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VARIATIONAL METHODS IN NONLINEAR PROBLEMS

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This lecture presents several methods for finding stationary points of functions defined on finite or (often) infinite dimensional spaces in case the functions are unbounded from above and below -- so that one seeks stationary points which are, in general, saddle points. Applications to the problem of finding nontrivial time-periodic solutions of a nonlinear string equation

$$u_{tt} - u_{xx} + g(u) = 0 \quad (1)$$

are presented. Here u is defined for $0 \leq x \leq \pi$, $t \in R$, and is to satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$ and to be periodic in t with given period T.

The author has recently published an expository article [10] devoted to variational and topological methods, and this lecture takes up some of the material of [10].

In treating variational problems on infinite dimensional spaces it is now rather standard to make use of a kind of compactness condition that was introduced by R. Palais and S. Smale often called the (P.S.) condition. For a real C^1 function f defined on a Banach space X it takes the form:

(P.S.): Any sequence $\{x_j\} \in X$ for which

(i) $|f(x_j)|$ is bounded,

and

(ii) $f'(x_j)$, as an element of the dual space X^*, tends strongly to zero,

has a strongly convergent subsequence $\{x_{j_i}\}$.

This is a rather strong condition. For instance if f satisfies (P.S.) and is bounded from below then f achieves its minimum.

The variational methods described here are all extensions of the Mountain Pass Lemma:

(MPL) Let f be a C^1 real function on X satisfying (PS). Assume that for some open neighborhood U of the origin in X we have

$$f(0) < c_1 \le f(u) \qquad \forall u \in \partial U ,$$

and assume that for some $u_0 \notin \bar{U}$,

$$f(u_0) < c_1 .$$

Then the following number c is a stationary value of f:

$$c = \inf_P \max_{u \in P} f(u) \ge c_1 ,$$

where P is any continuous path in X going from u_0 to 0, and we take the infimum with respect to all paths.

If $X = R^2$ and $f(x)$ represents the height of land above the point $u \in X$, then o lies in a valley U. The number c then represents the height of the lowest mountain pass crossing the mountain range ∂U from u_0 to 0.

This intuitively obvious lemma appeared for the first time in this form only in 1973 in a paper of Ambrosetti and Rabinowitz [1]. Together with its various extensions it has proved to be very useful. Here is an extension due to Rabinowitz [11] which we present in a slightly more general form of Ni [9].

(Gen'd MPL): Let f be a real C^1 function on X satisfying (PS). Let ϕ be a continuous map of the k-sphere S^k into X and assume

$$\max_{x \in S^k} f(\phi(x)) < c_1 . \tag{2}$$

Assume furthermore that for every continuous extension $h: B^{k+1} \to X$ of the map ϕ inside the unit ball B^{k+1} in R^{k+1} into the space X,

$$\max_{x \in B^{k+1}} f(h(x)) \geq c_1 . \tag{3}$$

Then

$$c = \inf_h \max_{x \in B^{k+1}} f(h(x)) \geq c_1$$

is a stationary value of f.

The conditions (2), (3) are difficult to verify in practice, and in [11] and [9] useful sufficient conditions are given. References to further extensions and related variational principles may be found in [10]. See in particular the papers there: [54], [29] and [8-10].

Turning to equation (1) we consider g monotone and satisfying $g(0) = 0$ and $|g|$ growing faster than linearly as $|u| \to \infty$ -- for example $g = u^3$. We suppose also that T/π is rational. In [11] Rabinowitz applied the Gen'd MPL to obtain nontrivial solutions for (1). Adapting a dual variational method of Ekeland [8], Brézis, Coron and Nirenberg [6] have given a different proof and extension of the results of Rabinowitz -- using the original (MPL). Further references may be found there and in [10]. Here are some more recent papers by Bahri and Brézis [5] and by Coron [7]. Further variational techniques have been devised by Bahri and Berestycki, see [2-4].

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