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LOWER BOUNDS FOR PSEUDO-DIFFERENTIAL OPERATORS

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In this lecture we shall discuss conditions under which a pseudo-differential operator of second order

$$(Au)(x) = \int e^{i<x,\xi>} a(x,\xi)\hat{u}(\xi)\,d\xi$$

with real symbol $a(x,\xi)$ will be bounded from below on $L^2(\mathbb{R}^n)$. That $a(x,\xi) \geqslant 0$ is a sufficient condition has now been known for some time (see [1]); on the other hand, if we keep in mind applications to subellipticity, it is important to allow $a(x,\xi)$ to take negative values. For this purpose, errors of the size of some Sobolev norm $\|u\|^2_{(\delta)}$ of very small order δ are negligible, and it suffices to establish estimates of the form

$$\mathrm{Re} < [A + C(1 - \Delta)^{\delta}]u,u> \geqslant 0 \tag{1}$$

Even so, the problem becomes considerably more difficult.

Earlier work of Melin [5] had led Hörmander [6] to study positivity under a condition on the Hessian of $a(x,\xi)$; however, the Heisenberg uncertainty principle and the invariance of (1) under canonical transformations suggest considering instead the symplectic geometry of the set

$$S = \{(x,\xi) \in T^*(\mathbb{R}^n) \ ; \ a(x,\xi) < 0\}$$

The following theorem provides a sufficient condition which is essentially also necessary.

Fix $0 < \delta << 1$ and for each $\hbar \in \mathbb{R}^+$ denote by $II_{\hbar}$ the cube

$$II_{\hbar} = \{(x,\xi) \in T^*(\mathbb{R}^n), |x| \leqslant \hbar \ , |\xi| \leqslant \hbar\}$$

<u>THEOREM</u> : There exist constants $0 < \hbar << \hbar^*$ and a family $\mathcal{F}$ of canonical transformations such that

(i) if S does not contain the image of $II_{\hbar}$ by any element of $\mathcal{F}$, then (1) will hold ;

(ii) conversely, (1) implies that the set $S^* = \{(x,\xi) \in T^*(\mathbb{R}^n);$ $a(x,\xi) + C(1+|\xi|^2)^{\delta} < 0\}$ does not contain the image of $II_{\hbar^*}$ by any element in $\mathcal{F}$.

The family $\mathcal{F}$ can be in fact constructed explicitly, and all its elements satisfy good bounds. This eliminates the difficulties that might arise from having to consider the infinite dimensional class of all canonical transformations.

The proof of the theorem is based on a new microlocalization procedure which we believe will prove to be a useful method on its own. Omitting the exact estimates involved, it can be roughly described as follows :

First, by using a classical $S^o_{\rho,\delta}$ partition of unity we may restrict our attention to a rectangular block centered at say (x^o,ξ_o), and of size $|\xi_o|^{-(\text{small power})} \times |\xi_o|^{1-(\text{small power})}$. The estimate (1) reduces to an L^2 bound from below for a pseudo-differential operator with symbol $p = |\xi_o|^{-(\text{small power})}(a(x,\xi))$ defined on the rectangular block. After a simple dilation $p(x,\xi)$ may then be viewed as a symbol of second order defined on a square of sides $M^{1/2} \times M^{1/2}$, where $M = |\xi_o|^{1-(\text{small power})}$. By definition, this is stage 0, to which we associate the canonical transformation Φ_o = Identity.

Assume now that we are at stage ℓ, where a rectangular block in phase space $II = \prod_{j=1}^{n} I_{x_j} \times I_{\xi_j}$ and a canonical transformation Φ_ℓ defined with good bounds on a dilate of II are given, satisfying the following properties :

(a) $$|I_{x_k}| = M_k^{1/2} \quad , \quad |I_{\xi_k}| = M_\ell M_k^{-1/2} \qquad (k \leqslant \ell)$$

$$|I_{x_k}| = |I_{\xi_k}| = M_\ell^{1/2} \qquad (k \geqslant \ell)$$

(b) If we set

$$\hat{p}_\mu = p \circ \Phi_\ell \qquad \text{for} \quad \mu = 0$$

$$\hat{p}_\mu(x_{\mu+1},\ldots,x_n, \xi_{\mu+1},\ldots, \xi_n) =$$

$$\frac{1}{\left|\prod_{j=1}^{\mu} I_{x_j}\right|} \int_{\prod_{j=1}^{\mu} I_{x_j}} (p \circ \Phi_\ell)(x_1,\ldots,x_n,0,\ldots,0, \xi_{\mu+1},\ldots, \xi_n)\,dx_1\ldots dx_\mu$$

for $1 \leqslant \mu \leqslant \ell$, then $\hat{p}_\mu$ satisfies good bounds on a dilate of $\prod_{j=\mu+1}^{n} (I_{x_j} \times I_{\xi_j})$ and can be written as

$$\hat{p}_\mu = M_{\mu+1}\,\xi_{\mu+1}^2 + \tilde{p}_\mu(x_{\mu+1},\dots,x_n,\ \xi_{\mu+2},\dots,\xi_n) \quad \text{when} \quad 1 \leqslant \mu < \ell$$

To pass from stage ℓ to stage $(\ell + 1)$, cut II and its adjacent congruent rectangles into 2^{2n} smaller rectangles, by cutting I_{x_k} and I_{ξ_k} into 2 equal subintervals when $k \geqslant \ell + 1$, I_{ξ_k} into 4 equal subintervals and retaining I_{x_k} when $k \leqslant \ell$. Observe that the product $|I_{x_j}| \times |I_{\xi_j}|$ remains independent of j. The arguments of [2] (Lemmas 3.2 and 3.3) show that repeating the process if necessary and stopping as soon as we can will yield a decomposition of the double of II into a union of rectangles $\bigcup \mathrm{II}_{(\lambda)}$, each of which will satisfy one of the following conditions.

(C_1) $\quad \min \{ \sum_{k\leqslant \ell} M_k^\lambda \xi_k^2,\ \xi_k \in I_{\xi_k}\ (k\leqslant \ell)\} \geqslant (\text{constant})(M_{\ell+1}^\lambda)^2$

(C_2) $\quad \hat{p}_\ell \geqslant (\text{constant})(M_{\ell+1}^\lambda)^2 \quad \text{on} \quad \prod_{k\geqslant \ell+1} I_{x_k} \times I_{\xi_k}$

(C_3) $\quad (M_{\ell+1}^\lambda)^2 \leqslant (\text{constant})$

(C_4) $\quad \min_{\prod_{k\geqslant \ell+1}(I_{x_k} \times I_{\xi_k})} \hat{p}_\ell \leqslant -(\text{constant})(M_{\ell+1}^\lambda)^2$

(C_5) $\quad \max_{|\alpha|+|\beta|=2} \|D_x^\alpha D_\xi^\beta \hat{p}_\ell\|_{L^\infty(\prod_{k\geqslant \ell+1} I_{x_k} \times I_{\xi_k})}$

$$\geqslant (\text{constant})(M_{\ell+1}^\lambda)^2 \prod_{k\geqslant \ell+1} |I_{x_k}|^{-\alpha_k} |I_{\xi_k}|^{-\beta_k}$$

(when $a(x,\xi)$ satisfies the condition in part (i) of the theorem, no rectangle of type (C_4) will arise). Among the new rectangles $\mathrm{II}_{(\lambda)}$ retain those satisfying (C_1) - (C_4), and define the final straightening Φ of p there to be the canonical transformation $\Phi_\ell \big|_{(\text{dilate of } \mathrm{II}_{(\lambda)})}$; as for those satisfying (C_5), we may choose a canonical transformation Ψ^λ defined with good bounds on a dilate of $\prod_{k\geqslant \ell+1} I_{x_k} \times I_{\xi_k}$ so that

$$\hat{p}_\ell \circ \Psi^\lambda (x_{\ell+1},\dots,x_n, \xi_{\ell+1},\dots, \xi_n) = M^\lambda_{\ell+1} \quad \xi^2_{\ell+1} + \tilde{p}_\ell (x_{\ell+1},\dots,x_n,\xi_{\ell+2},\dots,\xi_n)$$

Setting $\Phi_{\ell+1} = \Phi_\ell \circ (\mathrm{Id} \times \psi^\lambda)$ will bring us back to a situation entirely analogous to the one we started from at stage ℓ . Carrying out the same procedure repeatedly will ultimately yield a rectangle of the (C_1) - (C_4) type, or after n steps a canonical transformation Φ_n defined on a dilate of a rectangle II. Let

$$\hat{p}_n = \frac{1}{\left|\prod_{k=1}^{n} I_{x_k}\right|} \int_{\prod_{k=1}^{n} I_{x_k}} (p \circ \Phi_n)(x,0)\,dx$$

and decompose II and its adjoint congruent boxes by cutting I_{ξ_k} into 2 equal intervals, retaining the I_{x_k}'s, and stopping when either

$$|I_{x_k}|^2 \; |I_{\xi_k}|^2 \leqslant \max\{\text{large constant}, \hat{p}_n\}$$

or

$$\min_{\prod_k I_{\xi_k}} \left(\sum_k M_k \, \xi_k^2 \right) \geqslant (\text{constant}) |I_{x_k}|^2 \; |I_{\xi_k}|^2$$

In this manner we obtain new rectangles $\{II_{(\lambda)}\}$, to which we associate the corresponding final straightenings defined by $\Phi = \Phi^n|_{II_{(\lambda)}}$.

This completes the description of the microlocalization procedure to be carried out for any given symbol. Observe that phase space has been decomposed into an elaborate system of images by canonical transformations of rectangles of varying sizes, decomposition which is much more intricate than any of those previously known. The key fact, however, is that the patching up of estimates from stage (ℓ) to stage $(\ell+1)$ for each fixed ℓ can be accomplished by a well established calculus of pseudo-differential operators.

It thus suffices to establish positivity on each cube of the final straightening Φ. If the area of the corresponding block II is bounded, the hypothesis that S does not contain the image of II_K by canonical transformations implies that p is bounded from below. It is then possible to establish the desired estimate by making use of the spectral decomposition theorem of [2].

When the area of II is large, we argue by induction in the size of p. More precisely, for each $N \geqslant 0$ consider the final straightening corresponding to the symbol $2^{-N}p$. Estimates for the final straightening of $2^{-N}p$ for N large enough are trivial, and estimates for $2^{-N+1}p$ can be obtained from those for $2^{-N}p$, provided we appeal to the following Geometric Lemma, which constitutes the essential part of this work.

Geometric Lemma : Let $II^{[N]}$ be a rectangle in the final straightening of $2^{-N}p$ and $\Phi_{[N]}$ the associated canonical transformation. Assume that the size of $II^{[N]}$ is sufficiently large. If (x^*, ξ_*) is any point in a dilate of $II^{[N]}$, $\Phi_{[N+1]}$ the final straightening of $2^{-(N+1)}p$ near $\Phi_{[N]}(x^*, \xi_*)$, and $II^{[N+1]}$ the corresponding rectangle, then

(a) the size of $II^{[N+1]}$ is also large ;

(b) if we denote by $\imath_N$(Unit cube)$\mapsto II^{[N]}$, $\imath_{N+1}$: (Unit cube) $\longrightarrow II^{[N+1]}$ the natural changes of scales, the mapping

$$\imath^{-1}_{N+1}\ \Phi^{-1}_{[N+1]}\Phi_{[N]}\imath_N$$

and its inverse are well defined with good bounds on a ball of fixed radius around $\imath^{-1}_N(x^*,\xi_*)$ and its image.

Detailed proofs and applications can be found in [3] and [4].

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