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Séminaire Équations aux dérivées partielles (Polytechnique) (1983-1984), exp. n° 13, p. 1-12

<http://www.numdam.org/item?id=SEDP_1983-1984____A13_0>

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SEMINAIRE GOULAOUIC-MEYER-SCHWARTZ 1983-1984

ASYMPTOTIC BEHAVIOR OF THE SPECTRUM OF SINGULAR GREEN OPERATORS

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Exposé n° XIII

20 mars 1984

1. INTRODUCTION.

Singular Green operators are, roughly speaking, the messy terms that come up when one considers boundary value problems for elliptic differential and pseudo-differential operators on a C^∞ manifold $\overline{\Omega}$ (of dimension n) with boundary $\partial\Omega$; here the parametrices and compositions contain pseudo-differential terms, but also some other terms that are not pseudo-differential on Ω. Their systematic introduction is due to L. Boutet de Monvel [3]. Let us first give some examples.

Example 1. The operator

$$G_o = A_{Neu}^{-1} - A_{Dir}^{-1} \quad , \tag{1}$$

where A_{Neu} and A_{Dir} are the Neumann, resp. Dirichlet, realization of a strongly elliptic differential operator on $\overline{\Omega}$, is a simple kind of a singular Green operator (s.g.o.).

Example 2. Let P be a pseudo-differential operator (ps.d.o.) on an n-dimensional C^∞ manifold Σ in which $\overline{\Omega}$ is smoothly imbedded, and denote by P_Ω the operator on Ω induced from P by the formula

$$P_\Omega u = r^+ P e^+ u \quad ,$$

where r^+ and e^+ are the restriction, resp. "extension by zero" operators : $r^+ u = u_{|\Omega}$ for $u \in \mathcal{D}'(\Sigma)$; and $e^+ v$ equals v on Ω, zero on $\Sigma \setminus \Omega$, when v is a function on Ω. When P and Q are ps.d.o.s on Σ having the transmission property (Boutet [3]) at $\partial\Omega$, the "leftover" operator

$$L(P,Q) = (PQ)_\Omega - P_\Omega Q_\Omega$$

is a singular Green operator. One can in fact break it up in simpler parts. When Q is of order $\leqslant 0$, and the situation is localized to the case where $\Omega = \mathbb{R}^n_+$ with boundary $\partial\Omega = \{x \in \mathbb{R}^n \mid x_n = 0\}$, then

$$L(P,Q) = G^+(P) G^-(Q) \quad ,$$

where

$$G^+(P) = r^+Pe^-J$$

(2)

$$G^-(Q) = Jr^-Qe^+ = [G^+(Q^*)]^* \quad ;$$

here J stands for the reflection operator $J : u(x',x_n) \mapsto u(x',-x_n)$, and e^- and r^- are the extension by zero operator and restriction operator for $\mathbb{R}^n_- \subset \mathbb{R}^n$. Also $G^+(P)$ and $G^-(Q)$ are s.g.o.s (Grubb [10], [7]). When Q is of order >0, one must add some terms containing traces at $x_n = 0$.

Example 3 : Let T be a trace operator in the sense of [3] ; this class contains the operators of the form $\gamma_0 P$, where P is a ps.d.o. on Σ having the transmission property at $\partial\Omega$, and $\gamma_j u = (\frac{\partial}{\partial n})^j u_{|\partial\Omega}$. Let K be a Poisson operator [3] (this class consists of operators mapping functions on $\partial\Omega$ into functions on Ω, and they can be described as the adjoints of those trace operators that are defined on all of $L^2(\Omega)$). Then KT is a s.g.o., and of course also sums $\sum_{j=1}^{N} K_jT_j$ of such composites are s.g.o.s (example 1 is in fact of this kind).

Example 4 : When G is a s.g.o. and P is a ps.d.o. as above, then GP_Ω and $P_\Omega G$ are s.g.o.s When G' is another s.g.o., GG' is one.

Example 5 : Let Ω_- be a smooth bounded open subset of $\mathbb{R}^n$, let A be a uniformly elliptic invertible operator on $\mathbb{R}^n$, and let A_+ and A_- be invertible realizations on the exterior domain $\Omega_+ = \mathbb{R}^n \setminus \overline{\Omega}_-$, resp. the interior domain Ω_-, defined e.g. by Dirichlet or Neumann boundary conditions. Then the operator

(3)
$$G_1 = A^{-1} - A_+^{-1} \oplus A_-^{-1}$$

on $L^2(\mathbb{R}^n) = L^2(\Omega_+) \oplus L^2(\Omega_-)$ is a kind of s.g.o. consisting of terms of the form KT on Ω_+ and Ω_-, and terms $e^+r^+A^{-1}e^-r^-$ and $e^-r^-A^{-1}e^+r^+$ that resemble $G^+(A^{-1})$ in (2) relative to Ω_+ or Ω_-), apart from the reflection J.

The concept of singular Green operators has been generalized to other situations (e.g. parameter-dependent cases, with applications ot functional calculus and evolution equations, [7], [8]). Instead of going into that, we shall here concentrate on a new result for the original s.g.o.s (complete details in [10], announced in [9]) :

Theorem 1 : Let $\overline{\Omega}$ be a compact n-dimensional C^∞ manifold with boundary $\partial\Omega$, and let G be a polyhomogeneous singular Green operator on $\overline{\Omega}$ of order $-d<0$, continuous from $L^2(\Omega)$ to $H^d(\Omega)$. Then the characteristic values $s_k(G)=\lambda_k(G^*G)^{1/2}$ satisfy

$$s_k(G)k^{d/(n-1)} \longrightarrow c(g^o) \quad \text{for } k\to\infty \qquad , \tag{4}$$

where $c(g^o)$ is a constant derived from the principal symbol of G. In the self-adjoint case, the positive, resp. negative, eigenvalue sequences satisfy similar estimates

$$\lambda_k^{\pm}(G)k^{d/(n-1)} \longrightarrow c^{\pm}(g^o) \quad \text{for } k\to\infty \qquad , \tag{5}$$

The constants $c(g^o)$ and $c^{\pm}(g^o)$ will be explained further below. We also write (4) as $s_k(G)\sim c(g^o)k^{-d/(n-1)}$, etc.

The existence of an upper bound,

$$s_k(G)k^{d/(n-1)} \leqslant C \quad \text{for all } k \quad , \tag{6}$$

has been known for some time (cf. Grubb [6]), and (4)-(5) have been shown in various special cases before. For G_o as in (1), Birman showed (6) already in [1] for the second order case ; and a precise asymptotic estimate (with remainders, sharper than (4)-(5)) was obtained in Grubb [5], where it was shown that G_o is isometric with an elliptic ps.d.o. over $\partial\Omega$. Hmelnickii [11] showed (4)-(5) for some more general operators occuring in connection with elliptic differential boundary problems. Laptev [12] showed (4)-(5) for $G^+(P)$ as in (2), when P is a ps.d.o. of order -d, not necessarily having the transmission property.

An interesting feature of the estimates (4)-(5) is that they involve the boundary dimension n-1 instead of the interior dimension n (the continuity of G from $L^2(\Omega)$ to $H^d(\Omega)$ merely implies (6) with n instead of n-1). A nice consequence of the generality of Theorem 1 is that when G is a s.g.o. of order -d, covered by the earlier results, and P is any ps.d.o. of order -d' (having the transmission property), then

$$s_k(GP_\Omega) \sim c(g^o,p^o)k^{-(d+d')/(n-1)} \quad ;$$

in view of Example 4 ; it is not merely $O(k^{-d/(n-1)-d'/n})$, as one would get from product rules for s_k-numbers. -Note that G in Theorem 1 is not assumed to be "elliptic" in any sense.

2. DETAILS ON THE STRUCTURE OF s.g.o.s.

The proof of Theorem 1 builds on a very precise knowledge of the structure of s.g.o.s.

We only consider here s.g.o.s of class 0, they act on all of $L^2(\Omega)$ (whereas the more general s.g.o.s of class $r\geqslant 0$ contain additional terms $\sum_{0\leqslant j\leqslant r-1} K_j\gamma_j$ where the K_j are Poisson operators). For the situation where $\Omega = \mathbb{R}^n_+$, a singular Green operator of order d and class 0 is an operator of the form

$$(7)\qquad (Gu)(x',x_n) = (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}} e^{ix'.\xi'}\int_0^\infty \tilde{g}(x',\xi',x_n,y_n)u(\hat{\xi}',y_n)\,dy_n\,d\xi' \quad ,$$

where the symbol-kernel $\tilde{g}(x',\xi',x_n,y_n)$ lies in $\mathcal{S}(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+)$ (the restriction of $\mathcal{S}(\mathbb{R}^2)$ to $\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+$) as a function of (x_n,y_n) for each (x',ξ'), and is of type $S_{1,0}$, or polyhomogneneous, in the sense defined below. Here $c(x')$ stands for a continuous function depending on the indices, and $\langle\xi'\rangle = (1+|\xi'|^2)^{1/2}$.

<u>Definition 1</u> : $\tilde{g}$ <u>is of type</u> $S_{1,0}$ <u>and order</u> d, <u>when</u> g <u>is</u> C^∞ <u>in</u> $(x',\xi',x_n,y_n)\in\mathbb{R}^{2n-2}\times\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+$ <u>and there are estimates</u>

$$(8)\qquad \| x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_{x'}^\beta D_{\xi'}^\alpha \tilde{g}(x',\xi',x_n,y_n)\|_{L^2_{x_n,y_n}(\mathbb{R}_+\times\mathbb{R}_+)} \leqslant c(x')\langle\xi'\rangle^{d-k+k'-m+m'-|\alpha|} \quad ,$$

<u>for any set of indices</u> k, k', m, m', α, β.

The topology on the (Fréchet) space of such symbols is defined e.g. by the system of semi-norms

$$(9)\qquad |||\tilde{g}|||_{k,k',m,m',\alpha,\beta,K} = \sup_{x'\in K} c(x') \quad ,$$

where $c(x')$ is taken as the smallest value enterring in (8), and K runs through the compact subsets of $\mathbb{R}^{n-1}$.

<u>Definition 2</u> : $\tilde{g}$ <u>is polyhomogeneous of order</u> d, <u>when there is an asymptotic expansion</u>

$$\tilde{g}\sim g^o + \tilde{g}^{-1} + \ldots + \tilde{g}^{-j} + \ldots$$

in quasihomogeneous functions :

$$\tilde{g}^{-j}(x',t\xi',\tfrac{1}{t}x_n,\tfrac{1}{t}y_n) = t^{d-j+1}\,\tilde{g}^{-j}(x',\xi',x_n,y_n) \tag{10}$$

for $|\xi'|$ 1 and $t\geq 1$,

with $\tilde{g}-\sum_{j<M}\tilde{g}^{-j}$ *being of type* $S_{1,0}$ *and order* $d-M$ *for all* M.

For some purposes it suffices to consider symbol-kernels of type $S_{1,0}$, but Theorem 1 requires polyhomogeneity (or at least the existence of a homogeneous principal part).

In [3], the operators G are defined by their *symbols*, here the symbol $g(x',\xi',\xi_n,\eta_n)$ associated with the symbol-kernel $\tilde{g}(x',\xi',x_n,y_n)$ is the sesqui-Fourier transform

$$g(x',\xi',\xi_n,\eta_n) = \mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}\tilde{g}(x',\xi',x_n,y_n) \quad .$$

Then (8) looks more complicated, because $\mathcal{S}(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+)$ (extended by 0 on $\mathbb{R}^2\setminus(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+)$) is Fourier transformed to a fancier space $\mathcal{H}^+_{\xi_n}\hat{\otimes}\mathcal{H}^-_{\eta_n}$ of C^∞ functions with certain asymptotic properties, and the restriction operators from $\mathbb{R}^2$ to $\mathbb{R}_+\times\mathbb{R}_+$ carry into fancier projections. On the other hand, (10) carries into a simple homogeneity :

$$g^{-j}(x',t\xi',t\xi_n,t\eta_n)=t^{d-1-j}g^{-j}(x',\xi',\xi_n,\eta_n)$$

for $|\xi'|\geq 1$, $t\geq 1$.

In the polyhomogeneous case, G has a *principal symbol* g^0, resp. a *principal symbol-kernel* $\tilde{g}^0$.

We also define the *boundary symbol operator* $g(x',\xi',D_n)$ [resp. *principal boundary symbol operator* $g^0(x',\xi',D_n)$] by

$$g(x',\xi',D_n)u(x_n) = \int_0^\infty \tilde{g}(x',\xi',x_n,y_n)u(y_n)\,dy_n \quad ,$$

so that G can be regarded as a ps.d.o. in x', valued in the space of boundary symbol operators. Since the kernel is in $\mathcal{S}(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+)$, the operator $g(x',\xi',D_n)$ is compact on $L^2(\mathbb{R}_+)$ and belongs to any Schatten class C_p for $p>0$ (i.e. the sequence of characteristic values is in ℓ^p for any $p>0$).

We can now define the constant $c(g^o)$ in Theorem 1. It equals

$$c(g^o) = C(g^o)^{d/(n-1)} \quad , \quad \text{where}$$

(11)
$$C(g^o) = \frac{(2\pi)^{1-n}}{n-1} \int_{S^*(\partial\Omega)} \mathrm{tr}[(g^o(x',\xi',D_n)^* g^o(x',\xi',D_n))^{(n-1)/2d}] \, d\omega$$

or, equivalently (in view of the homogeneities),

(12)
$$C(g^o) = (2\pi)^{1-n} \int_{T^*(\partial\Omega)} N[1, g^o(x',\xi',D_n)^* g^o(x',\xi',D_n)] \, dx' \, d\xi' \quad ,$$

where $N[1,B]$ stands for the number of eigenvalues $\geqslant 1$ for B. When g^o is self-adjoint, one has

$$c^{\pm}(g^o) = \pm C(g^o)^{d/(n-1)} \quad , \quad \text{where}$$

(13)
$$C^{\pm}(g^o) = \frac{(2\pi)^{1-n}}{n-1} \int_{S^*(\Omega)} \Sigma(\pm\lambda_j^{\pm}(g^o(x',\xi',D_n))^{(n-1)/d} \, d\omega \quad ,$$

and there is a formula analogous to (12).

3. INDICATIONS OF THE PROOF.

We now list some of the ingredients in the proof of Theorem 1. For one thing, perturbation methods for eigenvalues estimates are used to a great extent, and the following plays a central rôle :

Lemma 1 : *Let B be a compact operator in a Hilbert space, let $\alpha > 0$ and $c_o \geqslant 0$, and assume that for any integer $M \geqslant 0$ there is a decomposition*

$$B = B_M + B'_M \quad ,$$

where B_M and B'_M satisfy, respectively,

(14)
$$s_k(B_M)k^{\alpha} \longrightarrow c_M \quad \text{for } k \to \infty \quad ,$$

(15)
$$s_k(B'_M)k^{\alpha} \leqslant \varepsilon_M \quad \text{for all } k \quad ,$$

where $c_M \to c_o$ and $\varepsilon_M \to 0$ for $M \to \infty$.

Then

(16) $$s_k(B)k^{\alpha} \longrightarrow c_o \quad \text{for } k\to\infty \quad .$$

Similar results hold for $\lambda_k^{\pm}(B)$ in the selfadjoint case.

It is a variant of the Weyl-Ky Fan theorem (cf. e.g. Gohberg-Krein [4, Chapter 2]), derived from the minimum-maximum principle.

Secondly, we need the following result for ps.d.o.s.

Lemma 2 : Let P be a polyhomogeneous $N\times N$-matrix formed ps.d.o. of order $-d'<0$ on a compact n'-dimensional C^{∞} manifold Ξ without boundary. Then P is a compact operator whose characteristic values satisfy

(17) $$s_k(P) \sim c(p^o)k^{-d'/n'} \qquad \text{for } k\to\infty \qquad ;$$

here $c(p^o) = C(p^o)^{d'/n'}$, where

(18) $$C(p^o) = \frac{(2\pi)^{-n'}}{n'} \int_{S(\Omega)} \operatorname{tr}[(p^o(x,\xi)^* \, p^o(x,\xi))^{n'/2d'}] \, d\omega \quad .$$

In the selfadjoint case, there are similar statements for the sequences $\lambda_k^{\pm}(P)$.

Note that P is not assumed elliptic here ; in the elliptic case there are more precise estimates by Hörmander and Ivrii. The result of Lemma 2 is shown, along with generalizations to anisotropic symbols, by Birman and Solomiak in [2]. (We observe that the result for $s_k(P)$ can be obtained easily from Seeley's principal estimate [13] for the elliptic case : write $P^*P = P^*P + \frac{1}{M}\Lambda - \frac{1}{M}\Lambda$, where Λ is a selfadjoint positive ps.d.o. on Ξ with symbol $\langle\xi\rangle^{-2d'}$; then $P^*P + \frac{1}{M}\Lambda$ is elliptic so that

$$s_k(P^*P + \frac{1}{M}\Lambda) \sim c(p^{o*}p^o + \frac{1}{M}|\xi|^{-2d'}I)k^{-2d'/n'} \quad ,$$

whereas $\sup_k s_k(\frac{1}{M}\Lambda)k^{2d'/n'}$ is $\mathcal{O}(\frac{1}{M})$ for $M\to\infty$.
Then (17) is obtained by an application of Lemma 1.

Thirdly, and most important for the proof, is the detailed knowledge of G. By a localization, the problem is carried over to the situation where G acts in $\overline{\mathbb{R}}^n_+$ and is multiplied on both sides by functions with compact support ; for simplicity we leave out the latter. We now expand $\tilde{g}$ in a double series of Laguerre functions. Actually, they are a variant of the usual Laguerre functions, defined for a parameter $\sigma > 0$ by

$$\varphi_k(x_n,\sigma) = (2\sigma)^{1/2}(\sigma - \partial_{x_n})^k \, (x_n^k e^{-\sigma x_n})/k! \quad \text{on } \mathbb{R}_+ \quad ,$$

for $k = 0,1,2,\ldots$; they form an orthonormal basis for $L^2(\mathbb{R}_+)$. (Their Fourier transforms

$$\hat{\varphi}_k(\xi_n,\sigma) = \mathcal{F}_{x_n \to \xi_n} \, (e^+ \varphi_k) = (2\sigma)^{1/2} \, \frac{(\sigma - i\xi_n)^k}{(\sigma + i\xi_n)^{k+1}}$$

are used in [3]). They are obviously in $\mathcal{S}(\overline{\mathbb{R}}_+)$, and have moreover the nice property

$$u \in \mathcal{S}(\overline{\mathbb{R}}_+) \iff u = \sum_{k \geq 0} b_k \varphi_k \quad , \quad \text{with } (b_k) \ \underline{\text{rapidly decreasing}} \quad ;$$

the rapid decrease means that $k^N b_k$ is bounded in k for all N, or, equivalently, $\sum_k (1+k)^N |b_k|^2 < \infty$ for all N.

Now we take $\sigma = \sigma(\xi')$ equal to $|\xi'|$ for $|\xi'| \geq 1$ and smooth and > 0 for all $\xi' \in \mathbb{R}^{n-1}$, and we write

$$(19) \qquad \tilde{g}(x',\xi',x_n,y_n) = \sum_{\ell,m \geq 0} c_{\ell m}(x',\xi') \, \varphi_\ell(x_n,\sigma) \, \varphi_m(y_n,\sigma) \quad .$$

It can be shown that the system of estimates (8) is equivalent with the following system of estimates

$$(20) \qquad \Big(\sum_{\ell,m \geq 0} (1+\ell)^N (1+m)^{N'} |D_{x'}^\beta D_{\xi'}^\alpha c_{\ell m}(x',\xi')|^2 \Big)^{1/2} \leq c(x') \langle \xi' \rangle^{d-|\alpha|}$$

for all sets of indices N, N', α, β. Note that each $c_{\ell m}(x',\xi')$ is a ps.d.o. symbol on $\mathbb{R}^{n-1}$ of order d, and the estimates (20) mean that the double sequence $(c_{\ell m})_{\ell,m \geq 0}$ is rapidly decreasing in the space of such symbols (s.g.o.s are introduced by double series in [3]).

Remark : It may be of interest to observe, that for a ps.d.o. of order $d \leq 0$ with constant coefficients and the transmission property, the symbolic operator $p(\xi',D_n)_\Omega$ on $\mathbb{R}_+$ is a Toeplitz operator with respect to the Laguerre system ; it is described by an infinite matrix $(a_{\ell-m}(\xi'))_{\ell,m \geq 0}$, where the sequence $(a_k)_{k \in \mathbb{Z}}$ is a rapidly decreasing sequence of ps.d.o. symbols on $\mathbb{R}^{n-1}$ of order d. The derived s.g.o. $G^+(P)$ (see (2)) is then described by the associated Hankel matrix $(a_{\ell+m+1}(\xi'))_{\ell,m \geq 0}$. The rapid decrease of the sequence $(a_k)_{k \in \mathbb{Z}}$ assures that the system $(c_{\ell m})_{\ell,m\geq 0} = (a_{\ell+m+1})_{\ell,m\geq 0}$ satisfies (20).

With the decomposition (19), the boundary symbol operator applies to $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$ as follows :

$$\begin{aligned} g(x',\xi',D_n)u &= \sum_{\ell,m\geq 0} c_{\ell m}\varphi_\ell\cdot(u,\varphi_m) \\ &= \sum_{m\geq 0} \tilde{k}_m(x',\xi',x_n)\cdot(u,\varphi_m) \quad , \end{aligned}$$

where the mapping $u \mapsto (u,\varphi_m) = \int_0^\infty u(y_n)\varphi_m(y_n,\sigma)dy_n$ is in fact a trace operator, and the multiplication by $\tilde{k}_m(x',\xi',x_n) = \sum_{\ell\geq 0} c_{\ell m}(x',\xi')\varphi_\ell(x_n,\sigma(\xi'))$ is a Poisson operator. Then we may view G as

$$G = \sum_{m\geq 0} K_m T_m \ , \tag{21}$$

where K_m is the Poisson operator with symbol-kernel $\tilde{k}_m(x',\xi',x_n)$ and T_m is the trace operator with symbol-kernel $\varphi_m(x_n,\sigma(\xi'))$. One here has moreover that

$$T_m T_\ell^* = \delta_{m\ell} I \qquad \text{on } \mathbb{R}^{n-1} \tag{22}$$

(where $\delta_{m\ell}$ is the Kronecker delta), thanks to the orthonormality of the Laguerre system. (Trace and Poisson operators are defined by formulas like (7), only with $g(x',\xi',D_n)$ replaced by the respective other boundary symbol operators relative to $\mathbb{R}_+$.)

The proof now uses in an essential way, that G is composed of operators going from Ω to $\partial\Omega$ and from $\partial\Omega$ back to Ω. In the localized situation, there is a complication due to the fact that even when G is truncated on both sides with compactly supported functions, T_m maps into functions with unbounded support in $\mathbb{R}^{n-1}$. Here one can obtain compactness either by inserting cut-off functions between K_m and T_m, or for instance by replacing $\mathbb{R}^{n-1}$ by a torus in the whole calculus (the latter gives better estimates in Theorem 2 below). Let us leave this problem out of the picture, and work with (21)-(22) as if we were back on the compact manifold $\overline{\Omega}$ with boundary $\Omega\partial$.

Write now, for each $M \geq 0$,

$$G = G_M + G'_M \quad , \quad \text{where}$$

(23)

$$G_M = \sum_{0\leq m<M} K_m T_m \quad \text{and} \quad G'_M = \sum_{m\geq M} K_m T_m \quad ,$$

then we shall show that Lemma 1 can be applied to this decomposition.

As for the G'_M term, one needs an estimate of the $s_k(G'_M)$ valid for all k. This is furnished by

Theorem 2 : Let G be a s.g.o. on $\bar{\Omega}$ of type $S_{1,0}$ and order $-d<0$, continuous from $L^2(\Omega)$ to $H^d(\Omega)$. The characteristic values satisfy

$$s_k(G)\, k^{d/(n-1)} \leqslant C(G) \quad , \tag{24}$$

where $C(G)$ is a constant estimated by a certain finite set of the symbol seminorms (9) in the localized situation.

We shall not describe the proof, which is based on some of the same ideas as the present proof (notably the decomposition (21)), combined with perturbation methods for characteristic numbers.

When $M\to\infty$, G'_M goes to 0 in all symbol seminorms, so (15) holds for G'_M with $\alpha = d/(n-1)$.

Now consider G_M. Let us write it

$$G_M = \mathcal{K}_M \mathcal{T}_M \quad , \quad \text{where}$$

$$\mathcal{K}_M = (K_o\ K_1\ \dots\ K_{m-1}) \quad , \quad \mathcal{T}_M = \begin{pmatrix} T_o \\ T_1 \\ \vdots \\ T_{m-1} \end{pmatrix} \quad .$$

Then we have for each k, using (22) and the well known identity $\lambda_k(B^*B) = \lambda_k(BB^*)$ for compact operators :

$$\begin{aligned} s_k(G_M)^2 &= \lambda_k(G_M^*G_M) = \lambda_k(G_MG_M^*) \\ &= \lambda_k(\mathcal{K}_M\mathcal{T}_M\mathcal{T}_M^*\mathcal{K}_M^*) = \lambda_k(\mathcal{K}_M\mathcal{K}_M^*) \\ &= \lambda_k(\mathcal{K}_M^*\mathcal{K}_M) = \lambda_k(\mathcal{P}_M) \quad , \end{aligned}$$

where $\mathcal{P}_M$ is the $M\times M$-matrix of operators on $\partial\Omega$

$$\mathcal{P}_M = (K_\ell^*K_m)_{0\leqslant \ell, m<M} \quad .$$

Here $K_\ell^*K_m$ is a ps.d.o. on $\partial\Omega$ of order $-2d$, by the rules of calculus, so we can apply Lemma 2 with $\Xi = \partial\Omega$; this gives

$$\lambda_k(P_M) \sim c(p_M^o)k^{-2d/(n-1)} \qquad \text{for } k\to\infty \quad .$$

Going through the same argument on the boundary symbol level, one finds that

$$\mathrm{tr}[(p_M^{o*}p_M^o)^{(n-1)/4d}] = \mathrm{tr}[(p_M^o)^{(n-1)/2d}] = \mathrm{tr}[(g_M^{o*}g_M^o)^{(n-1)/2d}] \ ,$$

which converges to $\mathrm{tr}[(g^{o*}g^o)^{(n-1)/2d}]$ for $M\to\infty$, uniformly for x' in a compact set and $|\xi'| = 1$, so that

$$c(p_M^o) \longrightarrow c(g^o) \qquad \text{for } M\to\infty \quad .$$

Then G_M satisfies (14), Lemma 1 can be applied, and the proof is complete.

The arguments for the $\lambda_k^{\pm}$ sequences follow the same lines.

The result applies in particular to each of the Examples 1-4. As for Example 5, we can write G_1 as a sum of two terms, where each satisfies estimates (4)-(5). The study of G_1 is then used to show that the operator

$$A^{-1} - A_+^{-1} \oplus 0$$

has the same spectral behavior as the <u>interior operator</u> A_-^{-1}, up to a remainder that is $\mathcal{O}(k^{-d/(n-1)+\varepsilon})$ (more details in [10]).

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