_{L^2(\partial D, d\sigma)}$$

To show that Lemma 2.4 implies the invertibility of the operators in question, we follow Verchota's [16] ideas. First of all the inequalities clearly show that $\frac{1}{2} I + (K^k)^*$ and $\frac{1}{2} I - (K^k)^*$ are one to one. A simple argument using the continuity of $(K^k)^*$ shows that these operators have closed range. We can therefore attach an index to these operators which might possibly be infinite. Now, for each t , $0 < t \leq 1$, we consider the Lipschitz domain D_t given by the graph of $t\varphi$. By the theorem of Coifman - McIntosh - Meyer ([2]), the operators $(K_t^k)^*$, corresponding to the domains D_t , are continuous in norm. At $t = 0$ we are in the case of the upper half plane, and therefore the index is 0 . Therefore the index is also 0 at $t = 1$, and the desired invertibility follows. We are indebted to A. McIntosh for pointing out to us this simple argument using the index, which simplifies our previous proof.

We therefore pass to the proof of Lemma 2.4. In order to do so, we will first explain the boundary conditions in (2) from the point of view of second order elliptic systems.

Let a_{ij}^{rs} , $1 \le r$, $s \le M$, $1 \le i$, $j \le n$ be constants satisfying the ellipticity condition

$$a_{ij}^{rs}\, \xi_i\, \xi_j\, \eta^r\, \eta^s \ge c|\xi|^2\, |\eta|^2 \quad ,$$

and the symmetry condition $a_{ij}^{rs} = a_{ji}^{sr}$. We consider vector valued functions $\vec{u} = (u^1,\dots,u^M)$ on $\mathbb{R}^n$, satisfying the divergence form system

$$\frac{\partial}{\partial X_i}\, a_{ij}^{rs}\, \frac{\partial}{\partial X_j}\, u^s = 0 \quad \text{in } D$$

From variational consideration, the most natural boundary conditions are Dirichlet conditions ($\vec{u}|_{\partial D} = \vec{f}$) or Neumann type condition

$\frac{\partial \vec{u}}{\partial \nu} = n_i\, a_{ij}^{rs}\, \frac{\partial}{\partial X_j}\, u^s\Big|_{\partial D} = f_r$. The interpretation of (2) in this framework is the following : given $k > 0$, there exist constants $a_{ij}^{rs}(k) = a_{ij}^{rs}$, $1 \le i$, $j \le 3$, $1 \le r$, $s \le 3$ satisfying the ellipticity and symmetry conditions, and such that $\mu\Delta\vec{u} + (\lambda+\mu)\nabla \operatorname{div} \vec{u} = 0$ in D if and only if $\frac{\partial}{\partial X_i}\, a_{ij}^{rs}\, \frac{\partial u^s}{\partial X_j} = 0$ in D , and with $T^k\, \vec{u} = \frac{\partial}{\partial \nu}\, \vec{u} = n_i\, a_{ij}^{rs}\, \frac{\partial}{\partial X_j}\, u^s$.

<u>Lemma</u> 2.5 : (The Rellich, Payne-Weinberger, Nečas identities (see [15], [14] and [13]). Let $\vec{h}$ be a constant vector in $\mathbb{R}^n$, and suppose that $\frac{\partial}{\partial X_i}\, a_{ij}^{rs}\, \frac{\partial}{\partial X_j}\, u^s = 0$ in D , $a_{ij}^{rs} = a_{ji}^{sr}$, and $\vec{u}$ and its derivatives are suitably small at ∞ . Then

$$\int_{\partial D} h_\ell\, n_\ell\, a_{ij}^{rs}\, \frac{\partial u^r}{\partial X_i}\, \frac{\partial u^s}{\partial X_j}\, d\sigma = 2\int_{\partial D} h_i\, \frac{\partial u^r}{\partial X_i}\, n_\ell\, a_{\ell j}^{rs}\, \frac{\partial u^s}{\partial X_j}\, d\sigma$$

<u>Proof</u> : Apply the divergence theorem to

$$\frac{\partial}{\partial X_\ell}\left[(h_\ell\, a_{ij}^{rs} - h_i\, a_{\ell j}^{rs} - h_j\, a_{i\ell}^{rs})\, \frac{\partial u^r}{\partial X_i}\, \frac{\partial u^s}{\partial X_j}\right] = 0$$

<u>Corollary</u> 2.6 : If $a_{ij}^{rs}\, \frac{\partial u^r}{\partial X_i}\, \frac{\partial u^s}{\partial X_j} \ge C \sum_r |\nabla u^r|^2$, then, $\frac{\partial \vec{u}}{\partial \nu} = n_i\, a_{ij}^{rs}\, \frac{\partial u^s}{\partial X_j}$ satisfies

$$\int_{\partial D} \left| \frac{\partial}{\partial v} \vec{u} \right|^2 \simeq \sum_r \int_{\partial D} |\nabla_t u^r|^2 \quad ,$$

where $\nabla_t u^r$ denotes the tangential components of the gradient of u^r , and the comparability constants depend only on the Lipschitz constant of ∂D .

Proof : Take $\vec{h} = e_n$. Because of the Lipschitz character of ∂D , $h_\ell n_\ell \geq C$. Then,

$$\sum_\Omega \int_{\partial D} |\nabla u^r|^2 \, d\sigma < C \int_{\partial D} h_\ell \, n_\ell \, a_{ij}^{rs} \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j} \, d\sigma =$$

$$= C \int_{\partial D} h_i \frac{\partial u^r}{\partial X_i} \, n_\ell \, a_{\ell j}^{rs} \frac{\partial u^s}{\partial X_j} \, d\sigma \leq$$

$$\leq C \left(\sum_\Omega \int_{\partial D} |\nabla u^r|^2 d\sigma \right)^{1/2} . \left(\int_{\partial D} \left| \frac{\partial \vec{u}}{\partial v} \right|^2 d\sigma \right)^{1/2} .$$

Thus, $\sum_\Omega \int_{\partial D} |\nabla_t u^r|^2 \, d\sigma \leq C \int_{\partial D} \left| \frac{\partial \vec{u}}{\partial v} \right|^2 d\sigma$.

For the opposite inequality, observe that, for each r,s,j fixed, the vector $h_i \, n_\ell \, a_{\ell j}^{rs} - h_\ell \, n_\ell \, a_{ij}^{rs}$ is perpendicular to n . Because of lemma 2.5,

$$\int_{\partial D} h_\ell \, n_\ell \, a_{ij}^{rs} \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j} \, d\sigma =$$

$$= 2 \int_{\partial D} (h_\ell \, n_\ell \, a_{ij}^{rs} - h_i \, n_\ell \, a_{\ell j}^{rs}) \frac{\partial u^\ell}{\partial X_i} \frac{\partial u^s}{\partial X_j} \, d\sigma$$

Hence, $\int_{\partial D} |\nabla \vec{u}|^2 \, d\sigma \leq C \left(\sum_\Omega \int_{\partial D} |\nabla_t u^r|^2 d\sigma \right)^{1/2} \left(\sum_\Omega \int_{\partial D} |\nabla u^r|^2 d\sigma \right)^{1/2}$,

and so ,

$$\int_{\partial D} \left| \frac{\partial \vec{u}}{\partial \gamma} \right|^2 d\sigma \leq C \int_{\partial D} |\nabla \vec{u}|^2 \, d\sigma \leq C \sum_\Omega \int_{\partial D} |\nabla_t u^r|^2 \, d\sigma \; .$$

Remark 2.7 : At this point we can explain the difference between problem (2) when $k \neq \mu$, and problem (3). In the case of problem (2), with $k \neq \mu$, $a_{ij}^{rs}(k)$ satisfy the hypothesis of corollary 2.6. On the other hand, when

$k = \mu$, $\mathcal{a}^{rs}_{ij} \frac{\partial u^s}{\partial X_i} \frac{\partial u^r}{\partial X_j} = \lambda(\operatorname{div} \vec{u})^2 + \frac{\mu}{2} \sum_{i,j} (\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j})^2$, which obviously does not satisfy the hypothesis of 2.6.

Proof of Lemma 2.4 : Let $\vec{u}(X) = S\vec{g}(X)$. We will apply corollary 2.6 to $\vec{u}$, which we can in the case $k \neq \mu$. We will do so in D and also in D- . First note that $T^k\vec{u} = \frac{\partial \vec{u}}{\partial \nu}$. Then note that because of Lemma 2.3, (d) $(\nabla_t u_j)_+ = (\nabla_t u_j)_-$. Therefore, $\int_{\partial D} |(T^k\vec{u})_+|^2 \, d\sigma \simeq \int_{\partial D} |(T^k\vec{u})_-|^2 \, d\sigma$. But, again using Lemma 2.3, (d) we see that Lemma 2.4 follows immediately. We have thus established Lemma 2.4 and hence Theorem 2.1.

Section 3 : Linear hydrostatics on a Lipschitz domain

We will continue utilizing the notation introduced in Section 2. We will discuss the so called Stokes problem of hydrostatics.

We seek a vector valued function $\vec{u} = (u_1, u_2, u_3)$ and a scalar valued function p satisfying

$$(4) \qquad \begin{cases} \Delta\vec{u} = \nabla p & \text{in } D \\ \operatorname{div} \vec{u} = 0 & \text{in } D \\ \vec{u}|_{\partial D} = \vec{f} \in L^2(\partial D, d\sigma) \end{cases}$$

Theorem 3.1 : There exists a unique solution of problem (4) in D , with $(\vec{u})^* \in L^2(\partial D, d\sigma)$, and $\vec{u}$ having non-tangential limit $\vec{f}(P)$, for almost every $P \in \partial D$. The solution $\vec{u}$ belongs to the Sobolev space $H^{1/2}(D)$.

In order to sketch the proof of Theorem 3.1 (which parallels that of Theorem 2.1), we introduce the matrix of fundamental solutions (see the book of Ladyzhenskaya, [12]) $\Gamma(X) = (\Gamma_{ij}(X))$, where $\Gamma_{ij}(X) = \frac{1}{8\pi} \frac{\delta_{ij}}{|X|} + \frac{1}{8\pi} \frac{X_i X_j}{|X|}$, and its corresponding pressure vector $\vec{q}(X)$ $(q^i(X))$, where $q^i(X) = \frac{X_i}{4\pi|X|^3}$. Our solution of (4) will be given in the form of a double layer potential

$$\vec{u}(X) = \mathcal{D}\vec{g}(X) = -\int_{\partial D} \{T'(Q)\Gamma(X-Q)\}\vec{g}(Q)\, d\sigma(Q) ,$$

where $(T'(Q)\Gamma(X-Q))_{i\ell} = \delta_{ij} q^{\ell}(X-Q) \quad n_j(Q) + \frac{\partial \Gamma_{i\ell}}{\partial Q_j}(X-Q) \quad n_j(Q)$.

We will also have use for the single layer potential

$$\vec{u}(X) = S\vec{g}(X) = \int_{\partial D} \Gamma(X-Q)\vec{g}(Q)\,d\sigma(Q)$$

<u>Lemma</u> 3.2 : Let $\mathcal{D}\vec{g}$, $S(\vec{g})$ be defined as above, with $\vec{g} \in L^2(\partial D, d\sigma)$ Then $\vec{u}(X) = \mathcal{D}(\vec{g})(X)$ solves

$$\Delta\vec{u} = \nabla p$$

$$\operatorname{div} \vec{u} = 0 \quad \text{in } D \text{ and } D_- .$$ Moreover

(a) $$\|(\mathcal{D}\vec{g})^*_+\|_{L^2(\partial D, d\sigma)} + \|(\mathcal{D}\vec{g})^*_-\|_{L^2(\partial D, d\sigma)} +$$

$$+ \|\mathcal{D}\vec{g}\|_{H^{1/2}(D)} \le C\|\vec{g}\|_{L^2(\partial D, d\sigma)}$$

(b) $$(\mathcal{D}g)^{\pm}(P) = \pm \frac{1}{2}\vec{g}(P) + \text{p.v.} -\!\!\!\!\!\!\int_{\partial D} \{T'(Q)\Gamma(P-Q)\}\vec{g}(Q)\,d\sigma(Q)$$

(c) $$\|(\nabla S\vec{g})^*_+\|_{L^2(\partial D, d\sigma)} + \|(\nabla S\vec{g})^*_-\|_{L^2(\partial D, d\sigma)} \le C\,\|\vec{g}\|_{L^2(\partial D, d\sigma)}$$

(d) $$\left(\frac{\partial}{\partial X_i}(S\vec{g})_j\right)^{\pm}(P) = \mp \left\{\frac{n_i(P)g_j(P)}{2} - \frac{n_i(P)\; n_j(P)}{2} < n(P), \vec{g}(P) > \right\}$$

$$+ \left(\text{p.v.} \int_{\partial D} \frac{\partial}{\partial P_i}\Gamma(P-Q)\vec{g}(Q)\;d\sigma(Q)\right)\Big\} ,$$ and

$$(TS\vec{g})^{\pm}(P) = \mp \frac{1}{2}\vec{g}(P) + \text{p.v.} \int_{\partial D} \{T(P)\Gamma(P-Q)\}\vec{g}(Q)\;d\sigma(Q) ,$$ where

$$(T(X)\Gamma(X-Q))_{i\ell} = n_j(X)\;\frac{\partial \Gamma_{i\ell}}{\partial X_j}(X-Q) - \delta_{ij}\;q^{\ell}(X-Q)\;n_j(X) .$$

The proof of Lemma 3.2 follows, as the one in Lemma 2.3 from [2]. See [12] for the case of smooth domains. Thus, the proof of Theorem 3.1 reduces to the invertibility in $L^2(\partial D, d\sigma)$ of the operator

$$\frac{1}{2} I + K \quad ,$$ where

$K\vec{g}(P) = \text{p.v.} -\!\!\!\!\!\!\int_{\partial D} \{T'(Q)\Gamma(P-Q)\}\;\vec{g}(Q)\;d\sigma(Q)$. As in section 2, this in turn follows from

Lemma 3.3 : There exists a constant C , which depends only on the Lipschitz constant of ∂D , such that, for all $\vec{g} \in L^2(\partial D, d\sigma)$,

$$\left\|\left(\frac{1}{2} I - K^*\right) \vec{g}\right\|_{L^2(\partial D, d\sigma)} \leq C \left\|\left(\frac{1}{2} I + K^*\right) \vec{g}\right\|_{L^2(\partial D, d\sigma)},$$

and

$$\left\|\left(\frac{1}{2} I + K^*\right) \vec{g}\right\|_{L^2(\partial D, d\sigma)} \leq C \left\|\left(\frac{1}{2} I - K^*\right) \vec{g}\right\|_{L^2(\partial D, d\sigma)}$$

We turn now to the proof of Lemma 3.3. The proof relies on two integral identities.

Lemma 3.4 : Let $\vec{h}$ be a constant vector in $\mathbb{R}^n$, and suppose that $\Delta \vec{u} = \nabla p$, div $\vec{u} = 0$ in D , and that $\vec{u}, p$ and their derivatives are suitably small at ∞ . Then,

$$\int_{\partial D} h_\ell \, n_\ell \cdot \frac{\partial u^s}{\partial X_j} \frac{\partial u^s}{\partial X_j} \, d\sigma = 2 \int_{\partial D} \frac{\partial u^s}{\partial n} . h_\ell \frac{\partial u^s}{\partial X_\ell} \, d\sigma - 2 \int_{\partial D} p . \, n_s h_\ell \frac{\partial u^s}{\partial X_\ell} \, d\sigma$$

Lemma 3.5 : Let $\vec{h}$, $\vec{u}$ and p be as in Lemma 3.4. Then,

$$\int_{\partial D} h_\ell n_\ell p^2 d\sigma = 2 \int_{\partial D} h_r \frac{\partial u^r}{\partial n} . p d\sigma - 2 \int_{\partial D} h_r \frac{\partial u^r}{\partial X_i} \frac{\partial u^i}{\partial n} \, d\sigma + 2 \int_{\partial D} h_r \, n_s \frac{\partial u^s}{\partial X_j} \frac{\partial u^r}{\partial X_j} \, d\sigma$$

The proofs of Lemmas 3.4 and 3.5 are simple applications of the properties of $\vec{u}$, p, and the divergence theorem.
An immediate consequence of Lemma 3.5 is

Corollary 3.6 : Let $\vec{u}$, p be as in Lemma 3.4 , D a Lipschitz domain. Then, $\int_{\partial D} p^2 \, d\sigma \leq C \int_{\partial D} |\nabla \vec{u}|^2 \, d\sigma$, where C depends only on the Lipschitz constant of ∂D .

A consequence of Corollary 3.6 and Lemma 3.4 is

Corollary 3.7 : Let $\vec{u}$, p be as in Lemma 3.4, D a Lipschitz domain. Then,

$$\int_{\partial D} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 d\sigma \simeq \sum_{\Omega} \int_{\partial D} |\nabla_t \, u^r|^2 \, d\sigma + \sum_j \int_{\partial D} \left| n_s \frac{\partial u^s}{\partial X_j} \right|^2 d\sigma ,$$

where, by definition $\frac{\partial \vec{u}}{\partial \nu} = \frac{\partial \vec{u}}{\partial n} - p . \, n.$

Proof : Lemma 3.4 clearly implies that

$$\int_{\partial D} |\nabla \vec{u}|^2 \, d\sigma \leq C \int_{\partial D} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 \, d\sigma$$

Arguing as in the proof of corollary 2.6, using Lemma 3.4, we see that

$$\int_{\partial D} |\nabla \vec{u}|^2 \, d\sigma < C(\sum_{\Omega} \int_{\partial D} |\nabla_t u^r|^2 \, d\sigma) + \left|\int_{\partial D} p \; n_s \, h_\ell \, \frac{\partial u^s}{\partial X_\ell} \, d\sigma\right| .$$

By corollary 3.6 , the right hand side is bounded by

$$C(\int_{\partial D} |\nabla \vec{u}|^2 \, d\sigma)^{1/2} \cdot (\sum_j \int_{\partial D} | n_s \frac{\partial u^s}{\partial X_j} |^2 \, d\sigma)^{1/2} .$$

The corollary now follows, using corollary 3.6 once more. We can now prove Lemma 3.3. Let $\vec{u} = S(\vec{g})$. Observe that (d) in Lemma 3.2 implies that both $\nabla_t \vec{u}$ and $n_s \frac{\partial u_s}{\partial X_j}$ are continuous across ∂D . Lemma 3.3 now follows from corollary 3.7, using the second part of (d) in Lemma 3.2.

References

[1] A.P. Calderón, Cauchy integrals on Lipchitz curves and related operators, Proc. Mat. Acad. Sc. U.S.A. 74 (1977), 1324-1327.

[2] R.R. Coifman, A. McIntosh and Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, Annals of Math 116 (1982), 361-387.

[3] B.E.J. Dahlberg, On estimates of harmonic measure, Arch. Rational Mech. Anal 65 (1977), 272-288.

[4] B.E.J. Dahlberg, On the Poisson integral for Lipschitz and C^1 domains, Studia Math.66 (1979), 13-24.

[5] B.E.J. Dahlberg, Weighted norm inequalities for the Lusin area integral and the non-tangential maximal functions for functions harmonic in a Lipschitz domain, Studia Math 67 (1980), 297-314.

[6] E. Fabes, M. Jodeit Jr. and N. Rivière, Potential technique for boundary value problems on C^1 domains, Acta Math. 141 , (1978), 165-186

[7] A. Gutieviez, Boundary value problem for linear elastostatics on C^1 domains, University of Minnesota preprint, 1980.

[8] D. Jerison and C. Kenig, An identity with applications to harmonic measure, Bull. AMS, Vol 2 (1980), 447-451.

[9] D. Jerison and C. Kenig, The Dirichlet problem in non-smooth domains, Annals of Math. 13 (1981), 367-382.

[10] D. Jerison and C. Kenig, The Neumann problem on Lipschitz domains, Bull. AMS, Vol.4 (1981), 203-207.

[11] V.D. Kupradze, Three dimensional problems of the mathematical theory of elasticity and thermoelasticity, North Holland, New-York, 1979.

[12] O. A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, Gordon and Breach, New York, 1963.

[13] J. Nečas, Les méthodes directes en théorie des équations elliptiques, Academia, Prague, 1967.

[14] L. Payne and H. Weinberger, New bounds for solutions of second order elliptic partial differential equations, Pacific J. of Math. 8 (1958), 551-573.

[15] F. Rellich, Darstelling der eigenwerte von $\Delta u + \lambda u$ durch ein randintegral, Math.Z. 46 (1940), 635-646.

[16] G. Verchota, Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains, University of Minnesota, 1982.

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