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THE RECONSTRUCTION THEOREM $\mathcal{M}^{\infty} = \mathcal{E}^{\infty} \otimes_{\varepsilon} \mathcal{M}_{reg}$.

par J. E. BJÖRK

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<u>Introduction</u> In |1| Kashiwara and Kawai made a far-recahing study of holonomic systems. A basic result is Theorem 5.2.1. in [1] which asserts that if $\mathcal{M}$ is a holonomic $\mathcal{E}$-module, where $\mathcal{E} = \mathcal{E}_X$ is the sheaf of micro-local differential operators of finite order defined outside the zero-section of the cotangent bundle of X, then the extended sheaf $\mathcal{E}^{\infty} \otimes_{\mathcal{E}} \mathcal{M}$ contains a unique $\mathcal{E}$-submodule $\mathcal{M}_{\mathrm{reg}}$ which has regular singularities and satisfies $\mathcal{E}^{\infty} \otimes_{\mathcal{E}} \mathcal{M}_{\mathrm{reg}} = \mathcal{E}^{\infty} \otimes_{\mathcal{E}} \mathcal{M}$. The proof of this result is quite involved and my aim is here to present the passage of the proof which appears in Chapter 4 of [1]. This part deals with prolongation properties of solutions to over-determined systems which arise from a given holonomic $\mathcal{E}$-module whose support is the conormal to a hypersurface S in the base manifold.

In [2] Kashiwara and Kawai offers a brief exposition of the proof too, so here Section 2 only repeats the steps which are necessary in order to understand how the proof is reduced to a specific problem about prolongations of solutions to an overdetermined system.

Section 3 - 5 contains the detailed analyis which finishes the proof. My presentation may appear to be quite different from the material in [1], but actually all the essential methods are already given in [1], the difference is that I have tried to avoid too much machinery based upon local cohomology, to make the proof more accesible to the analysists.

The material in Section 6-8 contain proofs of various results which were used before. They can be studied independently of the preceeding material, but of coure their content is motivated by the fact that it enable us to prove the Reconstruction Theorem.

<u>A Remark</u> The actual proof of the Reconstruction Theorem contains more results, see for example Theorem 4.1.1. in [1]. In particular Kashiwara and Kawai finds particular good filtrations on holonomic modules with R.S. and we refer to [11] for very interesting comments about this.

<u>The case of holonomic $\mathcal{D}$-modules</u>. In this case another method leads to the Reconstruction Theorem which we briefly discuss in Section [illegible].

1. The case of $\mathcal{D}$-modules

Before we begin the micro-local analysis, it should be pointed out that if we work with holonomic $\mathcal{D}$-modules, where $\mathcal{D} = \mathcal{D}_X$ is the sheaf of differential operators with analytic coefficients, then the Reconstruction Theorem can be attacked in a quite different way. This was for example done in [6, 8, 4]. We shall briefly recall how this goes.

First, let $D(X)_c$ be the derived category of bounded sheaf-complexes on X whose cohomology sheaves are constructible, i.e. they are locally constant on a complex analytic Whitney stratification $\{ X_\alpha \}$ and their stalks are finite dimensional complex vector spaces.

For a single constructible sheaf $\mathcal{F}$, the following basic result holds, where we put $\mathcal{O} = \mathcal{O}_X$ to simplify the notations.

1.1. Theorem $\mathcal{E}xt^j_{\mathbb{C}}(\mathcal{F}, \mathcal{O})$ are sheaves of $\mathcal{D}^\infty$-modules for each j 0 and using the natural action by $\mathcal{D}^\infty$ on $\mathcal{O}$ and each of them contains a distinguished sheaf $\mathcal{M}_j$ of $\mathcal{D}$-submodules which is holonomic and $\mathcal{D}^\infty \otimes_{\mathcal{D}} \mathcal{M}_j$ equals the $\mathcal{E}xt$-sheaf above.

This result was essentially proved in [5]. The sheaf $\mathcal{M}_j$ is found by the use of temperated cohomology as explained in [4]. It is also the starting point for the proof of the isomorphism of $D(X)_c$ and the derived category $D(\mathcal{D})_{h.r.}$ of bounded complexes of $\mathcal{D}$-modules whose cohomology sheaves are holonomic and have R.S.

Well, even if Theorem 1.1. is quite intuitive since the $\mathcal{D}^\infty$-module structure on these $\mathcal{E}xt$-sheaves is clear, the most convenient way to express $\mathcal{M}_j$, arise by the diagonal method. To be precise, in the product manifold X x X we let Δ be the diagonal and then we can consider the derived functors of Γ_Δ and of $\Gamma_{[\Delta]}$ respectively. A basis result is then

1.2. Grothendieck's Comparison Theorem which asserts that if $\mathcal{F}_\bullet$ is a complex in $D(X)_c$, then the canonical mapping from

$\mathbb{R}\Gamma_{[\Delta]}(\mathcal{F}_\bullet \otimes_{\mathbb{C}} \mathcal{O})$ into $\mathbb{R}\Gamma_\Delta(\mathcal{F}_\bullet \otimes_{\mathbb{C}} \mathcal{O})$ is injective and its image

is ample enough to generate the hypercohomology sheaves

$\mathbb{R}\Gamma_{\Delta}(\mathcal{F}_{\bullet} \otimes_{\mathbb{C}} \mathcal{O})$ as $\mathcal{D}^{\infty}$-modules.

1.3. Remark Saying this, the $\mathcal{D}^{\infty}$-module structure arises by Sato's cohomological description of $\mathcal{D}^{\infty}$ and of $\mathcal{D}'$, i.e. these sheaves on X are equal to the local cohomology sheaves $\mathcal{H}^{n}_{\Delta}(\hat{\Omega})$ and $\mathcal{H}^{n}_{[\Delta]}(\hat{\Omega})$ respectively, where $n = \dim(X)$ and $\hat{\Omega} = \Omega_X \otimes \mathcal{O}_{X \times X}$ and Ω_X is the sheaf of holomorphic n-forms.

Using a formula from Verdier's work in [10], we then have the isomorphism $\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{F}, \mathcal{O}) \simeq \mathbb{R}\Gamma_{\Delta}(\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{F}, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O})$ when $\mathcal{F}$ is a constructible sheaf, and using this isomorphism the holonomic $\mathcal{D}$-modules $\mathcal{M}_j$ from Theorem 1.1. arise when we take the hypercohomology sheaves in the complex $\mathbb{R}\Gamma_{[\Delta]}(\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{F}, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O})$.

1.4. The case when $\mathcal{M}$ is $\mathcal{D}$-holonomic If $\mathcal{M}$ is a holonomic sheaf of $\mathcal{D}$-modules, then $\mathcal{E}xt^{j}_{\mathcal{D}}(\mathcal{O}, \mathcal{M})$ are constructible sheaves for all j. This was proved in [6]. It is then combined with the Biduality Formula which asserts that $\mathcal{M}^{\infty} \simeq \mathbb{R}\Gamma_{\Delta}(\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{O}, \mathcal{M}) \otimes_{\mathbb{C}} \mathcal{O})$ holds. See [7] or [1, Theorem 1.4.9.] for a proof.

1.5. A formula for $\mathcal{M}_{reg}$ is now achieved. In fact, given the holonomic $\mathcal{D}$-module $\mathcal{M}$, the results in 2.2. and 2.4. imply that $\mathcal{M}_{reg} \subset \mathcal{M}^{\infty}$ is found by computing $\mathbb{R}\Gamma_{[\Delta]}(\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{O}, \mathcal{M}) \otimes_{\mathbb{C}} \mathcal{O})$.

We refer to [8] for a more detailed presentation of the material given above.

1.6. Some Remarks Even though the methods above lead to a distinguished holonomic $\mathcal{D}$-submodule of $\mathcal{M}^{\infty}$, it is not at all obvious that this sheaf has R.S. in the sense of [3]. However, it is a consequence of the Reconstruction Theorem, that the sheaf found in 1.5. has R.S. See for example Section 6-4 in [1].

1.7. Some explicit formulas Let us finish by saying that the formulas above can be made explicit. Here is an illuminating example.

1.8. The case when $\mathcal{M}$ has pure dimension. In general, if $\mathcal{M}$ is a holonomic $\mathcal{D}$-module we let $F_k(\mathcal{M})$ be the sheaf of sections in $\mathcal{M}$ whose supports are at most k-dimensional. Each $F_k(\mathcal{M})$ is then a holonomic $\mathcal{D}$-submodule of $\mathcal{M}$, and the quotients $F_k(\mathcal{M})/F_{k-1}(\mathcal{M})$ have pure dimension k, i.e. $\dim(\mathrm{supp}(\xi)) = k$ hold for any section in this sheaf.

If $\mathcal{M}$ now is holonomic and has a pure dimension k, then the Biduality Formula and the use of an associated spectral sequence gives

1.9. Proposition $\exists$ an exact sequence

$$0 \longrightarrow \mathcal{M}^{\infty} \longrightarrow \mathcal{E}xt^{n-k}_{\mathbb{C}}(\mathcal{E}xt^{n-k}_{\mathcal{D}}(\mathcal{M},\mathcal{O}),\mathcal{O}) \longrightarrow W_k \longrightarrow 0$$

where $\dim(\mathrm{supp}(W_k)) \leq k-2$.

See [1].Theorem 7.8. on page 74] for a similar result. The proof there can be repeated to give Proposition 1.9. above.

From this, we then find $\mathcal{M}_{reg}$ by intersecting $\mathcal{M}^{\infty}$ with the distinguishead R.S. submodule of $\mathcal{E}xt^{n-k}_{\mathbb{C}}(\mathcal{E}xt^{n-k}_{\mathcal{D}}(\mathcal{M},\mathcal{O}),\mathcal{O})$, so this shows the relevance of Theorem 1.1. Of course, the fact that it suffices to prove the Reconstruction Theorem for the $\mathcal{D}^{\infty}$-sheaves $\mathcal{E}xt^{j}_{\mathbb{C}}(\mathcal{F},\mathcal{O})$ when $\mathcal{F}$ is constructible, is a consequence of the isomorphism between $D(X)_c$ and $D(\mathcal{D})_{h.r.}$

1.10. The canonical filtration To finsh, we recall that the actual proof in [1] leads to the existence of a canonically defined good filtration of a holonomic $\mathcal{D}$-module with R.S. which may have topological consequences. See [11] and see Section 5-1 in [1] for the construction of this good filtration.

2. An outline of the proof

Following [1] we shall describe how the proof of the Reconstruction Theorem is reduced to a study of a certain over-determined system. So let X be a complex analytic manifold and let $\mathring{T}^*(X) = T^*(X) - T^*_X(X)$ be the complement of the zero-section of the cotangent bundle. Consider then a holonomic $\mathcal{E}_X$-module $\mathcal{M}$ which is defined in some open and conic subset Ω of $\mathring{T}^*(X)$. The support of the sheaf $\mathcal{M}$ is then a conic Lagrangian variety Λ, where Λ in general may have singular points. In any case, its regular part $\Lambda_{reg} = \Lambda \setminus \Lambda_{sing}$ is open and dense.

2.1. The existence of $\mathcal{M}_{reg}$ on Λ_{reg}. Using the Classification Theorem for holonomic $\mathcal{E}$-modules with non-singular support, both the existence and the uniqueness of $\mathcal{M}_{reg}$ on Λ_{reg} is easily proved. See [1:Section 1-3] for details.

2.2. The uniqueness of $\mathcal{M}_{reg}$. On Λ_{reg} we find the unique $\mathcal{E}$-submodule $\mathcal{M}_{reg}$ which then satisfies $\mathcal{M}^\infty = \mathcal{E}^\infty \otimes_{\mathcal{E}} \mathcal{M}_{reg}$ on Λ_{reg}. Using Theorem 1.2.1. in [1] which may be regarded as a kind of Hartog's Theorem for coherent $\mathcal{E}$-modules, the uniqueness of $\mathcal{M}_{reg}$ on the whole Lagrangian variety follows from its uniqueness on Λ_{reg}. Of course, this uniqueness is stated under the condition that we have found some coherent $\mathcal{E}$-submodule $\mathcal{N}$ of $\mathcal{M}^\infty$ which has R.S. and satisfies $\mathcal{M}^\infty = \mathcal{N}^\infty$. In fact, then $\mathcal{N} = \mathcal{M}_{reg}$ holds on Λ_{reg}, and the uniqueness of $\mathcal{N}$ is a consequence of the cited result, or more precisely of Corollary 1.2.3. in [1] which asserts:

2.3. Proposition Let $\mathcal{M}$ be holonomic and let $\mathcal{N}$ be a holonomic $\mathcal{E}$-submodule of $\mathcal{M}^\infty$ and let $Z \subset \Lambda$ be a subvariety with $\dim(Z) < \dim(\Lambda) = \dim(X)$. Then the following hold

(1) Any locally defined section $s \in \Gamma(U, \mathcal{M}^\infty)$ which belongs to $\mathcal{N}^\infty$ outside Z, belongs to $\Gamma(\ , \mathcal{N}^\infty)$

(2) Any $s \in \Gamma(U, \mathcal{M}^\infty)$ which belongs to $\mathcal{N}$ on $U - Z$ belongs to $\Gamma(U, \mathcal{N})$.

Summing up, $\mathcal{M}_{reg}$ exists on Λ_{reg} and is unique on Λ if it exists.

So for the proof of existence it is sufficient to prove that $\mathfrak{m}_{reg}$ exists locally, i.e. in a small conic neighborhood of a given point p_o on Λ_{sing}. To attain this we can assume that Λ has a <u>generic position</u> at the point p_o, i.e. that the fiber $\mathbb{C}^* p_o$ is isolated in the conic Lagrangian variety Λ. More precisely we obtain this as follows

2.4. <u>A geometric preparation</u> First, symplectic algebra shows that there exists a locally defined homogenous canonical transformation χ at p_o which maps the germ of Λ at p_o onto another germ of a Lagrangian variety at p_o which has generic position. See [1 :Section 1-6] for details.

Using a <u>contact transformation</u> attached to this canonical transformation we can then assume that Λ from the start has generic position at p_o. See for example [11: Theorem 6.1.] and notice also that the contact transformation is defined locally on $\mathcal{E}^\infty$, as explained in [11, Proposition 11.4. page 182]

2.5. <u>The hypersurface</u> $\pi(\Lambda) = S$. From now on we assume that Λ has a generic position at p_o. This implies that if π is the projection from $T^*(X)$ to the base manifold, then $\pi(\Omega \cap \Lambda) = S$ is a hypersurface in X, where Ω is a small conic neighborhood of p_o.

In addition to this the equality $\Lambda = T^*_S$ holds in a conic neighborhood of p_o, where T^*_S = the closure of the smooth conormal variety $T^*_{S_{reg}}$. The proof is easy and we refer to [1 :Section 4-2] for details.

2.6. <u>The $\mathcal{E}^\infty_{p_o}$-module</u> $\mathcal{C}$. Working locally we may assume that $X = \mathbb{C}^{n+1}$ where we use (x,t) as coordinates with $x = (x_1 \ldots x_n)$ and t is distinguished because we assume that the point $p_o = (0,0:0,dt)$. So here the base point $\pi(p_o)$ is the origin in $\mathbb{C}^{n+1}$ and the hypersurface $S = \pi(\Lambda)$ is defined in some polydisc $B(\delta,\varepsilon) = \{ (x,t) : |x| < \delta \text{ and } |t| < \varepsilon \}$.

The proof will now employ a certain $\mathcal{E}^\infty_{p_o}$-module which we begin to define. First, if $a > 0$ we put $W(a) = \{ (x,t) : \mathrm{Re}(t) < -a(|x| + |t|) \}$ and then

is the inductive limit of the quotient spaces $\mathcal{O}(W(a) \cap \Delta)/\mathcal{O}(\Delta)$ as $a \to 0$ and Δ are polydiscs which shrink to the origin in $\mathbb{C}^{n+1}$.

2.7. Remark Of course, this means that if $Z(a) = \{ (x,t) : \mathrm{Re}(t) \geq - a(|x| + |t|)\}$, then $\mathcal{C}$ is the stalk at (o,o) of the inductive limit of the local cohomology sheaves $\mathcal{H}^1_{Z_a}(\mathcal{O})$ where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{n+1}}$.

It can then be proved that $\mathcal{C}$ is a left $\mathcal{E}^\infty_{p_o}$-module. The module structure can be made explicit, . In [1 ,Section 3] a general construction of $\mathcal{E}^\infty_{p_o}$ -modules is made which contains the case above as a very special case.

2.8. The space Hom $_{\mathcal{E}^\infty_{p_o}}(\mathcal{M}^\infty_{p_o}, \mathcal{C})$ will now be studied. The crucial step towards the proof of the Reconstruction Theorem is to show that this is a finite dimensional complex vector space which in addition is ample enough to determine the stalk $\mathcal{M}^\infty_{p_o}$. Then, by the finite dimensionality and coherence, the sheaf $\mathcal{M}^\infty$ is determined in a small neighborhood of p_o too. Well, we have to clarify the content of this assertion. It goes as follows

2.9. $\mathcal{M} = \mathcal{E}/\mathcal{L}$ can be assumed from the start since holonomic $\mathcal{E}$-modules are locally cyclic. So here $\mathcal{L}$ is a coherent sheaf of left ideals in $\mathcal{E}$ and the equality $\Lambda = \mathrm{supp}(\mathcal{E}/\mathcal{L}) = \sigma(\mathcal{L})^{-1}(0)$ holds then, where $\sigma(\mathcal{L})^{-1}(0)$ is the set of common zeros of principal symbols of sections in the sheaf $\mathcal{L}$.

When $\mathcal{M} = \mathcal{E}/\mathcal{L}$ is cyclic it is clear that the space

$$\mathcal{C}_o = \text{Hom}_{\mathcal{E}^\infty_{p_o}}(\mathcal{M}^\infty_{p_o}, \mathcal{C}) \text{ can be realised it follows:}$$

2.10. A useful description. An element ξ in the space $(\mathcal{C}_o)$ above is represented by some holomorphic function $\varphi(x,t) \in \mathcal{O}(W(a) \cap \Delta)$ such that $R_1\varphi \ldots R_k\varphi$ all extend to $\mathcal{O}(\Delta)$, where $R_1 \ldots R_k$ is some finite set of generators for the left ideal $\mathcal{L}_{p_o}$.

In Section 3 we shall describe in more detail how generators in $\mathcal{L}_{p_o}$ are chosen. The assertion that Hom $_{\mathcal{E}^\infty_{p_o}}(\mathcal{M}^\infty_{p_o}, \mathcal{C})$ determines $\mathcal{M}^\infty_{p_o}$ means the following: If $Q \in \mathcal{E}^\infty_{p_o}$ is such that $Q\xi = 0$ in $\mathcal{C}$ for all ξ in the solution space $(\mathcal{C}_o)$ above, then Q belongs to the left ideal which $\mathcal{L}_{p_o}$

generates in $\mathcal{E}^{\infty}_{p_0}$.

2.11. A Remark Recall here that $\mathcal{E}^{\infty}$ is faithfully flat over $\mathcal{E}$. This is used in order to identify $\mathcal{M}^{\infty}$ with $\mathcal{E}^{\infty}/\mathcal{E}^{\infty}\mathcal{L}$ and so on.

2.12. How to use the result above As we have already said, it is the assertion that the solution space $\mathrm{Hom}_{\mathcal{E}^{\infty}_{p_0}}(\mathcal{M}^{\infty}_{p_0}, \mathcal{C})$ determines $\mathcal{M}^{\infty}_{p_0}$ which is the part of the proof to which the subsequent sections are devoted.

The proof of the reconstruction theorem is then done as follows:

2.13. The Imbedding Lemma First, assuming that the results from Sections 2.8. - 2.10. hold we can find some $a > 0$ and some small polydisc Δ and a k-tuple $\varphi_1 \dots \varphi_k$ in $\mathcal{O}(W(a) \cap \Delta)$ which give solutions and induce a basis for the k-dimensional complex vector space $\mathrm{Hom}_{\mathcal{E}^{\infty}_{p_0}}(\mathcal{M}^{\infty}_{p_0}, \mathcal{C})$.

A notable point should be mentioned here

2.14. Lemma If $a > 0$ then $W(a) \cap S \cap \Delta$ is empty if the polydisc Δ is sufficiently small.

This is easily proved, using the fact that $\mathbb{C}^{*}p_0$ by assumption is isolated in T^{*}_{S}. See also [1: Lemma 4.2.1.] for a proof.

Choosing Δ so small that $W(a) \cap \Delta \cap S$ is empty, it can then be proved that the holomorphic functions $\varphi_1 \dots \varphi_k$ above can be continued to (in general) multi-valued analytic functions in $\Delta \setminus S$.

In Section 8 we shall prove this existence of analytic continuations. Admitting it for the moment, we then get a local system Φ in $\Delta \setminus S$, whose stalks $\Phi_{(x,t)}$ = the $\mathbb{C}$-subspace of $\mathcal{O}_{(x,t)}$ generated by all local branches of the functions $\varphi_1 \dots \varphi_k$ respectively.

The multi-valued extensions of the φ-functions have finite determinatic so the stalks of Φ consist of finite dimensional $\mathbb{C}$-spaces. At this stage we make use of a result whose full proof requires Hironaka's Desingularisation and it goes as follows

2.15. Existence Lemma Given Φ as above there exists a section

$Q = Q(x,t,D_t,D_x)$ in $\Gamma(\Delta, \mathcal{D}^\infty)$ - where $\mathcal{D}^\infty$ is the sheaf of differential operators on $\mathbb{C}^{n+1}$ of infinite order- and another local system Ψ in $\Delta \setminus S$ whose stalks $\Psi_{(x,t)}$ again are $\mathbb{C}$-subspaces of $\mathcal{O}_{(x,t)}$ and they satisfy:

(1) The equality $Q\Psi = \Phi$ holds, i.e. Q operates on $\mathcal{O}$ in the usual way and maps the subspaces $\Psi_{(x,t)}$ onto $\Phi_{(x,t)}$ for all points (x,t) in $\Delta \setminus S$

(2) Ψ is the local system generated by local bracnehs of finitely many Nilsson class functions $\psi_1 \ldots \psi_m$ - all defined in $\Delta \setminus S$.

2.16. The sheaf $\mathcal{K} = \Psi^{-1}(0)$. Using the fact that Ψ arises from Nilsson class functions, the theory about holonomic $\mathcal{D}$-sheaves then shows that there exists a unique coherent sheaf $\mathcal{K}$ of left ideals in $\mathcal{D}$ which satisfies

(1) If $(x,t) \in \Delta \setminus S$ then the stalk $\mathcal{K}_{(x,t)} = \{ Q \in \mathcal{D}_{(x,t)} : Qg = 0$ in $\mathcal{O}_{(x,t)}$ for all germs g in the subspace $\Psi_{(x,t)} \}$

(2) The sheaf $\mathcal{D}/\mathcal{K}$ is holonomic and has no $\mathcal{O}$-torsion.

Well, its proof is not easy, we may refer to [11,Theorem 4.8.30 on page 270] . See also [5] and [1, Section 2]

2.17. The Imbedding Lemma We can introduce the holonomic $\mathcal{E}$-module $\mathcal{E} \otimes_{\mathcal{D}} (\mathcal{D}/\mathcal{K}) = \mathcal{N}$ and then the equality $Q\Psi = \Phi$ and $\mathcal{K} = \Psi^{-1}(0)$ on $\Delta - S$, imply that the $\mathcal{E}^\infty$-linear mapping from $\mathcal{M}^\infty = \mathcal{E}^\infty/\mathcal{E}^\infty\mathcal{L}$ into $\mathcal{N}^\infty$ determined by the right multiplication with the section Q is well defined and it is injective in a small neighborhood of p_0 because the solution space $\mathrm{Hom}_{\mathcal{E}^\infty_{p_0}}(\mathcal{M}^\infty_{p_0}, \mathcal{C})$ determines $\mathcal{M}^\infty_{p_0}$ in a neighborhood of p_0.

Well, here the holonomic $\mathcal{D}$-sheaf $\mathcal{D}/\mathcal{K}$ is of the so called Deligne type, and the result in 2.17. is the content of Theorem 4.1.1. in [1]. It is called the Imbedding Lemma since it shows that $\mathcal{M}^\infty$ can be imbedded into $\mathcal{E}^\infty \otimes_{\mathcal{D}} \mathcal{N}$ where $\mathcal{N}$ is $\mathcal{D}$-holonomic and of Deligne type along the hypersurface $S = \pi(\Lambda)$, where we aasumed that $\Lambda = \mathrm{supp}(\mathcal{M})$ has a generic position at p_0 from the start.

2.18. The final part of the proof Once the Imbedding Lemma has been proved, it can then be proved that $\mathcal{D}/\mathcal{K}$ contains a $\mathcal{D}$-submodule $\mathcal{N}_0$ so that the image of $\mathcal{M}^\infty$ in $\mathcal{E}\otimes_{\mathcal{D}}(\mathcal{D}/\mathcal{K})$ equals $\mathcal{E}^\infty \otimes_{\mathcal{D}} \mathcal{N}_0$. To prove this, Kawai and Kashiwara also makes use of the Imbedding Lemma applied to the dual holonomic $\mathcal{E}$-sheaf $\mathcal{M}^* = \mathcal{E}xt_{\mathcal{E}}^{n+1}(\mathcal{M}, \mathcal{E})$. The details can be found in Section 5-2 in [1].

2.19. The extension of $\mathcal{M}_{reg}$. If we have obtained the equality $\mathcal{M}^\infty = \mathcal{E}^\infty \otimes_{\mathcal{D}} \mathcal{N}_0$ where $\mathcal{N}_0$ is a holonomic $\mathcal{D}$-submodule of $\mathcal{D}/\mathcal{K}$ which has Deligne type, then the equality $\mathcal{M}_{reg} = \mathcal{E} \otimes \mathcal{N}_0$ easily follows on the open subset $T^*_{S_{reg}}$ of Λ.. Next, $\mathcal{E} \otimes \mathcal{N}_0$ is a holonomic $\mathcal{E}$-submodule of $\mathcal{M}^\infty$ defined on the whole variety Λ, so from the discussion in Section 2.2. we conclude that $\mathcal{E} \otimes \mathcal{N}_0$ is the required sheaf $\mathcal{M}_{reg}$!

2.20. A sticky point In general, if $\mathcal{N}$ is a holonomic $\mathcal{D}$-module of Deligne type along a hypersurface S it is not obvious that $\mathcal{E}\otimes_{\mathcal{D}}\mathcal{N}$ has R.S. in the sense of [3] because it involves a condition on all components of its characteristic variety, while the Deligne sheaf $\mathcal{N}$ a priori only has R.S. along the component of supp($\mathcal{E}\otimes\mathcal{N}$) given by $T^*_{S_{reg}}$. However, during the construction above the Deligne sheaf $\mathcal{D}/\mathcal{K}$ is quite special because its characteristic variety is small, i.e. it is the closure of the smooth conormal $T^*_{S_{reg}}$ so that $\mathcal{E} \otimes \mathcal{D}/\mathcal{K}$ has R.S. in the sense of [3].

A notable point is then, that using the Reconstruction Theorem, it can be proved that any holonomic $\mathcal{D}$-module of Deligne type has R.S. in the sense of [3]. See [1, Theorem 5.2.3] and observe that this important result cannot be proved until the whole micro-local calculus has been used to prove the Reconstruction Theorem for holonomic $\mathcal{E}$-modules. Of course, this is natural since the definition of R.S. is already of a micro-local nature.

Summing up, we have now finished a brief sketch of the proof, without any details. In the subsequent sections we shall give the details which lead to the assertions made in Section 2.8.-2.10.

3. The left ideal $\mathcal{L}_{p_o} \cap \mathcal{E}_{p_o}(x,t,D_t)\langle D_x\rangle$

We keep the notations from Section 2. So $\mathcal{M} = \mathcal{E}/\mathcal{L}$ and then supp($\mathcal{M}$) = $= \sigma(\mathcal{L})^{-1}(0) = T_S^*$ close to the point $p_o = (0,0,0,dt)$. The fact that the fiber $\mathbb{C}^* p_o$ is isolated in T_S^* easily implies that the hypersurface $S = p^{-1}(0)$ where $p(x,t)$ is a Weierstrass polynomial with respect to t, i.e. we have

$$p(x,t) = t^e + \rho_1(x)t^{e-1} + \dots + \rho_e(x)$$

3.1. The subring $\mathcal{E}_{p_o}(x,t,D_t)\langle D_x\rangle$ will be used in the sequel. An element there is given as a finite sum $\sum A_\alpha(x,t,D_t)D_x^\alpha$ extended over finitely many multi-indices $\alpha = (\alpha_1 \dots \alpha_n)$ while $A_\alpha(x,t,D_t)$ as indicated are germs in $\mathcal{E}_{p_o}$ which are independent of the D_x-variables.

Using the fact that $\mathbb{C}^* p_o$ is isolated in $\sigma(\mathcal{L})^{-1}(0)$, divisions in $\mathcal{E}_{p_o}$ show that the left ideal $\mathcal{L}_{p_o}$ is generated by elements which belong to the subring $\mathcal{E}_{p_o}(x,t,D_t)\langle D_x\rangle$.

This is useful, because now the solution space $\mathcal{H}om_{\mathcal{E}^\infty_{p_o}}(\mathcal{M}^\infty_{p_o}, \mathcal{C})$ is consists of elements ξ in $\mathcal{C}$ for which $R_1\xi = \dots = R_k\xi = 0$ where $R_1 \dots R_k$ is some finite subset of $\mathcal{L}_{p_o} \cap \mathcal{E}_{p_o}(x,t,D_t)\langle D_x\rangle$ which generate the left ideal $\mathcal{L}_{p_o}$ in the ring $\mathcal{E}_{p_o}$.

This simplifies the subsequent analysis because actions on $\mathcal{O}$ by elements in $\mathcal{E}_{p_o}(x,t,D_t)\langle D_x\rangle$ are rather easy to describe. We shall do this now because explicit formulas are needed later on.

3.2. The action on $\mathcal{O}$. Let $R = \sum A_\alpha(x,t,D_t)D_x^\alpha$ be given. Each germ $A_\alpha(x,t,D_t)$ can then be expanded with respect to D_t so we get

$$A_\alpha(x,t,D_t) = \sum_{j\geq 0} q_{j,\alpha}(x,t)D_t^j + \sum_{\nu=1}^{\infty} k_{\nu,\alpha}(x,t)D_t^{-\nu}$$

where the first sum is finite since $0 \leq j \leq \mathrm{ord}(A_\alpha)$ holds there. Collecting all these expansions we find that $R = Q + K$ where $Q = \sum q_{j,\alpha}(x,t)D_t^jD_x^\alpha$ is a section in $\mathcal{D}$, defined in some neighborhood of the origin, while

$K = \sum\sum k_{v,\alpha}(x,t)D_t^{-v}D_x^{\alpha}$ only contains negative powers of D_t.

3.3. <u>The kernels</u> $K_\alpha(x,t,u) = \sum_{\nu=1}^{\infty} k_{v,\alpha}(x,t)(t-u)^{v-1}/(v-1)!$ are now introduced in order to define the action by R on $\mathcal{O}$. Of course, the reason why these kernels are introduced is that negative powers of D_t should produce primitive functions with respect to t, and we may observe that if $\varphi(x,t)$ is a holomorphic function and if $v \geq 1$ and t_o is some given point then the integral $\int_{t_o}^{t} (t-u)^{v-1}/(v-1)!\,\varphi(x,u)du$ is the v-th primitive of φ with respect to t.

This suggests the following

3.4. <u>Definition</u> For a given germ φ in $\mathcal{O}_{(x_o,t_o)}$ we define

$$R\varphi(x,t) = Q(x,t,D_x,D_t)\varphi(x,t) + \sum_{\alpha}\int_{t_o}^{t} K_\alpha(x,t,u)D_x^{\alpha}\varphi(x,u)du$$

where the integrals are defined when (x,t) stays in a small polydisc Δ_o centered at (x_o,t_o) so that the germ φ belongs to $\mathcal{O}(\Delta_o)$. The integration is then taken along the straight line from t_o to t in the complex u-space.

3.5. <u>Remark about the convergence</u> Of course, the action by R on $\mathcal{O}_{(x,t)}$ is only defined when (x,t) is close to the origin. For example, we can find δ_o and ε_o so that all the coefficients of Q, and all the kernels $K_\alpha(x,t,u)$ are holomorphic when $|x| < \delta_o$ and both $|t|$ and $|u|$ are $< \varepsilon_o$.

Observe here that $K_\alpha(x,t,u)$ indeed are holomorphic in a neighborhood of the origin in the (x,t,u)-space because $\sum k_{\alpha,v}(x,t)D_t^{-v}$ belong to $\mathcal{E}_{p_o}$ which implies that there exists a polydisc Δ and constants A and B so that $\forall\, k_{v,\alpha} \in \mathcal{O}(\Delta)$ and the sup-norms $|k_{v,\alpha}|_\Delta \leq A(v!)B^v$ for all v.

<u>Summing up</u>, if $R_1 \ldots R_k$ is a finite subset of $\mathcal{E}_{p_o}(x,t,D_t) \langle D_x \rangle$ then there exists a polydisc Δ so that $R_1 \ldots R_k$ define $\mathbb{C}$-linear operators on the stalks $\mathcal{O}_{(x,t)}$ for all points (x,t) inside Δ.

4. The local solution spaces $\mathcal{E}_j(x_o)$

From now on we fix generators $R_1 \dots R_k$ which belong to $\mathcal{E}_{p_o}(x,t,D_t)\langle D_x\rangle$ for the left ideal $\mathcal{L}_{p_o}$. By coherence it then follows that

$\mathcal{L} = \mathcal{E}R_1 + \dots + \mathcal{E}R_k$ holds in a conic neighborhood of p_o, and since $\mathrm{supp}(\mathcal{E}/\mathcal{L}) = T^*_S$ we can then find a small polydisc $B(\delta_o,\varepsilon_o) = \{ (x,t) : |x| < \delta_o \text{ and } |t| < \varepsilon_o \}$ such that the following hold:

(1) $\mathcal{L} = \mathcal{E}R_1 + \dots + \mathcal{E}R_k$ holds on T^*_S where $S = \pi(\Lambda) \cap B(\delta_o,\varepsilon_o)$

(2) Both δ_o and ε_o are chosen so small that $R_1 \dots R_k$ operate on $\mathcal{O}$ inside $B(\delta_o,\varepsilon_o)$ as described in Section 3.

In addition to this we shall need some elementary facts about the hypersurface S. Recall that $S = p^{-1}(0)$ where $p(x,t)$ is Weierstrass with respect to t. As usual we then choose $\delta_o << \varepsilon_o$ and in $B(\delta_o) = \{ x : |x| < \delta_o \}$ we find a hypersurface Z = the local of the discriminant of p, such that the projection $(x,t) \to x$ is an e-fold covering of (S-Z) onto $B(\delta_o)-Z$. The following notations will be used to describe this

<u>Notations</u> Put $B^*(\delta_o) = B(\delta_o)-Z$ and if $x \in B^*(\delta_o)$ then we get the roots $\alpha_1(x) \dots \alpha_m(x)$ which give the points $(x,\alpha_j(x))$ on S, and actually they all belong to the regular part S_{reg}.

4.1. <u>The points</u> $\mathcal{T}_j(x_o)$ on $T^*_{S_{reg}}$. When x_o is given in $B^*(\delta_o)$ then the smooth conormal $T^*_{S_{reg}}$ contains the points $\mathcal{T}_j(x_o) = (x_o,\alpha_j(x_o) : dp(x_o,\alpha_j(x_o))$ and using the classification theorem for holonomic $\mathcal{E}$-modules with smooth support, applied to $\mathcal{M} = \mathcal{E}/\mathcal{L}$ at the points $\mathcal{T}_j(x_o)$, we arrive at the result to be discussed below. First we need more notations.

4.2. <u>The solution spaces</u> $\mathcal{E}_j(x_o : t_j : \Delta_j)$. Let $x_o \in B^*(\delta_o)$ and let Δ_j be a small polydisc centered at $(x_o,\alpha_j(x_o))$ and let t_j be a point in the complex t-space such that (x_o,t_j) is outside S. Now $R_1 \dots R_k$ operate on $\mathcal{O}_{(x_o,t_j)}$ and we say that a germ φ there is a local solution if all the functions $R_i\varphi(x,t) = Q_i\varphi(x,t) + \sum_\alpha \int_{t_j}^{t} K_{i,\alpha}(x,t,u)D_x^\alpha\varphi(x,u)$ extend to Δ_j, i.e. if they belong to $\mathcal{O}(\Delta_j)$.

Of course, if φ already belongs to $\mathcal{O}(\Delta_j)$ then $R_i\varphi \in \mathcal{O}(\Delta_j)$ for all $1 \leq i \leq k$. So we are only interested in the <u>non-trivial solutions</u>, which consists of the quotient space of all solutions/trivial solutions. This is a complex vector space which we denote by $\mathcal{C}_j(x_o : t_j : \Delta_j)$ and they are defined provided Δ_j is a sufficiently small polydisc centered at $(x_o, \alpha_j(x_o))$, i.e. it suffices to know that $\Delta_j \subset B(\delta_o, \varepsilon_o)$.

Well, this is just a definition. The result which can be proved is now the following

4.3. <u>Proposition</u> Given x_o in $B^*(\delta_o)$ and some $1 \leq j \leq e$ there exists a polydisc $\Delta_j^{\bullet}$ centered at $(x_o, \alpha_j(x_o))$ such that the following is true for every polydisc $\Delta_j \subset \Delta_j^o$ and any point t_j chosen so that (x_o, t_j) is in $\Delta_j - S$ *we get:*

The solution space $\mathcal{C}_j(x_o : t_j : \Delta_j)$ is finite dimensional and its dimension is independent of both Δ_j and t_j and equals the multiplicity of the stalk of $\mathcal{E}/\mathcal{L}$ at the point $\mathcal{T}_j(x_o)$ defined in 4.1.

4.4. <u>Remark</u> The assertion above uses different notations as in | 1 | where our solution spaces $\mathcal{C}_j(x_o : t_j : \Delta_j)$ are interpreted from the vector space $Hom_{\mathcal{E}^\infty_{\mathcal{T}_j(x_o)}}(\mathcal{M}^\infty_{\mathcal{T}_j(x_o)}, \mathcal{C}^{\mathbb{R}}_{S|r_j})$ where $r_j = (x_o, \alpha_j(x_o))$. However, the actual proof of the classification theorem and the fact that the sheaf $\mathcal{L}$ is generated by sections $R_1 \dots R_k$ which belong to $\mathcal{E}(x,t,D_t) \langle D_x \rangle$ implies that the "naive" solution spaces $\mathcal{C}_j(x_o : t_j, \Delta_j)$ introduced above are the good micro-local solution spaces <u>which are ample enough to determine the stalk</u> $\mathcal{M}^\infty_{\mathcal{T}_j(x_o)}$ also.

To be precise, rather than working with $\mathcal{E}/\mathcal{L}$ we may work with the sheaf $\mathcal{R}/(\mathcal{R}R_1 + \dots + \mathcal{R}R_k)$ where $\mathcal{R} = \mathcal{E}(x,t,D_t) \langle D_x \rangle$, and the claim that the solution spaces $\mathcal{C}_j(x_o : t_j, \Delta_j)$ are ample means this

4.5. Proposition If $R \in \mathcal{E}^{\infty}_{p_0}(x,t,D_t) \langle D_x \rangle$ so that R is a finite sum $\sum A_\alpha(x,t,D_t)D_x^\alpha$ where $A_\alpha \in \mathcal{E}^{\infty}_{p_0}(x,t,D_t)$ then R operates on $\mathcal{O}$ inside $B(\delta_0,\varepsilon_0)$ if δ_0 and ε_0 are small. If R then is the zero mapping on a solution space $\mathcal{L}_j(x_0:t_j,\Delta_j)$- where Δ_j is some small polydisc centered at $(x_0,\alpha_j(x_0))$, then the image of R in $\mathcal{E}^{\infty}_{\gamma_j(x_0)}$ belongs to the left ideal which $\mathcal{L}$ generates there. Finally, this conclusion holds for each $1 \leq j \leq e$.

Well, we do not try to provide detailed proofs of the assertions made so far. As indicated they are rather easy, once the classifiration theorem has been proved. To finish we give

4.6. Definition When $x_0 \in B^*(\delta_0)$ and when $1 \leq j \leq e$ is given, then $\mathcal{L}_j(x_0)$ is the inductive limit of the solution spaces $\mathcal{L}_j(x_0:t_j,\Delta_j)$ as Δ_j shrink to $(x_0,\alpha_j(x_0))$ and $(x_0,t_j) \in \Delta_j - S$.

Using the version of Hartog's Theorem, as explained in Section 2.3. together with Proposition 4.5. and divisions in the stalk $\mathcal{E}^{\infty}_{p_0}$ which reduce any given germ P there to a germ which belongs to

$$\mathcal{E}^{\infty}_{p_0}\mathcal{L}_{p_0} + \mathcal{E}^{\infty}_{p_0}(x,t,D_t) \langle D_x \rangle$$

and the fact that if x_0 is given then the e-tuple of points $(x_0,\alpha_j(x_0))$: $1 \leq j \leq e$ provide points on every component of S_{reg}, implies

4.7. Proposition The direct sum $\mathcal{L}_1(x_0) \oplus \ldots \oplus \mathcal{L}_e(x_0)$ is a finite dimensional complex vector space, and as $x_0 \to 0$ these spaces are ample enough to determine $\mathcal{M}^{\infty}_{p_0}$.

Returning to the notations used in Section 2.7., it remains now to show that there exists a surjective mapping from $Hom_{\mathcal{E}^{\infty}_{p_0}}(\mathcal{M}^{\infty}_{p_0}, \mathcal{C})$ onto $\oplus \mathcal{L}_j(x_0)$. provided that x_0 is sufficiently close to the origin.

In other words, we must prove that any e-tuple of local solutions can be obtained from a "global solution" . The remaining sections are devoted to the proof of this. In [1] the surjectivity is proved by means of a quite general machinery. See for example [1 ,Proposition 4.4.1]. Here we try to supply a more self-contained proof.

5. The passage from local to global solutions

Now we enter a more detailed analysis. Recall that generators $R_1 \dots R_k$ have been introduced. The fact that the fiber $\mathbb{C}^{\mathbf{z}} p_o$ is isolated in $\sigma(\mathcal{L})^{-1}(0)$ can be used to show that the k-tuple $R_1 \dots R_k$ contains special operators $P_o, Q_1 \dots Q_n$ which we describe below.

5.1. $P_o = P_o(x,t,D_t)$ has order zero and $\sigma(P) = p^w$ for some $w \geq 1$. So introducing a kernel for the part of order -1, we can write

$$P_o(x,t,D_t) = p(x,t)^w + \int K(x,t,u)du$$

5.2. To each $1 \leq i \leq n$, the operator $Q_i = D_{x_i}^{\omega} + \sum_{\nu=1}^{\omega} A_v(x,t,D_t) D_{x_i}^{\omega - v}$ where $\mathrm{ord}(A_v) \leq v$ for each $1 \leq v \leq \omega$ and ω is some positive integer.

5.3. Remark The existence of operators $P_o, Q_1 \dots Q_n$ in the stalk $\mathcal{L}_{p_o}$ follows by division theorems in $\mathcal{E}_{p_o}$. See for example [1 ;Section 3-5] for a similar construction, where the notations differ from ours since the sheaf $\mathcal{M}$ is not assumed to be cyclic from the start.

5.4. Semi-local solutions

Recall that we always consider a polydisc $B(\delta_o : \varepsilon_o)$ where actions by $R_1 \dots R_k$ on $\mathcal{O}$ exist. In general we shall let $\delta_o << \varepsilon_o$. In particular we can assume that δ_o and ε_o have been chosen such that if $t_o = -\varepsilon_o/4$ say, then (x,t_o) is outside the hypersurface S for all $|x| < \delta_o$. In fact, δ_o may even be chosen so small that when $x_o \in B^{\mathbf{z}}(\delta_o)$ then the roots $\alpha_j(x_o)$ all have absolute value $< \varepsilon_o/4$. Having made this choice we give

5.5. Definition A germ φ in $\mathcal{O}_{(x_o,t_o)}$ is called a semi-local solution if there exists some $\delta > 0$ such that $R_i\varphi$ are holomorphic in the polydisc

$B(x_o,t_o:\delta,\varepsilon_o/2) = \{ (x,t) : |x-x_o| < \delta \text{ and } |t-t_o| < \varepsilon_o/2 \}$.

<u>A remark</u> Since $x_o \in B^{*}(\delta_o)$ here we observe that if δ is sufficiently small then there also exists some $\varepsilon > 0$ such that the open sets

$$W_j = \{ (x,t) : |x-x_o| < \delta \text{ and } |t-\alpha_j(x)| < \varepsilon \} \text{ are pairwise disjoint,}$$

which amonuts to say that $S \cap B(x_o:t_o:\delta,\varepsilon_o/2)$ is decomposed into e components given by the equations $t = \alpha_j(x)$.

5.6. <u>The analytic continuation</u> Given a semi-local solution φ, the fact that $P_o\varphi \in \mathcal{O}(B(x_o,t_o:\delta:\varepsilon_o/2)$ implies that φ extends to a multi-valued analytic function on $B(x_o,t_o:\delta,\varepsilon_o/2) - S$. Indeed, this is a consequence of the analysis to be given in Section 8. See also Proposition 6.10.

Admitting this, a semi-local solution <u>produces local solutions</u>. In fact, to each $1 \leq j \leq e$ we choose some path γ_j in the complex t-plane whose initial point $\gamma_j(0) = t_o$ while the end-point $\gamma_j(1) = t_j$ is close to the root $\alpha_j(x_o)$. At the same time $(x_o, \gamma_j(s))$ stay outside S for all $0 \leq s \leq 1$ where the path γ_j is a continuous mapping $s \to \gamma_j(s)$.

Given φ, we then take its analytic continuation along the path Γ_j where $\Gamma_j(s) = (x_o, \gamma_j(s))$ and arrive at the germ $(\varphi)_{\Gamma_j}$ in $\mathcal{O}_{(x_o,t_j)}$ which then gives a local solution, i.e. an element in $\mathcal{L}_j(x_o)$. So for an e-tuple of paths $\gamma_1 \dots \gamma_e$ chosen as above, we get the e-tuple of local solutions $(\varphi)_{\Gamma_1} \oplus \dots \oplus (\varphi)_{\Gamma_e}$ in the space $\oplus \mathcal{L}_j(x_o)$. The material to be presented in Section 6.12 gives then

5.7. <u>Proposition</u> Given any e-tuple $(\xi_1 \dots \xi_e)$ in $\mathcal{L}_j(x_o)$, there exist semi-local solutions $\varphi_1 \dots \varphi_e$ and paths $\gamma_1 \dots \gamma_e$ so that $\xi_j = (\varphi_j)_{\Gamma_j}$ for each j.

5.8. The passage from semi-local to global solutions

Recall that the family $R_1 \dots R_k$ contains the operators $Q_1 \dots Q_n$. They are of a form which enable us to apply the Cauchy-Kowalevsky Theorem and show that a semi-local solution φ can be replaced by a global solution without changing the images in $\oplus\, \mathcal{C}_j(x_o)$. The result we need for this is

5.9. Proposition There exist positive constants δ_1 and K- which only depend on the operators $Q_1 \dots Q_n$, such that the following is true:

If $\varphi \in \mathcal{O}_{(x_o,t_o)}$ is a semi-local solution then there exists another germ $\tilde{\varphi}$ in $\mathcal{O}_{(x_o,t_o)}$ which satisfies

(1) $\varphi - \tilde{\varphi}$ extends to $\mathcal{O}(B(x_o,t_o:\delta,\varepsilon_o/2K)$ for some $\varepsilon > 0$

(2) $R_v\tilde{\varphi}$ belong to $\mathcal{O}(B(x_o,t_o:\delta,\varepsilon_o/2K)$ for all $1 \le v \le k$.

(3) The germ $\tilde{\varphi}$ is holomorphic in a polydisc $B(x_o,t_o:\delta_1,\varepsilon)$ where $\varepsilon > 0$ in general is a small number.

Conclusions The result above holds for any point (x_o,t_o) which is sufficiently close to the origin. Keeping δ_1 and K as above we make a good choice. For example, we choose $|t_o| < \varepsilon_o/4K$ and choose δ_o so small that the roots $|\alpha_j(x)| < \varepsilon_o/4K$ for all $1 \le j \le e$ and all x in $B^*(\delta_o)$. Then (1) in Proposition 5.9. implies that the images of φ and $\tilde{\varphi}$ in $\oplus\, \mathcal{C}_j(x_o)$ are equal because the polydisc $B(x_o,t_o:\delta,\varepsilon_o/2K)$ contains the points $(x_o,\alpha_j(x_o))$.

Next, since the operators R_v all belong to $\mathcal{E}_{p_o}(x,t,D_t) < D_x >$ it is clear that (3) implies that $R_v\tilde{\varphi}$ also are holomorphic in the polydisc $B(x_o,t_o:\delta_1,\varepsilon)$.

Well, then we combine this with (2) and a classical result- due to Reinhardt- which implies that the functions $R_v\tilde{\varphi}$ actually are holomorphic in the polydisc $B(x_o:t_o:\delta_1/2,\varepsilon_o/4K)$. Finally, with δ_1 and K fixed here we may assume that $|x_o| < \delta_1/2$ and that $|t_o| < \varepsilon_o/4K$ so that $R_v\tilde{\varphi}$ extend to holomorphic functions in a neighborhood of the origin and this means that $\tilde{\varphi}$ is a global solution.

Of course, to finish the proof we then have to show that the germ $\tilde{\varphi}$ extends to a multi-valued function in $\Delta - S$ for some polydisc Δ centered at the origin. This analytic extension is not at all trivial to achieve, the proof uses method similar to those in the work by <u>Nilsson</u>. See |12| and [14] and it was also used by Kawai and Kashiwara in [1]. In Section 8 we shall describe how the analytic continuation of $\tilde{\varphi}$ is proved.

<u>Summing up</u>, the material so far has finished our account of the proof of the Reconstruction Theorem, where 3 essential details have been omitted, namely Proposition 5.7. and 5.9. and the fact that global solutions, which <u>a priori</u> consist of germs φ for which $R_v\varphi$ extend to $\mathcal{O}(\Delta)$, extend by themselves to multi-valued functions in $\Delta - S$.

The remaining sections contain material which supply proofs of these assertions.

6. The integral operator $p^w + \int K(x,t,u)$

To simplify the subsequent notations we shall replace x_o by 0 and t_o by 0 and normalize ε_o to be = 1. The assumptions below reflect the situation which occurs in Section 5.4.

So we consider a function $p(x,t) = \prod (t-\alpha_j(x))$ where the roots $\alpha_1(x) \dots \alpha_e(x)$ are distinct for all $|x| \leq \delta_o$ and also $0 < |\alpha_j(x)| < 1$ hold then.

Let also $K(x,t,u)$ be holomorphic in a neighborhood of the closed polydisc where $|x| \leq \delta_o$ and both $|t|$ and $|u|$ are ≤ 1.

To a given postitive integer w we then study the operator $p^w + \int K$ and we begin to study

6.1. <u>The freezed equations</u> With $|x_o| < \delta_o$ given we consider the operator $\mathcal{P}_{x_o}$ which maps a germ $\varphi(t)$ at the origin in the complex t-plane to the germ $\mathcal{P}_{x_o}\varphi(t) = p(x_o,t)\varphi(t) + \int_o^t K(x_o,t,u)\varphi(u)du$

We say that a germ φ is a solution if $\mathcal{P}_{x_o}\varphi$ extends to a continuous function on the closed disc $D = \{ t:|t| \leq 1 \}$ which in addition is holomorphic in the interior D. In other words, φ is a solution when $\mathcal{P}_{x_o}\varphi$ belongs to the Banach space A(D), where A(D) is the famous disc algebra.

Of course, if the germ φ already belongs to A(D), then φ is a trivial solution. So we give

6.2. <u>Definition</u> The space $\mathcal{L}_{x_o}$ = solutions/trivial solutions, is called the space of non-trivial solutions to $\mathcal{P}_{x_o}$.

Now we can prove

6.3. <u>Proposition</u> $\mathcal{L}_{x_o}$ is an ω-dimensional complex vector space, where ω = we.

The proof is an easy consequence of the following two preliminary results.

6.4. <u>Lemma</u> $\mathcal{P}_{x_o}$ is bijective on the space of germs at the origin.

6.5. <u>Lemma</u> The operator $\mathcal{K}_{x_o} g = \int_o^t K(x_o,t,u)g(u)du$ is compact on A(D).

We leave out the easy proofs .

Proof of Proposition 6.3. The operator $g \to p^W(x_0,t)g(t)$ on $A(D)$ is obviously injective, while its cokernel is of dimension $= \omega$. Since $\mathcal{K}_{x_0}$ is a compact pertubation by Lemma 6.5. , it follows that $\mathcal{P}_{x_0}$ has index ω as an operator on $A(D)$. It is easily seen that its kernel is zero and hence its cokernel is ω-dimensional. Choose then $h_1 \dots h_\omega$ in $A(D)$ so that

$$A(D) = \mathrm{Im}(\ \ _{x_0}) \oplus \mathbb{C}h_1 \oplus \dots \oplus \mathbb{C}h_\omega \text{ holds.}$$

Using Lemma 6.4. we get unique germs φ_j satisfying $\mathcal{P}_{x_0}\varphi_j = h_j$ in $C\{t\}$ and then it is easily seen that $\varphi_1 \dots \varphi_\omega$ is a $\mathbb{C}$-basis of $\mathcal{L}_{x_0}$.

6.6. Equations with parameters Since solutions to the freezed equations have been found in an effective way, we expect that they can be obtained in such a way that their dependence on x is analytic. Well, the result below shows that this is so

6.7. Theorem There exists some $\varepsilon > 0$ and holomorphic functions $\varphi_1(x,t) \dots \varphi_\omega(x,t)$,all defined in the polydisc $B(\delta_0,\varepsilon)$ such that:
$\forall\, x_0$ with $|x_0| < \delta \Rightarrow$ the functions $\varphi_1(x_0,t) \dots \varphi\ (x_0,t)$ is a basis for the non-trivial solution space $\mathcal{L}_{x_0}$.

Proof Suppose first that we have found functions $H_1(x,t) \dots H_\omega(x,t)$ which are holomorphic when $|x| < \delta_0$ and $|t| < 1$ and continuous when $|t| = 1$ such that $A(D) = \mathrm{Im}(\mathcal{P}_{x_0}) \oplus H_1(x_0,t) \oplus \dots \oplus H\ (x_0,t)$ hold for every x_0.

Then we find the function $\varphi_j(x,t)$ by solving the equation

$\mathcal{P}_x\varphi_j(x,t) = H_j(x,t)$ and Theorem 6.7. follows. It remains only to see why the family $H_1 \dots H_\omega$ exists. For the unpertubed operator p^W , the ω-dimensional cokernel spaces are generated by the holomorphic functions $G_{v,j}(x,t) = p^W(x,t)/(t-\alpha_j(x))^v$ where $1 \leq j \leq e$ and $1 \leq v \leq e$. The existence of $H_1 \dots H_\omega$ for the pertubed operators follows easily then. In fact, the standard proofs of the Index Theorem for compactly pertubed linear operators on Banach spaces, gives the result below which is applied to conclude that $H_1 \dots H_\omega$ exist.

6.8. Proposition Let $x \to T_x$ be a holomorphic function, defined in a neighborhood of the polydisc $|x| \leq \delta$, with values in $L(B,B)$ = the space of bounded linear operators on a Banach space. Assume that T_x are injective for all x and that there exists an integer ω and B-valued holomorphic functions $G_1 \dots G_\omega$ such that the decomposition $B = \mathrm{Im}(T_x) \oplus \mathbb{C}G_1(x) \oplus \dots \oplus \mathbb{C}G_\omega(x)$ hold for all x. If now $x \to K_x$ is holomorphic, where K_x are compact operators on B and where $T_x + K_x$ are injective for all x, then there exist B-valued holomorphic functions $H_1 \dots H_\omega$ such that $B = \mathrm{Im}(T_x+K_x) \oplus \mathbb{C}H_j(x)$ hold for all x.

6.9. Remarks about Theorem 6.7. Consider one of the φ-functions in Theorem 6.7. For example, put $\varphi = \varphi_1$ to simplify the notations. If we first introduce the positive number $\mu = \inf \{ |\alpha_j(x)| : 1 \leq j \leq e \text{ and } |x| \leq \delta_0 \}$ which by assumption is positive, then the fact that

$p(x,t)\varphi(x,t) + \int_0^t K(x,t,u)\varphi(x,u)du$ belongs to $B(\delta_0:1)$ and that $p(x,t) \neq$ when $|t| < \mu$ and $|x| \leq \delta_0$, easily implies that $\varphi(x,t)$ extends to a holomorphic function in the polydisc $B(\delta_0,\mu) = \{ (x,t) : |x| < \delta_0 \text{ and } |t| < \mu \}$.

In fact, this follows from an estimate of the Taylor expansion $\varphi(x,t) = \Sigma\, \varphi_j(x)t^j$ and the fact that the kernel $K \in \mathcal{O}(B(\delta_0:1,1))$. The question arises if we can continue φ to a larger subset of the polydisc $B(\delta_0,1)$.

This turns put to be true. For the proof we use the assumption that the roots $\alpha_1(x) \dots \alpha_e(x)$ are distinct and the result is

6.10. Proposition Each function φ_j from Theorem 6.7. extends to a (in general) multi-valued analytic function on $B(\delta_0,1) - p^{-1}(0)$.

6.11. Remark The proof is actually not as trivial as one may expect, using the easy observation that if x_0 is fixed. the fact that $\varphi(x_0,t)$ is a solution to the freezed equation, and an easy 1-dimensional analysis then implies that $t \to \varphi(x_0,t)$ extends to a multi-valued function in the punctured disc $\{ t : |t| < 1 \text{ and } t \neq \alpha_j(x_0) \text{ for all } j \}$.

In fact, even though φ continues analytically along the paths described above, we cannot conclude that φ then prolonges to a multi-valued function on $B(\delta_0,1) - p^{-1}(0)$ because we must verify the existence of analytic continuations along other paths too. However, using some elementary facts in homotopy theory, this can be proved and we shall describe this method in the proof of Theorem 8.4 which can be used to prove Proposition 6.10. too.

6.12. <u>How to deduce Proposition</u> 5.7. Let us first remark that the solution spaces $\mathcal{L}_{x_0}$ to the freezed equations have as many non-trivial solutions as we could hope for. So this means that Theorem 6.7. really gives the optimal number of non-trivial solutions. This has the following consequence.

Let $|x_0| < \delta_0$ and some $1 \leq j \leq e$ be given ana suppose that $\Delta_j = B(x_0,\alpha_j(x_0):\delta,\varepsilon)$ is a small polydisc centered at $(x_0,\alpha_j(x_0))$ so that the intersection $\Delta_j \cap p^{-1}(0)$ is given by the equation $t = \alpha_j(x)$. Let t_j be a point $|t_j-\alpha_j(x_0)<\varepsilon|$ and $(x,t) \notin p^{-1}(0)$ when $|x-x_0|<\delta$ and le $f \in \mathcal{O}_{(x_0,t_j)}$ satisfying: The germs $p(x,t)^w f(x,t) + \int_{t_j}^{t} K(x,t,u)f(x,u)du$ extends to the whole polydisc Δ_j. So f is like <u>a local solution</u> as described in Section 4.2.

Let then x_1 be given with $|x_1-x_0| < \delta$. A 1-dimensional analysis and the fact that $\varphi_1(x_1,t) \ldots \varphi_\omega(x_1,t)$ is a basis for $\mathcal{L}_{x_1}$, implies that there exist unique complex scalars $c_1(x_1) \ldots c_\omega(x_1)$ such that:

(1) $f(x_1,t) - \Sigma\, c_j(x_1)\tilde{\varphi}_j(x_1,t)$ is holomorphic in the disc $|t-\alpha_j(x_0)| < \varepsilon$ where $\tilde{\varphi}_1 \ldots \tilde{\varphi}_\omega$ are local branches of the multi-valued extensions of $\varphi_1 \ldots \varphi_\omega$ at the point (x_0,t_j) - all obtained by a continuation along <u>a fixed path</u> γ from $(x_0,0)$ to (x_0,t_j).

Recall also that t_j is chosen so that (x,t_j) stays outside $p^{-1}(0)$ when $|x-x_0| < \delta$. Hence $\tilde{\varphi}_1 \ldots \tilde{\varphi}_\omega$ exist when $|x-x_0| < \delta$ and $t=t_j$ so the scalars $c_1(x) \ldots c_\omega(x)$ can be determined for all $|x-x_0| < \delta$, and by their uniqueness they will be holomorphic in the polydisc $|x-x_0| < \delta$.

Summing up, we have found

6.13. Proposition If $f \in \mathcal{O}_{(x_o,t_j)}$ is a local solution then there exist holomorphic functions $c_1(x) \dots c_\omega(x)$ defined in some polydisc $B(x_o:\delta)$ so that $f(x,t) - \sum_{\nu=1}^{\omega} c_\nu(x)\tilde{\varphi}_\nu(x,t)$ belongs to $\mathcal{O}(B(x_o,\alpha_j(x_o):\delta,\varepsilon)$

A Remark During the preceeding discussion we were choosing analytic extensions of the ω-tuple $\varphi_1 \dots \varphi_\omega$. The choice of a path γ does determine the ω-tuple $c_1(x) \dots c_\omega(x)$ then. However, the fact that $c_1(x) \dots c_\omega(x)$ exist does not depend on the chosen path γ from $(x_o,0)$ to (x_o,t_j). The reason is that the ω-tuple $(\varphi_1 \dots \varphi_\omega)$ determines a local system of rank ω in $B(\delta_o,1) - p^{-1}(0)$.

The precise meaning of this assertion is

6.14. Proposition Let γ and λ be two paths in $B(\delta_o,1) - p^{-1}(0)$ having $\gamma(o) = \lambda(o) = (x_o,o)$ and $\gamma(1) = \lambda(1) = (x_1,t_1)$. By analytic contonuations along γ and along λ respectively, we get ω-tuples of germs in $\mathcal{O}_{(x_1,t_1)}$ denoted by $\{ (\varphi_j)_\gamma \}$ and $\{ (\varphi_j)_\lambda \}$ respectively. Then there exists an invertible matrix $A(x)$ of size (ω, ω) with coefficients in the local ring $\mathcal{O}_{x_1}$, such that the vector $(\varphi_j)_\lambda = A(x)(\varphi_j)_\gamma$ in $\mathcal{O}^{\omega}_{x_1,t_1}$.

We leave out the proof, which again is an easy consequence of 1-dimensional analysis. Finally, at this stage we leave it to the reader to deduce Proposition 5.7. from the detailed material above.

7. A Cauchy-Kowalevsky Theorem

The operators $Q_1 \dots Q_n$, introduced in Section 5.2. are of the so called Cauchy-Kowalevsky type with respect to $D_{x_1} \dots D_{x_n}$ respectively. Proposition 5.9. is an easy consequence of the Cauchy-Kowalevsky Theorem, applied to each of the operators $Q_1 \dots Q_n$ and a classical result about analytic continuations in Reinhardt domains. The details are as follows.

7.1. The Cauchy-Kowalevsky Theorem for each Q_i will first be recalled.

7.2. Proposition $\exists$ constants K and δ_1 such that the following two results hold for each $1 \leq i \leq n$

(1) If $g \in \mathcal{O}(B(x_o, t_o : \delta, \varepsilon)$ where $|x_o| + \delta < \delta_o$ and $|t_o| + \varepsilon < \varepsilon_o$ then the inhomogenous equation $Q_i h = g$ is solvable in $B(x_o, t_o : \delta, \varepsilon/K)$

(2) If $\varphi \in \mathcal{O}_{(x_o, t_o)}$ satisfies $Q_i \varphi = 0$ in $\mathcal{O}_{(x_o, t_o)}$ then φ extends to a polydisc whose x_i-radius is δ_1, while the radius with respect to other coordinates $x_1 \dots x_i \dots x_n$ and t remain small.

Applying Proposition 7.2. n times, i.e. to each $Q_1 \dots Q_n$ we get

7.3. Corollary Let φ be a germ in $\mathcal{O}_{(x_o, t_o)}$ such that $Q_i \varphi \in \mathcal{O}(B(x_o, t_o : \delta, \varepsilon_o/2))$ for all i. Then there exists another germ $\tilde{\varphi}$ in $\mathcal{O}_{(x_o, t_o)}$ satisfying

(1) $\varphi - \tilde{\varphi}$ is holomorphic in $B(x_o, t_o : \delta, \varepsilon_o/2K^n)$

(2) The germ $\tilde{\varphi}(x,t)$ is holomorphic in $B(x_o, t_o : \delta_1, \varepsilon)$ for some small $\varepsilon > 0$

(3) $Q_i \tilde{\varphi}$ are holomorphic in $B(x_o, t_o : \delta, \varepsilon_o/2K^n)$ for all i.

A Remark Notice here that δ is a small number, i.e. we study functions which from start are defined in small neighborhoods of x_o.

Sketch of proof First the inhomogenous equation $Q_1 g = Q_1 \varphi$ is solved so that g belongs to $\mathcal{O}(B(x_o, t_o : \delta, \varepsilon_o/2K)$. Put then $\varphi_1 = \varphi - g$ which then is x_1-holomorphic in a disc of radious δ_1 by (2) in Proposition 7.2. Then the inhomogenous equation $Q_2 h = \varphi_1$ is solved and we put $\varphi_2 = \varphi_1 - h_1$, and so on. The function $\varphi_n = \tilde{\varphi}$ satisfies (1)-(3) in the Corollary.

7.4. <u>An improvement of</u> (3) in Corollary 7.3. For a given i the function $Q_i\varphi$ is holomorphic in a Reinhardt domain centered at (x_o,t_o), i.e. in the union $B(x_o,t_o:\delta_1,\varepsilon) \cup B(x_o,t_o:\delta,\varepsilon_o/2K^n)$ and this implies that these functions extend to be holomorphic in the polydisc $B(x_o,t_o,\delta_1/2,\varepsilon_o/4K^n)$

7.5. <u>How to deduce Proposition</u> 5.9. Well this is obvious because the operators $Q_1 \dots Q_n$ are present in the family $R_1 \dots R_k$. Of course, the constant K in Proposition 5.9. should now be chosen as $1/2K^n$ with K as in Corollary 7.3.

Notice here that (2) in Proposition 5.9. follows from (1) and the fact that φ from start is a semi-local solution. So the "adding of equations" does not change anything in Proposition 5.9., simply because the semi-local solution φ was given.

8 Analytic continuations

In this section we shall study analytic continuations of certain integrals. We are going to use methods similar to those employed by Nilsson in [12] and [14]. Let us begin with the following set-up.

8.1. $p(x,t) = t^e + \rho_1(x)t^{e-1} + \dots + \rho_e(x)$ is a reduced Weierstrass polynomial, where $\rho_1 \dots \rho_e$ are holomorphic in a neighborhood of $|x| \leq \delta_o$

The roots $\alpha_1(x) \dots \alpha_e(x)$ have absolute value < 1, and we also assume that t_o is a given point such that $t_o \neq \alpha_j(x)$ for all j and all $|x| \leq \delta_o$.

8.2. The set $B^* = B(\delta_o, 1) - (S \cup Z)$ where $S = p^{-1}(0)$ and Z is the locus of the discriminant of p, i.e. $Z = \{ x : \text{At least two roots } \alpha_j(x) \text{ are equal}\}$

8.3. The kernel $K(x,t,u)$ is holomorphic in $|x| \leq \delta_o$ and both $|t|$ and $|u| \leq 1$.

Out aim is then to prove

8.4. Theorem Let $\varphi_o \in \mathcal{O}_{(x_o,t_o)}$ for some $|x_o| < \delta$ and assume that φ_o can be continued to a multi-valued analytic function Φ in B^*. Then the germ at (x_o,t_o) defined by

$$\psi_o(x,t) = \int_{t_o}^{t} K(x,t,u)\varphi_o(x,u)du$$ also extends to a multi-valued analytic function Ψ in B^*. Finally, if Φ belongs to the Nilsson class, so does Ψ.

The proof of the existence of the multi-valued extension of φ_o is an easy consequence of the following elementary result.

8.5. A Homotopy Lemma Let γ be a path in B^* having (x_o,t_o) as initial point. Write $\gamma(s) = (\gamma_x(s), \gamma_t(s))$ for $0 \leq s \leq 1$ and let us put $x_1 = \gamma_x(1)$ and $t_1 = \gamma_x(1)$ so that (x_1,t_1) is the end-point of γ.

Then we get the path Γ in B^* with $\Gamma(s) = (\gamma_x(s), t_o)$ which has (x_1,t_o) as end-point. With these notations we have:

Claim $\exists$ a path ρ in the punctured t-disc $= \{ t : t \neq \alpha_j(x_1) \}$ which moves from t_o to t_1, such that γ is homotopic to the composed path $\hat{\rho} \circ \Gamma$ in B^*, where $\hat{\rho}(s) = (x_1, \rho(s))$ for all $0 \leq s \leq 1$.

Proof of Theorem 8.4. Using the homotopy lemma, the existence of the multi-valued extension Φ easily follows. If γ is a path in B^* with $\gamma(o) = (x_o,t_o)$ then the germ $(\varphi_o)_\gamma$ which arises by analytic continuation along is given by the following sum:

$$\int_{t_o}^{t_1} K(x,t,u)\tilde{\varphi}(x,u)du + \int_{t_1}^{t} K(x,t,u)(\varphi_o)_\gamma (x,u)du$$

where $(\varphi_o)_\gamma$ is the germ at the end point $\gamma(1) = (x_1,t_1)$ arising by the analytic continuation of φ_c along γ. In the first integral, the integration in the complex u-plane is along the path ρ from 8.5. and during this integration we have considered the continuation of φ_o, first along the path Γ from (x_o,t_o) to (x_1,t_o) and then followed along the path $\hat{\rho}$.

The case when Φ belongs to the Nilsson class Using Definition 4.3.4. in [11, page 255], the proof that Ψ belongs to the Nilsson class if Φ does is achieved as follows. Given the end-point (x_1,t_1) of a path γ we define the number $\mu(x_1,t_1) = \inf \{ |t-\alpha_j(x_1)|, |\alpha_j(x_1)-\alpha_v(x_1)| : v \neq j$ and both v and j from 1 to $e \}$.

Then the path ρ in 8.5. can be chosen so that its distance from the roots $\alpha_1(x_1) \dots \alpha_e(x_1)$ stays $\geq \mu(x_1,t_1)/2$ and then the temperated grwoth condition on Ψ can be derived using similar arguments as in [12]. A notable point here, is that if the growth of the Nilsson class function Φ is some number $\rho \geq 0$, i.e. if $\mathrm{dist}((x,t),Z \cup S)^\rho \, |\Phi(x,t)|$ is locally bonded - see [11, Definition 4.3.4.] for the precise meaning of the order of growth of a Nilsson class function, then the order of growth of Ψ is $\leq \rho\rho'$, where ρ' is a fixed number determined so that $\mu(x_1,t_1) \geq A\mathrm{dist}((x,t),Z \cup S)^{\rho'}$ hold for all (x,t) in B^*, where A is some positive constant. This can for example be used to give an alternative proof of Theorem 5.1.1. in [1].

8.5. The extension of local solutions

Recall that in Section 2 we have claimed that a global solution can be continued to a multi-valued analytic function in $B(\delta_o,\varepsilon_o) - S$ when $\delta_o << \varepsilon_o$ and ε_o is sufficiently small. The extension to the complement of $S \cup Z$ is easily obtained using the proof of Theorem 8.4., but it is not obvious why Z can be removed. In [1] this is proved on page 4-5-5 and here we shall present an alternative proof, using our set-up.

So let φ_o be a global solution. This means that we can find positive numbers $\delta_o < \varepsilon_o$ such that the following conditions hold:

(1) $\exists$ a fixed point t_o with $|t_o| < \varepsilon_o$ such that (x,t) is outside $S = p^{-1}(0)$ when $|x| < \delta_o$

(2) $\varphi_o(x,t)$ is holomorphic in some polydisc $B(x_o,t_o:\delta_o,\varepsilon)$ with ε small

(3) $p^{w}(x,t)\varphi_o(x,t) + \int_{t_o}^{t} K(x,t,u)\varphi_o(x,u)du$ extends to a holomorphic function in $B(\delta_o,\varepsilon_o)$

<u>A notation</u> We put $B^{*} = B(\delta_o,\varepsilon_o) - (S \cup Z)$ where Z is the locus of the discriminant of p.

8.6. <u>Remark</u> Of course, the local solution φ_o actually satisfies more equations, i.e. $R_v\varphi_o$ extend to $B(\delta_o,\varepsilon_o)$ for all v. However, to prove that φ_o extends to $B(\delta_o,\varepsilon_o) - S$ it suffices to use the equation (3), where the reader may observe that use of the other equations already has been made in the passage from semi-local to global solutions, which implies that φ_o can be assumed to satisfy (2) from the start.

<u>Proof of the extension to</u> $B(\delta_o,\varepsilon_o) - S$. We are going to use the ω-tuple $\varphi_1 \dots \varphi_\omega$ from Section 6. These functions also satisfy (2) and (3) above, and as we have already said (2) and (3) and the method used to prove Theorem 8.4. implies that $\varphi_o,\varphi_1 \dots \varphi_\omega$ all extend to B^{*} and it remains to get rid of Z.

To prove this we shall first study the actual continuation inside B^{*}. So let γ be a path in B^{*} where $\gamma(o) = (x_o,t_o)$ and $\gamma(1) = (x_1,t_1)$. Then we find that the germ $(\varphi_o)_\gamma$ at (x_1,t_1)- arising by the analytic continu-

-ation along γ satisfies

Formula 1. $p^{w}(\varphi_0)_{\gamma}(x,t) + \int_{t_1}^{t} K(x,t,u)(\varphi_0)_{\gamma}(x,u)du + \Psi(x,t)$

holds in $\mathcal{O}_{(x_1,t_1)}$, where $\Psi(x,t) = \int_{t_0}^{t_1} K(x,t,u)\varphi_0(x,u)du$ and the integration takes place along a path ρ from t_0 to t_1 which stays outside the roots $\alpha_j(x_1)$, as explained in the proof of Theorem 8.4.

Since $\Psi(x,t)$ is defined with t_0 and t_1 as fixed end-points in the integral, and since the kernel $K(x,t,u)$ is holomorphic from start, it follows that $\Psi(x,t)$ is holomorphic in a polydisc $|x-x_1| < \delta$ and $|t| < \varepsilon_0$ already, i.e. the freezed functions $t \to \Psi(x,t)$ belong to the Banach space $A(D)$ for all $|x-x_1| < \delta$, where δ is small and $D = \{t : |t| \leq \varepsilon_0\}$.

Using Formula 1 and the material from Section 6.12-13, we then find

Formula 2. $\exists$ unique germs $c_1(x) \ldots c_{\omega}(x)$ in $\mathcal{O}_{x_1}$ and a function $H(x,t)$ which is holomorphic when $|t| \leq \delta_0$ and $|x-x_1| < \delta$, such that

$$(\varphi_0)_{\gamma}(x,t) = c_1(x)(\varphi_1)_{\rho}(x,t) + \ldots + c_{\omega}(x)(\varphi_1)_{\rho}(x,t) + H(x,t)$$

holds in $\mathcal{O}_{(x_1,t_1)}$, where $(\varphi_j)_{\rho}$ are the local branches which arise by analytic continuation of φ_j which is performed as in Section 6.12. That is, first $\varphi_j(x,t)$ is holomorphic in a polydisc $|x| \leq \delta_0$ and $|t-t_0| < \varepsilon$ - see 6.9. Remark- and staying there we consider the germ of φ_j at (x_1,t_0) which then is continued along the path $\hat{\rho}$ with $\hat{\rho}(s) = (x_1, \rho(s))$ and where $s \to \rho(s)$ moves in the punctured t-disc from t_0 to t_1- avoiding the roots $\alpha_j(x_1)$. Here ρ is the same path used to define Ψ.

Proof continued Of course, in Formula 2, the germs $c_1 \ldots c_{\omega}$ in $\mathcal{O}_{x_1}$ and the function $H(x,t)$ both depend on the path γ.

If we write $\gamma(s) = (\gamma_x(s), \gamma_t(s))$, then $c_1 \ldots c_{\omega}$ are continued analytically along the path $s \to \gamma_x(s)$ from x_0 to x_1. However, this analytic continuation is trivial, because from the start, the given germ φ_0 at (x_0,t_0) already satisfies (2)- so that

$\varphi_o(x,t) = C_1(x)\varphi_1(x,t) + \dots + C_\omega(x)\varphi_\omega(x,t) + H_o(x,t)$ holds in $\mathcal{O}_{(x_o,t_o)}$ where C_j already are holomorphic in $|x| < \delta_o$. In other words, $c_1 \dots c_\omega$ are simply analytic extensions of given holomorphic functions in $|x| \leq \delta_o$. So $c_j = C_j$ holds all the time. In the same way, the germ $H(x,t)$ arises by analytic continuation of H_o, so it is holomorphic everywhere too. Summing up, we have achieved

<u>Sublemma</u> 3 $(\varphi_o)_\gamma(x,t) = C_1(x)(\varphi_1)_\rho(x,t) + \dots + C_\omega(x,t)(\varphi_\omega)_\rho(x,t) + H(x,t)$ hold, where $C_1 \dots C_\omega$ and H are holomorphic in $B(\delta_o,\varepsilon_o)$.

At this stage the continuation to $B(\delta_o,\varepsilon_o) - S$ is evident, because Sublemma 3 asserts that the analytic continuation of φ_o along any path γ in B^* is achieved by analytic continuations of $\varphi_1 \dots \varphi_\omega$ along the simpler paths ρ which arise from γ in view of the Homotopy Lemma 8.5. So at this stage the continuation to $B(\delta_o,\varepsilon_o) - S$ follows from the observation already made in Section 6, namely that the freezed functions $t \to \varphi_j(x_1,t)$ extend along any path ρ from t_o inside the punctured disc where $t \neq \alpha_j(x_1)$, and that this is true <u>even if multiple zeros occur</u>.

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