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REGULARITY OF THE $\bar{\partial}$-NEUMANN PROBLEM

by

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Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a smooth boundary $b\Omega$. If $z_1,\dots,z_n$ are coordinates in $\mathbb{C}^n$ we set $x_j = \mathrm{Re}(z_j), y_j = \mathrm{Im}(z_j)$ and, when u is a differentiable function, we define :

(1)
$$u_{z_j} = \frac{1}{2}\,(u_{x_j} - \sqrt{-1}\; u_{y_j})$$
$$u_{\bar{z}_j} = \frac{1}{2}\,(u_{x_j} + \sqrt{+1}\; u_{y_j})\ .$$

Given $\alpha_1,\dots,\alpha_n$ on Ω we will study the system

(2)
$$u_{\bar{z}_j} = \alpha_j \text{ for } j = 1,\dots,n.$$

We will assume that the α_j satisfy the compatibility conditions

(3)
$$\alpha_{j\bar{z}_k} = \alpha_{k\bar{z}_j}\ .$$

We shall also assume that $b\,\Omega$ is pseudo-convex, that is if r is a real C^∞ function defined in a neighborhood of $b\,\Omega$ such that $dr \neq 0$, $r < 0$ in Ω and $r > 0$ outside of $\bar{\Omega}$, if $P \in b\,\Omega$ and if $(\zeta_1,\dots,\zeta_n) \in \mathbb{C}^n$ satisfy

(4)
$$\Sigma\, r_{z_i}(P)\, \zeta_i = 0$$

then we have

(5)
$$\Sigma\, r_{z_i \bar{z}_j}(P)\, \zeta_i\, \bar{\zeta}_j \geq 0.$$

<u>Definition</u> : If $u \in L_2(\Omega)$ we define the <u>singular support</u> of u, denoted by s.s.(u), to be the subset of $\bar{\Omega}$ defined as follows. The point $P \notin$ s.s.(u) if and only if there exists a neighborhood U of P such that the restriction of u to $U \cap \bar{\Omega}$ is in $C^\infty(U \cap \bar{\Omega})$.

This lecture will be concerned with the following two problems.

<u>Problem I.</u> : Given $\alpha_j \in L_2(\Omega)$, $j = 1,\dots,n$ satisfying (3) does

there exist a $u \in L_2(\Omega)$ such that

$$\text{s.s.}(u) \subset \bigcup_j \text{s.s.}(\alpha_j) \quad ? \tag{5}$$

<u>Problem II.</u> : Let $\mathcal{H}(\Omega)$ denote the subspace of $L_2(\Omega)$ consisting of holomorphic functions. Let u be the unique solution of (2) such that $u \perp \mathcal{H}(\Omega)$. Does u satisfy (5) ?

Observe that the system (2) is elliptic thus $\Omega \cap \text{s.s.}(u)$ is contained in the singular support of the α_j. Hence problem I is only interesting for points in $b\,\Omega$. It is easy to construct a holomorphic function h which is singular at $b\,\Omega$, then if u satisfies (2) so does $u + h$, so that (5) cannot hold for all solutions of (2). In case the right side of (5) is empty we have the following result.

<u>Theorem</u> : If Ω is pseudo-convex and if $\alpha_j \in C^\infty(\overline{\Omega})$, $j = 1,\dots,n$ satisfying (3), then there exists $u \in C^\infty(\overline{\Omega})$ which satisfies (2).

It is at present unknown whether, under the hypotheses of the above theorem, the solution of (2) which is orthogonal to $\mathcal{H}(\Omega)$ is also in $C^\infty(\overline{\Omega})$. One of the reasons for the solution orthogonal to $\mathcal{H}(\Omega)$ is because the Bergman projection $B : L_2(\Omega) \longrightarrow \mathcal{H}(\Omega)$ can be expressed in terms of it. Namely, if $f \in L_2(\Omega)$, if the $f_{z_j} \in L_2(\Omega)$ and if $u \perp \mathcal{H}(\Omega)$ satisfies $u_{z_j} = f_{z_j}$ for $j = 1,\dots,n$ then clearly Bf, the orthogonal projection of f in $\mathcal{H}(\Omega)$, is given by :

$$Bf = f - u. \tag{6}$$

This (5) implies $\text{s.s.}B(f) \subset \text{s.s.}(f)$. By a result of Bell and Ligocka regularity properties of B imply boundary regularity of biholomorphic maps.

It is known that the solution of Problem I depends on the geometry of Ω. The following results of Catlin show this .

<u>Theorem</u> (Catlin) : If Ω is pseudo-convex and if V is a complex-analytic curve contained in $b\,\Omega$ then there exists α_j satisfying (3) such that (5) is not satisfied for any solution of (2).

Theorem (Catlin) : There exists a pseudo-convex domain $\Omega \subset \mathbb{C}^3$ such that $b\,\Omega$ does not contain any complex curves and such that there exist α_j as above, so that (5) does not hold.

The $\bar\partial$-Neumann problem enables one to express the solution of II in terms of a boundary value problem. In particular we have the notion of subellipticity of this problem defined as follows.

Definition : The $\bar\partial$-Neumann problem on Ω is subelliptic at $x_o \in b\,\Omega$ if there exists a neighborhood U of x_o and constants $\varepsilon > 0$, $c > 0$ such that for all n-tuples $(\varphi_1,\dots,\varphi_n)$ with $\varphi_j \in C_o^\infty(U \cap \bar\Omega)$ and

(7) $$\sum r_{z_j} \varphi_j = 0 \text{ on } b\,\Omega$$

we have

(8) $$\sum \|\varphi_j\|_\varepsilon^2 \leq C\Big(\sum_1^n \|\varphi_{j\bar z_k} - \varphi_{k\bar z_j}\|^2 + \|\sum_1^n \varphi_{j\,z_j}\|^2\Big) ,$$

where $\|\ \|_\varepsilon$ denotes the ε-Sobolev-norm.

The quadratic from defined by the left side of (8) is connected with the solution u of II as follows. If $(\alpha_1,\dots,\alpha_n)$ satisfies (3) and $(\varphi_1,\dots,\varphi_n)$ with $\varphi_j \in C^\infty(\bar\Omega)$ satisfies (7) and

(9) $$\sum_{j,k} (\varphi_{j\bar z_k} - \varphi_{k\bar z_j}, \Psi_{j\bar z_k} - \Psi_{k\bar z_j}) + (\sum_j \varphi_{j z_j}, \sum_k \Psi_{k\bar z_k})$$
$$= \sum (\alpha_j, \Psi_j) ,$$

for all $(\Psi_1,\dots,\Psi_n)$ with $\Psi_j \in C^\infty(\bar\Omega)$ and $\sum r_{z_j} \Psi_j = 0$ on $b\,\Omega$. Then

(10) $$u = - \sum_j \varphi_{j z_j}$$

is the unique solution of (2) which is orthogonal to $\mathcal{H}(\Omega)$.

Proposition : If the $\bar\partial$-Neumann is subelliptic at $x_o \in b\,\Omega$, if $(\alpha_1,\dots,\alpha_n)$ satisfies (3) and if $\alpha_j \in H^s_{loc}(x_o)$ then $u \in H^{s+\varepsilon}_{loc}(x_o)$, where $u \perp \mathcal{H}(\Omega)$ and u satisfies (2). In particular we have

(11) $$U \cap \text{s.s.}(u) \subset U \cap \bigcup_j \text{s.s.}(\alpha_j).$$

Hence if the $\bar{\partial}$-Neumann problem is subelliptic at all points in $b\,\Omega$ then II is settled affirmatively. From Catlin's result it then follows that subellipticity cannot hold at $x_o \in b\,\Omega$ if there is a complex-analytic curve through x_o contained in $b\,\Omega$. Catlin also obtained a quantitative measure of the dependence of subelliplicity on how close curves through x_o can be to $b\,\Omega$.

<u>Theorem (Catlin)</u> : Suppose Ω is pseudo-convex, $x_o \in b\,\Omega$ and (8) holds for some $\varepsilon > 0$. Suppose further thatthere is a complex-analytic curve V with $x_o \in V$, which has the property that there exists a neighborhood U of x_o and constants $C > 0$, $\eta > 0$ such that

$$(12) \qquad |r(z)| \leq C|z - x_o|^{\eta}$$

for all $z \in U \cap V$. Then $\varepsilon \leq \frac{1}{\eta}$.

<u>Definition</u> : Suppose $x_o \in b\,\Omega$, let $\mathcal{V}(x_o) = \mathcal{V}$ denote the set of germs of complex analytic curves through x_o. The order of x_o denoted by $\mathcal{O}(x_o)$ is defined by

$$(13) \qquad \mathcal{O}(x_o) = \sup_{V \in \mathcal{V}} \mathcal{O}(x_o, V) ,$$

where $\mathcal{O}(x_o,V)$,the order of contact of V to $b\,\Omega$ at x_o, is defined by

$$(14) \quad \mathcal{O}(x_o,V) = \sup\{\eta \mid \exists \text{ a neighborhood } U \text{ of } x_o \text{ and } C > 0 \text{ so that (12) holds for all } z \in U \cap V\}.$$

Denoting by $\mathcal{W}$ the set of germs of non-singular complex-analytic curves at x_o we define reg $\mathcal{O}(x_o)$ the <u>regular order</u> of x_o by

$$(15) \qquad \text{reg } \mathcal{O}(x_o,V) = \sup_{V \in \mathcal{W}} \mathcal{O}(x_o,V).$$

It is clear that for $n = 1$ we have $\mathcal{O}(x_o)=\text{reg } \mathcal{O}(x_o) = 1$ and that if $n > 1$ then $\mathcal{O}(x_o) \geq \text{reg } \mathcal{O}(x_o) \geq 2$. Furthermore, it is easy to show that for $n = 2$ and Ω pseudo-convex we have $\mathcal{O}(x_o)=\text{reg } \mathcal{O}(x_o)$. However for $\Omega \subset \mathbb{C}^3$, we have the following example (studied by Bloom and Graham), let Ω be given by

$$(16) \qquad r(z) = \text{Re}(z_3) + |z_1^2 - z_2^3|^2 + \exp(-1/|z|^2)$$

then for $x_o \neq 0$ we have $\mathcal{O}(x_o)$ = reg $\mathcal{O}(x_o) = 2$ and when $x_o = 0$, we have reg $\mathcal{O}(o) = 6$ and $\mathcal{O}(o) = \infty$.

D'Angelo has shown that for pseudo-convex domains if reg $\mathcal{O}(x_o) \leq 4$ than $\mathcal{O}(x_o)$ = reg $\mathcal{O}(x_o)$.

Returning to subellipticity for pseudo-convex domains the convex of Catlin's theorem holds if $n \leq 2$ and if $\mathcal{O}(x_o)=2$, that is in those cases when $\mathcal{O}(x_o) < \infty$, then subellipticity holds at x_o for all $\varepsilon \leq \frac{1}{\mathcal{O}(x_o)}$. A recent example of d'Angelo shows that this is not true in general. Let $\Omega \subset \mathbb{C}^n$ for $n \geq 3$ defined by :

$$r(z) = \mathrm{Re}(z_n) + \sum_{j=1}^{n-2} |z_j^m - z_n z_{j+1}|^2 + |z_{n-1}^m|^2 , \tag{17}$$

for this domain $\mathcal{O}(x_o) = 2m$ when $x_o = (0,\ldots,0)$, $\mathcal{O}(x_o) = 2$ when $z_{n-1}(x_o) \neq 0$ and $\mathcal{O}(x_o) = (2m)^n$ when $x_o = (0,\ldots,0,i\delta)$ whenever $\delta \in \mathbb{R} - \{0\}$. Catlin shows that subellipticity at $x_o = (0,\ldots,0)$ holds for $\varepsilon \leq (2m)^{-n}$ but does not hold for $\varepsilon > (2m)^{-n}$. Using the phenomena involved in d'Angelo's example Catlin has constructed for each ε_o, with $0 < \varepsilon_o \leq \frac{1}{4}$, a domain such that subellipticity holds with $\varepsilon = \varepsilon_o$ but does not hold for $\varepsilon > \varepsilon_o$. Furthermore he constructs a domain such that subellipticity holds for all $\varepsilon < \varepsilon_o$ but does not hold for $\varepsilon = \varepsilon_o$.

The major problem now is whether $\mathcal{O}(x_o) < \infty$ implies subellipticity on pseudo-convex domain. Here we will briefly describe the sufficient conditions that are known.

<u>Definition</u> : If $x_o \in b\,\Omega$ we define a succession of ideals $I_1(x_o) \subset I_2(x_o) \subset \ldots \subset I_k(x_o) \subset C^\infty(x_o)$, where $C^\infty(x_o)$ denotes the ring of germs of C^∞ functions at x_o. We define by

$$\lambda = \det \begin{pmatrix} 0 & r_{z_1} & \cdots\cdots & r_{z_n} \\ r_{\bar z_1} & r_{z_1 \bar z_1} & \cdots\cdots & r_{z_n \bar z_1} \\ \vdots & \vdots & & \vdots \\ r_{\bar z_n} & r_{z_1 \bar z_n} & \cdots\cdots & r_{z_n \bar z_n} \end{pmatrix} \tag{18}$$

Let $I_1(x_o) = \sqrt[\mathbb{R}]{(\lambda)}$, where (λ) denotes the ideal generated by λ and $\sqrt[\mathbb{R}]{\ }$ is the real radical defined as follows. If $J \subset C^\infty(x_o)$ is an ideal then $\sqrt[\mathbb{R}]{J} = \{f \in C^\infty(x_o) \mid \exists\, m$ and $g \in J$ with $|f|^m \leq |g|\}$. To define $I_k(x_o)$ consider for any $f^{(1)},\dots,f^{(n+1)} \in I_{k-1}(x_o)$ the $(n+1) \times 2(n+1)$ matrix

$$(19) \qquad \begin{pmatrix} 0 & r_{z_1} & \cdots & r_{z_n} \\ r_{\bar z_1} & r_{z_1\bar z_1} & \cdots & r_{z_n\bar z_1} \\ \vdots & \vdots & & \\ r_{\bar z_n} & r_{z_1\bar z_n} & \cdots & r_{z_n\bar z_n} \\ 0 & f^{(1)}_{z_1} & \cdots & f^{(1)}_{z_n} \\ 0 & f^{(n+1)}_{z_1} & \cdots & f^{(n+1)}_{z_n} \end{pmatrix}$$

Let $D^k(f^{(1)},\dots,f^{(n+1)})$ denote the set of all determinants of $(n+1)\times(n+1)$ minors of the above matrix. Then we define $I_k(x_o)$ by

$$(20) \qquad I_k(x_o) = \sqrt[\mathbb{R}]{(I_{k-1}(x_o), \cup D^k(f^{(1)},\dots,f^{(n+1)}))}\ ,$$

where the union is taken over all $(n+1)$-tuples in $I_{k-1}(x_o)$.

<u>Theorem</u> : If Ω is pseudo-convex and if $1 \in I_k(x_o)$ the subellipticity holds at x_o .

In case r is real-analytic in a neighborhood of x_o it can be shown (using a result of Diederich and Fornaess) that for pseudo-convex domain the condition $1 \in I_k(x_o)$ for some k is equivalent to the non-existence of complex-analytic curves through x_o which lie in $b\,\Omega$.

Recent result of d'Angelo give a new analysis of the notion of order of x_o and in many cases give sharp bounds for the $\mathcal{O}(x)$ in terms of $\mathcal{O}(x_o)$ when x is near x_o.

References : The recent results of Catlin and d'Angelo are not yet published. For the other references and a more detailed survey see : "Several complex variables from the point of view of linear partial differential equations" by J.J. Kohn in Proc. Conf. on PDE and Diff. Geom. 1980, Beijing. Acad. Sinica.

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