Séminaire Équations aux dérivées partielles – École Polytechnique

D. Jerison

The Sobolev inequality on the Heisenberg group and the Yamabe problem on CR manifolds (d'après travaux avec J. M. Lee)

Séminaire Équations aux dérivées partielles (Polytechnique) (1983-1984), exp. nº 18, p. 1-13

<http://www.numdam.org/item?id=SEDP_1983-1984____A18_0>

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

91128 PALAISEAU CEDEX - FRANCE

Tél. (6) 941.82.00 - Poste N°
Télex : ECOLEX 691596 F

SEMINAIRE GOULAOUIC-MEYER-SCHWARTZ 1983-1984

THE SOBOLEV INEQUALITY ON THE HEISENBERG GROUP AND THE YAMABE PROBLEM ON CR MANIFOLDS

(d'après travaux avec J.M. Lee)

par D. JERISON

Exposé n°XVIII

20 Mars 1984

We begin by recalling the classical Sobolev inequality and the Yamabe problem on Riemannian manifolds.

The Sobolev inequality in $\mathbb{R}^m$, $m \geq 3$, is

$$\left(\int_{\mathbb{R}^m} |\varphi(x))^q dx\right)^{2/q} < C \int_{\mathbb{R}^m} |\nabla \varphi(x)|^2 \, dx \quad \text{for all } \varphi \in C_o^\infty(\mathbb{R}^m)$$

The change of variable by dilation $\varphi(x) \longrightarrow \varphi(\delta x)$, $\delta > 0$, shows that the only exponent q for which the inequality can hold is $1/q = 1/2 - 1/m$. The best constant C in the Sobolev inequality can be calculated by considering the variational problem

$$\inf \left\{\int_{\mathbb{R}^m} |\nabla \varphi|^2 \Big/ \left(\int_{\mathbb{R}^m} |\varphi|^q\right)^{2/q} : \nabla \varphi \in L^2(\mathbb{R}^m) \ ; \ \varphi \in L^q(\mathbb{R}^m)\right\}$$

Extremal functions exist and satisfy the Euler-Lagrange equation

$$(1) \qquad \Delta\varphi = \mu\varphi^{q-1} \quad \text{on } \mathbb{R}^m \quad ,$$

Indeed, the extremal μ constant functions are translations and dilations of the function $C(1+|x|^2)^{(m-2)/2}$ (See [1, 11]).

However, the problem has more symmetries than dilation and translation. For instance, the Kelvin transform $\varphi(x) \longrightarrow |x|^{2-m}\varphi(x/|x|^2)$ preserves both numerator and denominator in the infinum above. This is a special case of a more general phenomenon of conformal invariance. Stereographic projection is a conformal transformation of the sphere S^m to $\mathbb{R}^m$. The corresponding extremal problem on the sphere is rotation invariant and the Kelvin transform in $\mathbb{R}^m$ corresponds to reflection a-cross the equator of S^m .

Let (M,g) be a Riemannian manifold of dimension $m \geq 3$, with scalar curvature K and Laplace-Beltrami quator Δ . If $\tilde{K}$ is the scalar curvature associated with the metric $\tilde{g} = \varphi^{q-2}g(\varphi \in C^\infty(M), \varphi > 0)$ then $\tilde{K}$ is given by

$$a_m \, \Delta\varphi + K\varphi = \tilde{K} \, \varphi^{q-1} \ , a_m = 4(m-1)/(m-2) \ .$$

A key feature of this formula is that the exponent $q = 2m/(m-2)$ is the same as the exponent q in the Sobolev inequality.

The (Riemannian) Yamabe problem is to find a metric of constant scalar curvature among all conformal multiples of a given metric. Thus the problem is equivalent to solving

(2) $$a_m \Delta\varphi + K\varphi = \mu\, \varphi^{q-1}$$

for some $\varphi \in C^\infty(M)$, $\varphi > 0$, μ constant. In the case of $\mathbb{R}^m$, $K = 0$, so that this is the same as equation (1). (On the other hand, $\mathbb{R}^m$ and S^m are conformally equivalent, so solving equation (1) is the same as solving (2) in the case of S^m)

Equation (2) is the Euler-Lagrange equation for the variational problem

(3) $$\mu(M,g) = \inf\{\int_M (a_m < d\varphi, d\varphi >_g + K\varphi^2)\, dV_g \;:\; \int_M |\varphi|^q\, dV_g = 1\}$$

Here $< \, , >_g$ denotes the (dual) metric induced by g on 1-forms and dV_g the volume element for g.

The best result to date is due to T. Aubin [1]

Theorem Let (M,g) be a compact, Riemannian manifold of dimension ≥ 3

(a) $\mu(M,g)$ depends only on the conformal class of g

(b) $\mu(M,g) \leq \mu(S^m$, standard metric) $= \mu(\mathbb{R}^m$, standard metric) $= \mu_o$

(c) If $\mu(M,g) < \mu_o$, then the infinimum in (3) is attained by a positive, C^∞ solution to (2). Thus the Yamabe problem is solved.

(d) If $m \geq 6$, then $\mu(M,g) < \mu_o$ unless M is everywhere locally conformally equivalent to $\mathbb{R}^m$.

When $m < 6$, much less is known about when $\mu(M,g) < \mu_o$.

In this article I would like to discuss an analogous theorem in the setting of CR manifolds. I would also like to point out why it would be useful to solve explicitly the problem on the Heisenberg group analogous to (1) and to announce a partial result in that direction. I will be describing joint work with John M. Lee ; details will appear elsewhere.

There is a far-reaching analogy between conformal and CR geometry. To illustrate this as vivially as possible, we summarize in a table the most important parallels. This will serve as an outline of the discussion that follows.

Conformal Geometry	CR Geometry
Riemannian manifold (M,g)	Pseudohermitian manifold (N,θ)
Euclidean space $\mathbb{R}^m$	Heisenberg group $\mathbb{H}^n$
m-sphere S^m in $\mathbb{R}^{m+1}$	2n+1 sphere S^{2n+1} in $\mathbb{C}^{n+1}$
Stereographic projection	Cayley transform
Riemannian normal coordinates	Folland-Stein normal coordinates
Scalar curvature K	Webster scalar curvature R
Laplace-Beltrami operator Δ	Sublaplacian Δ_b (Re $\square_b$ on functions)
Sobolev spaces L_k^r	Folland-Stein spaces S_k^r
Sobolev embedding $L_1^2 \subset L^q$	Sobolev embedding $S_1^2 \subset L^p$,
$\frac{1}{q} = \frac{1}{2} - \frac{1}{m}$	$\frac{1}{p} = \frac{1}{2} - \frac{1}{2n+2}$.
Conformal change :	Change of pseudohermitian structure
$\tilde{g} = \varphi^{q-2} g$	$\tilde{\theta} = u^{p-2}\,\theta$
Conformal invariant $\mu(M,g)$	CR invariant $\lambda(N,\theta)$
Yamabe equation : $a_m\Delta\varphi+K\varphi=\mu^{q-1}$	CR Yamabe equation : $b_n\Delta_b u+Ru=\lambda u^{p-1}$.

The Heisenberg group $\mathbb{H}^n$ is the Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with coordinates $(z,t) = (z_1,\dots,z_n,t)$ and whose group law is given by

$$(z,t)\;(z',t') = (z+t',t+t' + 2\mathrm{Im}\sum_{j=1}^{n} z_j\;\overline{z'_j})$$

We will also denote elements of $\mathbb{H}^n$ by x and y. For $x = (z,t)$ the dilations $\delta x = (\delta z, \delta^2 t)$ preserve the group law : $\delta(xy) = (\delta x)(\delta y)$. The vector fields $Z_j = \partial/\partial z_j + i\bar{z}_j \, \partial/\partial t$ are left-invariant and homogeneous of degree -1 with respect to dilations. The Haar measure $d\mu$ for the group is just Euclidean measure.

Consider the Sobolev-type inequality

$$(*) \qquad \left(\int_{\mathbb{H}^n} |u(x)|^p d\mu(x)\right)^{2/p} \leq C \int_{\mathbb{H}^n} \sum_{j=1}^{n} |Z_j u(x)|^2 \, d\mu(x)$$

for real-valued functions u. The change of variable by dilation (noting that $d\mu(\delta x) = \delta^{2n+2} d\mu(x)$) shows that the only possible exponent p is given by $1/p = 1/2 - 1/(2n+2)$. Denote

$$\mathcal{L} = -\frac{1}{2} \sum Z_j \bar{Z}_j + \bar{Z}_j Z_j$$

The Euler-Lagrange equation associated to the extremal problem for $(*)$ is

$$(**) \qquad \mathcal{L}\, u = \lambda u^{p-1} \quad \text{on } \mathbb{H}^n \, ; \ \lambda \ \text{constant.}$$

The analogue of stereographic projection is as follows. The Heisenberg group is identified with the boundary of the Siegel upper half space

$$\mathcal{D} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} w > |z|^2\} \quad \text{by the mapping}$$

$(z,t) \longmapsto (z,w)$, $w = t + i|z|^2$. Moreover, the domain $\mathcal{D}$ is biholomorphic to the unit ball in $\mathbb{C}^{n+1}$ by the Cayley transform :

$$w = i\left(\frac{1-\zeta_{n+1}}{1+\zeta_{n+1}}\right) \; ; \; z_k = \zeta_k/(1+\zeta_{n+1}) \quad k = 1,\dots,n \, .$$

where $\zeta \in \mathbb{C}^{n+1}$, $|\zeta| \leq 1$. When restricted to the boundary this transformation gives what we will call a CR equivalence between S^{2n+1} and $\mathbb{H}^n$.

A CR structure on a manifold N of dimension $2n+1$ is given by a complex n-dimensional sub-bundle $T_{1,0}$ of the complexified tangent bundle $\mathbb{C}TN$ of N, satisfying $T_{1,0} \cap \bar{T}_{1,0} = \{0\}$. We will assume that the CR structure is integrable, i.e. the Lie bracket $[T_{1,0}, T_{1,0}] \subset T_{1,0}$, and that there is a global non-vanishing real 1-form θ annihilating

$T_{1,0}$. The Levi form is a hermitian form on $T_{1,0}$ given by $<z,w>_\theta = -\, id\theta(Z,\overline{W})$ for $Z, W \in T_{1,0}$. We will also assume strict pseudoconvexity, i.e., that the Levi form is positive definite. A hypersurface in $\mathbb{C}^{n+1}$ of real codimension 1 has a natural integrable CR structure given by the holomorphic vectors of $\mathbb{C}^{n+1}$ tangent to the hypersurface. In particular $S^{2n+1} \subset \mathbb{C}^{n+1}$ has a natural CR structure. Local biholomorphisms of $\mathbb{C}^{n+1}$ preserve the CR structure.

A pseudohermitian structure on a CR manifold N is a choice of the 1-form θ. With this choice, N has a natural volume form $\theta \wedge d\theta^n$ (non zero because N is strictly pseudoconvex). Dual to the Levi form is an inner product on real 1-forms ω, given locally by

$$< \omega , \omega >_\theta = \sum_{j=1}^{n} |\omega(W_j)|^2$$

in which $W_1, \dots, W_n$ form an orthonormal basis for $T_{1,0}$ with respect to the Levi form. The sublaplacian Δ_b is defined on functions by

$$\int_N (\Delta_b u) v \;\theta\wedge d\theta^n = \int_N < du, dv >_\theta \;\theta \wedge d\theta^n \quad , \text{ for all } v \in C_o^\infty(N).$$

The operator Δ_b is subelliptic. In fact, it is the real part of the Kohn-Spencer Laplacian $\Box_b$ on functions (cf.[7]).

S. Webster has defined a scalar curvature R associated with a pseudohermitian structure [14]. Under a change of pseudohermitian structure $\tilde{\theta} = u^{p-2}\,\theta$ with $p = 2 + 2/n$, R transforms as calculated in [7]

(4) $$(b_n \Delta_b + R)u = \tilde{R}\, u^{p-1} \quad , \; b_n = 2(n+1)/n$$

A key point here is that the value of p is the same as the only possible exponent p in (*). We will comment on this at length in the proof of Theorem 1(a) which contains (4) as a special case.
Hence the problem of finding a pseudohermitian structure with constant scalar curvature on a CR manifold N is equivalent to solving the equation

(5) $$b_n\, \Delta_b\, u + Ru = \lambda\, u^{p-1}$$

for $u \in C^\infty(N)$, $u > 0$ and λ constant. Equation (5) is the Euler-Lagrange equation for the variational problem

$$(6)\qquad \lambda(N,\theta) = \inf \{\int_N (b_n \langle du,du\rangle_\theta + Ru^2)\theta\wedge d\theta^n \;:\; \int_N |u|^p\, \theta\wedge d\theta^n = 1\}$$

In the case of $\mathbb{H}^n$ we put $T_{1,0} = \operatorname{span}\{Z_1,\dots,Z_n\}$.

The real 1-form $\theta_o = dt + \sum_{j=1}^n iz_j\, d\bar{z}_j - i\bar{z}_j\, dz_j$ is orthogonal to $T_{1,0}$ and the vectors $Z_1,\dots,Z_n$ form on orthonormal basis with respect to the Levi form for θ_o . The operator Δ_b associated to θ_o is $\mathcal{L}$ and the scalar curvature for $\mathbb{H}^n$ with pseudohermitian structure θ_o is zero. Finally $\theta_o \wedge d\theta_o^n$ is a multiple of the standard volume form. Thus the problem of finding the best constant in (*) is equivalent to the variational problem

$$(*')\ \lambda(\mathbb{H}^n,\theta_o) = \inf\{b_n\int_{\mathbb{H}^n} \sum_{j=1}^n |Z_j u|^2\, \theta_o\wedge d\theta_o^n \;:\; \int_{\mathbb{H}^n} |u|^p \theta_o\wedge d\theta_o^n = 1,\ u \text{ real-valued}\}$$

Our main theorem is

<u>Theorem</u> 1 <u>Let</u> N <u>be a compact</u>, <u>strictly pseudoconvex integrable</u> CR <u>manifold of dimension</u> 2n+1 .

(a) $\lambda(N,\theta)$ <u>depends only on the</u> CR <u>structure</u> , <u>not the choice of</u> 1-<u>form</u> θ .

(b) $\lambda(N,\theta) \le \lambda(S^{2n+1}$, standard structure$) = \lambda(\mathbb{H}^n,\theta_o) \equiv \lambda_o$.

(c) <u>If</u> $\lambda(N,\theta) < \lambda_o$, <u>then the infimum in</u> (6) <u>is attained by a</u> C^∞ , <u>positive solution to</u> (5). <u>Thus the pseudohermitian structure</u> $\tilde{\theta} = u^{p-2}\theta$ <u>has constant scalar curvature</u> $\tilde{R} = \lambda(N,\theta)$.

S.S. Chern and R. Hamilton, while studying contact structure on CR manifolds, have independently obtained a result which is equivalent to part (c) in the case n = 1 and $\lambda(N,\theta) \le 0$.

Notice that we do not have an analogue to part (d) of Aubin's theorem. The reason is that we do not yet have the complete solution to the problem of identifying extremals in problem (*').
Here is a partial result.

<u>Theorem</u> 2 <u>The extremals to problem</u> (*') <u>exist</u>, <u>and are positive</u> C^∞ <u>solutions to</u> (**). <u>Moreover</u>, <u>if we assume in addition that the solution depends</u>

<u>only on</u> $|z|$ <u>and</u> t , <u>then it must take the form</u> $c|t+i|z|^2+\alpha|^{-n}$ <u>for some</u> $c > 0$ <u>and some</u> $\alpha \in \mathbb{C}$ <u>such that</u> $\operatorname{Im}\alpha > 0$.

We conjecture that these are all the solutions up to left translation on the Heisenberg group.

<u>Main ideas of the proofs</u>

Let $t \in C^\infty(M)$, $t > 0$. Under the conformal change $\tilde{g} = t^{q-2} g$, we have

$$(7) \qquad (a_m \tilde{\Delta} + \tilde{k})\tilde{\varphi} = t^{1-q}(a_m\Delta + K)\varphi \quad \text{for } \tilde{\varphi} = t^{-1}\varphi \ .$$

It follows immediately that $\mu(M,g) = \mu(M,\tilde{g})$ in other words part (a) of Aubin's theorem is proved. Likewise on a CR manifold N the transformation law

$$(8) \qquad (b_n\tilde{\Delta}_b + \tilde{R})\tilde{u} = t^{1-p}(b_n\Delta_b + R)u, \ \underline{\text{with}} \ \tilde{\theta} = t^{p-2}\,\theta \ \underline{\text{and}} \ \tilde{u} = t^{-1}\,u \ .$$

Shows that $\lambda(N,\theta) = \lambda(N,\tilde{\theta})$. This formula follows from transformation laws for R and Δ_b computed by J.M. Lee in [7]. Indeed, the case $u = t$ is (4) .

We first recognized that there must be a transformation law like (7) by approaching it from another point of view. This point of view not only sheds light on the significance of the invariance, but also allows us to see in advance that the critical exponent p of the Sobolev inequality (*) will match the exponent of the transformation law (8). In [4], C. Fefferman constructed a circle bundle C over N and a Lorentz metric g on C such that a change in pseudohermitian structure on N corresponds to a conformal change in g . It turns out that the projection of C onto N carries the (Laplace-Beltrami) wave operator to Δ_b and the scalar curvature of g to a multiple of R (See [7] for proofs). Thus the transformation law (8) follows from the corresponding law (7) on C , once we realize that with $m = \dim C = 2n+2$. The critical exponents p and q are equal.

The remainder of the proof of theorem 1 depends fundamentally on the work of Folland and Stein [5].

Recall that $Z_1,\dots,Z_n$ are homogeneous of degree -1 and that θ_o is homogeneous of degree 2 with respect to the dilations of $\mathbb{H}^n$. Folland and Stein normal coordinates can be described as follows :

<u>Lemma</u> 1 <u>Let</u> (N,θ) <u>be a pseudohermitian manifold</u>. <u>Every point of</u> N <u>has an open neighborhood</u> Ω <u>with an orthonormal frame</u> $W_1,\dots,W_n$ <u>for</u> $T_{1,0}N$ <u>defined on</u> Ω <u>and</u> a C^∞ <u>mapping</u> $\Phi_x : \Omega \longrightarrow \mathbb{H}^n$ $(x \in \Omega)$ <u>such that</u> $\Phi_x(x)=(0,0)$ <u>and</u>

$$(\Phi_x)_* W_j = Z_j + (\text{terms of degree} \geq 0).$$

$$(\Phi_x)_* \theta = \theta_o + (\text{terms of degree} \geq 3)$$

We can now prove Theorem 1 b). Denote

$$\mathcal{A}_\theta(v) = \int_N (b_n < dv,dv >_\theta + Rv^2)\theta \wedge d\theta^n \quad \text{and } \mathcal{B}_\theta(v) = \int_N |v|^q\, \theta \wedge d\theta^n .$$

One first shows that for any $\varepsilon > 0$ there is a compactly supported function u (in fact $u \in C_o^\infty(\mathbb{H}^n)$) such that $\mathcal{B}_{\theta_o}(u) = 1$ and $\mathcal{A}_{\theta_o}(u) < \lambda_o + \varepsilon$. The function $u_\delta(x) = \delta^{-n} u(\delta^{-1}x)$ satisfies $\mathcal{B}_{\theta_o}(u_\delta) = \mathcal{B}_{\theta_o}(u) = 1$ and $\mathcal{A}_{\theta_o}(u_\delta) = \mathcal{A}_\theta(u) < \lambda_o + \varepsilon$. Moreover the support of u_δ tends to zero as $\delta \longrightarrow 0$. Lemma 1 shows that $V_\delta(y) = u_\delta(\Phi_x(y))$ satisfies $\mathcal{A}_\theta(v_\delta) \longrightarrow \mathcal{A}_\theta(u)$ and $\mathcal{B}_\theta(v_\delta) \longrightarrow 1$ as $\delta \longrightarrow 0$, which proves Theorem 1 b). A refinement of this argument with an explicit formula for the extremal on $\mathbb{H}^n$ should yield an analogue to part d) of Aubin's theorem.

In order to prove Theorem 1 c) we will need to state several regularity theorems of Folland and Stein.

We will keep the notations of Lemma 1 and let $X_j = \operatorname{Re} W_j$, $X_{j+n} = \operatorname{Im} W_j$ $j = 1,\dots,n$. Denote $X^\alpha = X_{\alpha_1}\dots X_{\alpha_k}$ where $\alpha = (\alpha_1,\dots,\alpha_k)$, $1 \leq \alpha_j \leq 2n$, $\ell(\alpha) = k$. The Folland-Stein Sobolev spaces are defined by

$$S_k^p(\Omega) = \{f \in L^1_{loc}(\Omega) : X^\alpha f \in L^p(\Omega)\}$$

A natural distance function is defined by

$\rho(x,y) = |\Phi_x(y)|$, where $|(z,t)| = (|z|^4 + t^2)^{1/4}$. For any non-integer $\beta > 0$ we let k be the integer such that $k < \beta < k+1$ and denote Folland-

Stein Lipschitz classes by

$$\Gamma_\beta(\Omega) = \{f : \sup_{x,y\in\Omega} \max_{\ell(\alpha)\le k} \frac{|X^\alpha f(x) - X^\alpha f(y)|}{\rho(x,y)^{\beta-k}} < \infty \} .$$

Finally, ordinary Lipschitz classes (for $\Omega \subset \mathbb{R}^{2n+1}$) are defined by

$$\Lambda_\beta(\Omega) = \{f : \sup_{x,y\in\Omega} \sup_{|\alpha|\le k} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{\|x-y\|^{\beta-k}} < \infty \} .$$

With the help of a partition of unity we can, of course, define $S^p(N)$, $\Gamma_\beta(N)$, $\Lambda_\beta(N)$.

__Lemma__ 2 [5] __For__ $p > 0$ __non-integer__, k __an integer__ ≥ 1 , __and__ r , $1 < r < \infty$,

$S^r_k \subset \Gamma_\beta$ where $\frac{1}{r} = \frac{k-\beta}{2n+2}$

$\Gamma_\beta \subset \Lambda_{\beta/2}$

__If__ $\Delta_b u \in L^r$ __then__ $u \in S^r_2$ __and if__ $\Delta_b u \in \Gamma_\beta$ __then__ $u \in \Gamma_{\beta+2}$.

(In order to compare Γ_β and $\Lambda_{\beta/2}$ we have identified Ω with its image under Φ_x for some fixed $\chi \in \Omega$.)

__Lemma__ 3 __Suppose that__ $f \in L^s(\Omega)$ __for some__ $s > n+1$ __that__ $u \in S^2_1(\Omega)$, $u \ge 0$, __and__ $\Delta_b u + fu = 0$ __in the weak sense on__ Ω , __then__ u __is bounded above and below by positive constants on any compact subset__ of Ω .

(This Harnack inequality is analogous to the one used by Trudinger [12]. It is proved, as is Trudinger's , by Moser's technique.)

The basic approach to a variational problem is to take a minimizing sequence and to find a convergent subsequence. The problem is that the exponent p is exactly the exponent for which the inclusion $S^2_1(N) \subset L^p(N)$, while true, is not compact. Aubin overcame this difficulty in the Riemannian case with the help of the observation that the best constant in the Sobolev inequality is the same for all compact m-manifolds in the following sense : if $\mu_o = \mu(S^m$, standend metric) then for any M and any $\varepsilon > 0$

$$(\mu-\varepsilon)\left(\int_M |f|^q dV_g\right)^{2/q} < a_m \int_M \langle df,df\rangle_g dV_g + C_{M,\varepsilon} \int_M |f|^2 dV_g .$$

Subsequently, Aubin's approach has been considerably simplified and extended to a large number of elliptic problems. (See Brezis-Nirenberg [2], P.L. Lions [8] and the surveys [3,9] where many further references can be found.)

This leads to the natural conjecture :

Conjecture. *If $\lambda_o = \lambda(\mathbb{H}^n, \theta_o)$, then for any compact $(2n+1)$-dimensional pseudohermitian manifold (N,θ) and any $\varepsilon > 0$ there exists $C = C(N,\varepsilon)$ such that.*

$$(\lambda_o - \varepsilon)\left(\int_N |f|^p \theta\wedge d\theta^n\right)^{2/p} \leq \int_N (b_n\langle df, df\rangle_\theta + C|f|^2)\theta\wedge d\theta^n \quad \textit{for all } f \in S_1^2(N) .$$

If true, this conjecture would lead to a much simpler proof of Theorem 1 c). A partition of unity reduces the conjecture to a local question. Unfortunately, Folland-Stein normal coordinates are not analogous to Riemannian normal coordinates in one important respect : the norm $\int \langle df, df\rangle_\theta$ inherited from N is not comparable to the norm $\int \langle df, df\rangle_{\theta_o}$ of $\mathbb{H}^n$ even in very small neighborhoods of the origin.

Nevertheless, Theorem 1 c) can be proved by a method analogous to a proof of Aubin's theorem due to K. Uhlenbeck [13]. The point is that the regularity estimates in Lemmas 2 and 3 are uniform in a family of (non-comparable) CR structures uniform not only in the base point x of in the coordinate charts Φ_x but also under dilation.

We now sketch the proof. Consider the variational problems

$$\lambda_r = \inf\left\{\int_N (b_n\langle du, du\rangle_\theta + Ru^2)\theta\wedge d\theta^n : \int_N |u|^r \theta\wedge d\theta^n = 1\right\} .$$

For $r < p$ one can show a positive, C^∞ extremal u_r exists and satisfies $b_n\Delta_b u_r + R u_r = \lambda_r u^{r-1}$. It is not hard to show that $\lambda_r \longrightarrow \lambda_p = \lambda(N,\theta)$ as $r \longrightarrow p$.

Case 1. $\langle du_{r_j}, du_{r_j}\rangle_\theta$ is bounded on N uniformly as $r_j \longrightarrow p$. In this case one can choose a subsequence convergent in $S_1^2(N)$ (and $L^s(N)$ for any $s < \infty$) norm. Thus the limit function satisfies the equation (5) and the constraint $\mathcal{B}_\theta(u) = 1$. By repeated application of Lemmas 2 and 3, we find that $u \in C^\infty(N)$, $u > 0$.

<u>Case 2</u>. $\max_N < du_r , du_r >_\theta \longrightarrow \infty$ as $r \longrightarrow p$.

We will show that this contradicts the strict inequality $\lambda_p = \lambda(N,\theta) < \lambda(\mathbb{H}^n , \theta_o)$.

Choose $x_v \in N$ such that

$\max_N < du_r , du_r >_\theta = < du_r(x_r), du_r(x_r) >_\theta$. We will consider u_r in local coordinates given by Φ_{x_r} . Thus u_r is identified with a function in a neighborhood of the origin in $\mathbb{H}^n$ and x_r is identified with the origin. Define $h_r(z,t) = \delta^{2/(r-2)} u_r(\delta z, \delta^2 t)$ and choose $f = f(r)$ such that $< dh_r(0), dh_r(0) >_{\theta_o} = 1$. Note that $\delta \longrightarrow 0$ as $r \longrightarrow p$. Moreover this change of variable yields an equation for h_r of the form

$$\mathcal{L}_r h_r + R_r \delta^2 h_r = \lambda_r h_r^{r-1}$$

where $\mathcal{L}_r$ tends toward $\mathcal{L}$ (the sublaplacian of $\mathbb{H}^n$) as $r \longrightarrow p$ and R_r is the scalar curvature of (N,θ) after suitable change of variable depending on x and δ . The crucial point is that the norm inequalities associated to Lemmas 2 and 3 for the CR structure associated $\mathcal{L}_r$ are uniform as $r \longrightarrow p$. Because $\delta \longrightarrow 0$, the domain of definition of h_r tends to the entire space $\mathbb{H}^n$ as $r \longrightarrow p$. Moreover, $< d\, h_r(x), d\, h_r(x) >_\theta < 1$ since the maximum value at the origin is 1 . It follows from the uniform versions of Lemmas 2 and 3 that a subsequence of the sequence h_r converges uniformly in, say, $\Lambda_{1+\varepsilon}$ on compact subsets of $\mathbb{H}^n$ to a limit h satisfying $\mathcal{L}h = \lambda p\, h^{p-1}$ on all of $\mathbb{H}^n$. One can check that $\mathcal{A}_{\theta_o}(h) < \infty$ and $\mathcal{B}_{\theta_o}(h) \leq 1$. It follows from integration by parts that h gives the bound $\lambda(\mathbb{H}^n , \theta_o) \leq \lambda_p$, a contradiction.

We conclude by describing the idea of the proof of Theorem 2 . (For Riemannian case, see Obata [10] and Gidas, Ni, Nirenberg [6].) Given a positive C^∞ solution to $\mathcal{L}u = \lambda u^{p-1}$ depending only on $|z|$ and t we define a collection of functions h_j of $|z|$, t and u and its first and second derivatives and prove a formula of form

$\sum h_j^2 = \operatorname{div}(\text{something})$.

We conclude by integration by parts that each $h_j = 0$. These additional equations are enough to specify u .

References :

[1] T. Aubin, "Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire", J. Math. pures et appl.,55 (1976), 269-296.

[2] H. Brezis, L. Nirenberg, "Positive solutions of non linear elliptic equations involving critical Sobolev exponents. Comm. P.A.M. (1983) 437-477.

[3] H. Brezis, Proceedings, AMS Summer school in non-linear functional analysis (Berkeley 1983).

[4] C. Fefferman, "Monge-Ampère equations, the Bergman Kernel, and geometry of pseudoconvex domains", Annals of Math., 103 (1976), 395-416 ; correction, Annals of Math., 104 (1976), 393-394.

[5] G.B. Folland and E.M. Stein, "Estimates for the $\bar{\partial}_b$-complex and analysis on the Heisenberg group", Comm. Pure and Appl. Math. 27 (1974), 429-522.

[6] B. Gidas, W.M. Ni, L. Nirenberg, "Symmetry and related properties via the maximum principle" Comm. Math. Phys.68 (1979) 204-243.

[7] J.M. Lee, "The Fefferman metric and pseudohermitian invariants", in preparation.

[8] P.L. Lions, "Applications de la méthode de concentration-compacité à l'existence de fonctions extrêmals" C.R. Acad. Sci Paris 296 (1983) 645-648.

[9] P.L. Lions, "The concentration-compactness principle in the calculus of variations. The limit case Parts I and II" preprint.

[10] M. Obata, "The conjectures on conformal transformations of Riemannian manifolds", J. Diff. Geom.,6 (1971), 247-258.

[11] G. Talenti, "Best constant in Sobolev inequality", Ann. Mat. Pura Appl. 110 (1976), 353-372.

[12] N.S. Trudinger "Remarks concerning the conformal deformation of Riemannian structures on compact manifolds". Ann. Sc. Norm. Sup Pisa, 22 (1968) 265-274.

[13] K. Uhlenbeck, lectures given at Harvard University, 1983.

[14] S.M. Webster, "Pseudohermitian structures on a real hypersurface", J. Diff. Geom.,13 (1978), 25-41.