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ON THE LAGUERRE CALCULUS OF LEFT-INVARIANT CONVOLUTION (PSEUDO-DIFFERENTIAL) OPERATORS ON THE HEISENBERG GROUP

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1. INTRODUCTION AND THE MAIN RESULT.

The purpose of this article is to derive a multiplicative symbolic calculus for left invariant zero order homogeneous pseudo-differential operators on the Heisenberg group, H_1. At the end, I shall indicate that the restriction "zero order" can be discarded and, with some obvious modifications so can "homogeneity". But, for the sake of clarity, I shall restrict the discussion to operators appearing in my first sentence.

The simplest noncommutative nilpotent Lie group is the, so called, Heisenberg group, H_1, with underlying manifold $\mathbb{R}^3 = \{(x_1,x_2,t)\}$ and with the group law

$$(1.1)\quad (x_1,x_2,t)\,(x_1',x_2',t) = (x_1+x_1' \,,\, x_2+x_2' \,,\, t+t' + 2[x_2x_1' - x_1x_2']) \quad .$$

(1.1) should be looked upon as the non abelian analogue of Euclidean translation on $\mathbb{R}^3$. Note that $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ and writing $z = x_1+ix_2$ the Heisenberg group law can be written in the following symbolic form

$$(1.2)\qquad (z,t).(z',t') = (z+z',t+t' + 2\,\mathrm{Im}\, z\bar{z}') \quad .$$

Here (z,t) stands for $(z,\bar{z},t)$ or (x_1,x_2,t). With this convention, I shall continue using the notation (z,t) for points of $\mathbb{C} \times \mathbb{R}$. The unit of H_1 is $(z,t) = (0,0)$ and

$$(z,t)^{-1} = (-z,-t) \quad .$$

Given functions, $\varphi,\psi \in C_0^\infty(\mathbb{R}^3)$ the H-convolution (Heisenberg convolution) is usually defined by

$$(1.3)\qquad \varphi *_H \psi(u) = \int_{\mathbb{R}^3} \varphi(v^{-1}u)\,\psi(v)\,dv \;,$$

where $u = (z,t)$, $v = (w,s)$ and $dv = dy_1\,dy_2\,ds$, with $w = y_1+iy_2$, is Lebesgue measure on $\mathbb{R}^3$. Set

$$(1.4)\qquad T_{u_1}\,\varphi(u_2) = \varphi(u_1,u_2) \;;$$

i.e. T_{u_1} is left translation with respect to u_1 . Then

$$(1.5) \qquad \int_{\mathbb{R}^3} \varphi(v^{-1} u_2)\, \psi(u_1 v)\, dv = \int_{\mathbb{R}^3} \varphi(v^{-1} [u_1 u_2])\, \psi(v)\, dv \; ,$$

since $d(u_1^{-1} v) = dv$ follows easily from the definition of the Heisenberg translation. Thus one has

$$(1.6) \qquad (\varphi *_H (T_{u'} \psi))(u) = (T_{u'}(\varphi *_H \psi))(u) \quad .$$

In other words the convolution commutes with left translations on H_1 , or the H-convolution product is associative,

$$(1.7) \qquad \varphi *_H (\psi *_H \chi) = (\varphi *_H \psi) *_H \chi \quad .$$

Now I am ready to introduce principal value convolution operators on H_1 . These are the analogues of Mikhlin-Calderon-Zygmund principal value convolution operators on $\mathbb{R}^2$ (or $\mathbb{R}^n$, in general).

Let

$$(1.8) \qquad r(z,t) = (rz \; , r^2 t) \quad , \qquad r > 0$$

denote the Heisenberg dilation. F is said to be H-homogeneous of degree m on H_1 if

$$(1.9) \qquad f(rz, r^2 t) = r^m f(z,t) \; , \quad r > 0 \quad .$$

Next one introduces a norm in H_1 by

$$(1.10) \qquad |(z,t)| = (|z|^4 + t^2)^{1/4} \quad ,$$

which is H-homogeneous of degree 1. The distance, $d(u,v)$, of the points $u,v \in H_1$ is defined to be

$$(1.11) \qquad d(v,u) = d(u^{-1} v, 0) = |u^{-1} v| \quad .$$

Suppose $G \in C^{\infty}(\mathbb{R}^3\backslash 0)$ is H-homogeneous of degree γ. Then G is integrable near the origin if $\gamma > -4$. This article is mainly concerned with convolution operators on H_1, induced by functions which are H-homogeneous of the critical degree, -4.

<u>1.12 Definition</u> : Let $F \in C^{\infty}(\mathbb{R}^3\backslash 0)$, H-homogeneous of degree -4. F is said to have principal value zero if

$$(1.13) \qquad \int_{d(u,0)=1} f \, d\sigma = 0 \quad ,$$

where $d\sigma$ is the induced measure on the Heisenberg unit ball, $d((z,t),0) = 1$.

The basic result concerning principal value convolution operators on H_1 is the following - see [3].

<u>1.14 Proposition</u> : Let $F \in C^{\infty}(\mathbb{R}^3\backslash 0)$, H-homogeneous of degree -4 with principal value zero. Then F induces a "principal value convolution operator", given by

$$(1.15) \qquad F *_H \varphi(u) = \lim_{\varepsilon \to 0} \int_{d(u,v)>\varepsilon} F(v^{-1}u) \; \varphi(v) \; dv$$

on functions $\varphi \in C_0^{\infty}(\mathbb{R}^3)$. F can be extented to a bounded operator

$$(1.16) \qquad F : L^2(H_1) \to L^2(H_1) \quad .$$

In particular, principal value convolution operators can be composed. Furthermore, their composition yields another principal value convolution operator.

For the rest of this article I shall denote principal value convolution operators by capital letters, F, G, etc.. and their composition, simply by $F *_H G$.

The Euclidean Fourier transform, $\hat{\varphi}$, of $\varphi \in C_0^{\infty}(\mathbb{R}^3)$ is given by

$$(1.17) \qquad \hat{\varphi}(\xi,\tau) = \int_{\mathbb{R}^3} e^{-i<\xi,x>-i\tau t} \; \varphi(x,t) \; dx \; dt \quad ,$$

where $x = (x_1,x_2)$, $\xi = (\xi_1,\xi_2)$, $dx = dx_1\, dx_2$ and $<\xi,x> = \xi_1 x_1 + \xi_2 x_2$. Its inverse is

$$(1.18) \qquad \varphi(x,t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i<x,\xi>+it\tau}\, \hat{\varphi}(\xi,\tau)\, d\xi\, d\tau \quad ,$$

with $d\xi = d\xi_1\, d\xi_2$. If $F \in C^\infty(\mathbb{R}^3\setminus 0)$, H-homogeneous of degree -4 with vanishing principal value, then $\hat{F}$ exists as a tempered distribution. Furthermore $\hat{F}(\xi,\tau) \in C^\infty(\mathbb{R}^3\setminus 0)$ and H-homogeneous of degree zero ([1],[13]). The following result can be found in [1],[7] and [13].

<u>1.19 Proposition</u> : Let F induce a left-invariant principal value convolution operator on H_1. Then $F *_H \varphi$ has the following representation as a pseudo-differential operator

$$(1.20) \qquad F *_H \varphi(x,t) = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i<x,\xi>+it\tau} \,.$$

$$. \hat{F}(\xi_1-2x_2\tau, \xi_2+2x_1\tau, \tau)\, \hat{\varphi}(\xi,\tau)\, d\xi\, d\tau \ ,$$

where $\varphi \in C_o^\infty(\mathbb{R}^3)$.

The best known example of a left invariant principal value convolution operator on H_1 is induced by the singular Cauchy-Szegö kernel, S, given by

$$(1.21) \qquad S(z,t) = \frac{1}{\pi^2(|z|^2-it)^2} \quad .$$

If H_1 is viewed as the boundary of the generalized upper half-plane, $\mathcal{D} = \{\mathrm{Im}\, z_2 > |z_1|^2\} \in \mathbb{C}^2$, then $S *_H \varphi$, $\varphi \in L^2(H_1)$, is the projection of $L^2(H_1)$ on to the boundary values of the Hardy space, $H_2(\mathcal{D})$, of holomorphic functions of $\mathcal{D}$. A simple calculation yields the Fourier transform, $\hat{S}$, of S, namely

$$(1.22) \qquad \hat{S}(\xi,\tau) = \begin{cases} 2e^{-\frac{1}{2}\frac{|\xi|^2}{2\tau}} & ,\tau > 0 \ , \\ 0 & ,\tau < 0 \end{cases}$$

Since S is a projection, one has

$$(1.23) \qquad S *_H S = S \ , \quad \text{or} \quad S^2 = S \ .$$

This does not follow from classical symbol multiplication, as can be seen from

$$\left[2e^{-\frac{|\xi|^2}{4\tau}}\right]^2 \neq 2e^{-\frac{|\xi|^2}{4\tau}} \ , \ \tau > 0 \ .$$

Of course, one expects this, since left-invariant convolution operators on H_1 do not, in general, commute, hence, there can be no commutative symbolic calculus for them. It is useful to prove (1.23) by an explicit calculation because it yields the first clue to finding a multiplicative symbolic calculus, given by infinite matrix symbols (Theorem 1.47), for such operators. I shall carry out this calculation in section 3.

S turns out to be the simplest of a large number of "basic operators" on H_1 induced by Laguerre functions. Laguerre functions have already been used in the study of the "twisted convolution", or, equivalently, the "Heisenberg convolution", for several decades. More precisely, one defines the generalized Laguerre polynomials, $L_k^{(p)}$, $k,p = 0,1,2,\ldots$ via the following generating function formula

$$(1.24) \qquad \sum_{n=0}^{\infty} L_n^{(p)}(x)\, z^n = \frac{1}{(1-z)^{p+1}}\, e^{-\frac{xz}{1-z}} \ .$$

Then

$$(1.25) \qquad \ell_n^{(p)}(x) = \left[\frac{\Gamma(n+1)}{\Gamma(n+p+1)}\right]^{1/2} x^{p/2}\, L_n^{(p)}(x)\, e^{-x/2}$$

are known as the "Laguerre functions", where $x \geq 0$ and $p,n = 0,1,2,\ldots$ It is well known, that

$$(1.26) \qquad \ell_o^{(p)}(x) \ , \ \ell_1^{(p)}(x) \ , \ \ell_2^{(p)}(x) \ , \ \ldots$$

is a complete orthonormal set of functions in $L^2(0,\infty)$ for each $p = 0,1,2,\ldots$ (see [15]).

1.27 Definition : I define the exponential Laguerre functions, $\hat{\mathcal{L}}_k^{(p)}(\xi_1,\xi_2)$, on $\mathbb{R}^2$ by

$$\hat{\mathcal{L}}_k^{(p)}(\xi_1,\xi_2) = 2(-i)^p\,(-1)^k\,\ell_h^{(p)}(|\xi|^2)\,e^{ip\theta} \tag{1.28}$$

and

$$\hat{\mathcal{L}}_k^{(-p)}(\xi_1,\xi_2) = \overline{\hat{\mathcal{L}}_k^{(p)}(\xi_1,\xi_2)} \quad , \tag{1.29}$$

where $k,p = 0,1,2\ldots$ and $\xi = (\xi_1,\xi_2) = |\xi|\,e^{i\theta}$.

Suppose we are given $\hat{F}(\xi,\tau)$, H-homogeneous of degree 0, i.e.

$$\hat{F}(\rho\xi,\rho^2\tau) = \hat{F}(\xi,\tau) \quad , \quad (\xi,\tau) \in \mathbb{C} \times \mathbb{R} \quad .$$

The homogeneity permits us to write $\hat{F}$ as a direct sum of two functions on $\mathbb{C}$, $\hat{F}_+$ for $\tau>0$ and $\hat{F}_-$ for $\tau<0$. Namely

$$\hat{F}(\xi,\tau) = \hat{F}\left(\frac{\xi}{\sqrt{2|\tau|}}, \frac{1}{2}\operatorname{sgn}\tau\right) = \hat{F}_+\left(\frac{\xi}{\sqrt{2|\tau|}}\right) \oplus \hat{F}_-\left(\frac{\xi}{\sqrt{2|\tau|}}\right) , \tag{1.30}$$

where

$$\hat{F}_+ = \begin{cases} \hat{F}\left(\dfrac{\xi}{\sqrt{2|\tau|}}, \dfrac{1}{2}\right) & \text{if } \tau>0 \\ 0 & \text{if } \tau<0, \end{cases} \tag{1.31}$$

and

$$\hat{F}_- = \begin{cases} 0 & \text{if } \tau>0 \\ \hat{F}\left(\dfrac{\xi}{\sqrt{2|\tau|}}, -\dfrac{1}{2}\right) & \text{if } \tau<0 \quad . \end{cases} \tag{1.32}$$

In particular, one has, formally, the decomposition

$$F = F_+ \oplus F_- \quad , \tag{1.33}$$

where

$$F_+(z,t) = \frac{1}{(2\pi)^3}\int_0^\infty e^{it\tau}\,d\tau \int_{\mathbb{R}^2} e^{i\langle\xi,x\rangle}\,\hat{F}_+\left(\frac{\xi}{\sqrt{2|\tau|}}\right) d\xi \quad , \tag{1.34}$$

and

$$(1.35)\qquad F_-(z,t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{o} e^{it\tau}\, d\tau \int_{\mathbb{R}^2} e^{i<\xi,x>}\, \hat{F}_-\left(\frac{\xi}{\sqrt{2|\tau|}}\right) d\xi \quad ,$$

with $z = x_1+ix_2$. Thus Proposition 1.19 implies that

$$(1.36)\qquad F_+ *_H F_- = 0 \quad .$$

$\hat{F}_+$ and $\hat{F}_-$ can be expanded in an exponential Laguerre series, namely, with $\xi = (\xi_1,\xi_2)$,

$$(1.37)\qquad \hat{F}_\pm(\xi) = \sum_{p=-\infty}^{\infty}\ \sum_{k=o}^{\infty} \hat{F}^{(p)}_{\pm,k}\ \hat{\mathfrak{L}}^{(p)}_k(\xi) \quad .$$

This expansion is unique. Given a function $\hat{G} = \hat{G}(\xi_1,\xi_2)$, I collect the coefficients of its expansion in an exponential Laguerre series in the form of an infinite matrix. As a matter os fact I define two infinite matrices, $\hat{\mathfrak{L}}_+(\hat{G})$ and $\hat{\mathfrak{L}}_-(\hat{G})$. First, $\hat{\mathfrak{L}}_+(\hat{G})$ can be written as a sum of matrices, each of whose non zero elements occur in a single (sub -, or super-) diagonal. Thus

$$(1.38)\quad \hat{\mathfrak{L}}_+(\hat{G}) = \ldots + \hat{\mathfrak{L}}^{(-2)}_+(\hat{G}) + \hat{\mathfrak{L}}^{(-1)}_+(\hat{G}) + \hat{\mathfrak{L}}^{(o)}_+(\hat{G}) + \hat{\mathfrak{L}}^{(1)}_+(\hat{G}) + \hat{\mathfrak{L}}^{(2)}_+(\hat{G}) + \ldots ,$$

where, if $p = 0,1,2,\ldots$

$$(1.39)\qquad \hat{\mathfrak{L}}^{(p)}_+ = \mathrm{diag}^{(p)}\ (\hat{G}^{(p)}_o\ ,\ \hat{G}^{(p)}_1\ ,\ \hat{G}^{(p)}_2 ,\ \ldots\ ) \ ,$$

i.e. $\hat{\mathfrak{L}}^{(p)}_+$ is the matrix, whose p-th superdiagonal is

$$(1.40)\qquad (\hat{G}^{(p)}_o\ ,\ \hat{G}^{(p)}_1\ ,\ \hat{G}^{(p)}_2\ ,\ \ldots) \ ,$$

and the rest of its elements are zero.
Similarly,

$$\hat{\mathfrak{L}}^{(-p)}_+(\hat{G}) = \mathrm{diag}_p\ (\hat{G}^{(-p)}_o\ ,\ \hat{G}^{(-p)}_1\ ,\ \hat{G}^{(-p)}_2 ,\ldots),$$

i.e. $\hat{\mathfrak{L}}^{(-p)}_+(\hat{G})$ is the matrix, whose p-th subdiagonal is

$$(\hat{G}_o^{(-p)}, \hat{G}_1^{(-p)}, \hat{G}_2^{(-p)}, \ldots), \tag{1.41}$$

and the rest of its elements vanish. Hence, in its usual form, $\hat{\mathcal{L}}_+(\hat{G})$ looks like

$$\hat{\mathcal{L}}_+(\hat{G}) = \begin{bmatrix} \hat{G}_o^{(o)} & \hat{G}_o^{(1)} & \hat{G}_o^{(2)} & \hat{G}_o^{(3)} & \ldots \\ \hat{G}_o^{(-1)} & \hat{G}_1^{(o)} & \hat{G}_1^{(1)} & \hat{G}_1^{(2)} & \ldots \\ \hat{G}_o^{(-2)} & \hat{G}_1^{(-1)} & \hat{G}_2^{(o)} & \hat{G}_2^{(1)} & \ldots \\ \hat{G}_o^{(-3)} & \hat{G}_1^{(-2)} & \hat{G}_2^{(-1)} & \hat{G}_3^{(o)} & \ldots \\ & & \ldots & & \end{bmatrix} \tag{1.42}$$

I note that the upper and lower induces of $\hat{G}_k^{(p)}$ do not represent the usual position as matrix elements. The upper index represents the sub- or super- diagonal and the lower index its position in that diagonal. It is not difficult to see that in the (i,j)-th position one has the element

$$\hat{G}_{\min(i,j)-1}^{(j-i)} \quad , \qquad i,j=1,2,\ldots \tag{1.43}$$

Now I define $\hat{\mathcal{L}}_-(\hat{G})$ as the transpose of $\hat{\mathcal{L}}_+(\hat{G})$, i.e.

$$\hat{\mathcal{L}}_-(\hat{G}) = \hat{\mathcal{L}}_+(\hat{G})^t \quad . \tag{1.44}$$

<u>1.45 Definition</u> : Let $F(z,t) \in C^\infty(\mathbb{C} \times \mathbb{R} \setminus 0)$ be H-homogeneous of degree -4 with zero principal value. Equivalently, its Fourier transform $\hat{F}(\xi,\tau) \in C^\infty(\mathbb{R}^3 \setminus 0)$ is H-homogeneous of degree 0. I define the Laguerre matrix $\hat{\mathcal{L}}(\hat{F})$ by

$$\hat{\mathcal{L}}(\hat{F}) = \hat{\mathcal{L}}_+(\hat{F}_+) \oplus \hat{\mathcal{L}}_-(\hat{F}_-) \quad . \tag{1.46}$$

It turns out that the Laguerre matrix plays the same role for principal value convolution operators on H_1 that is usually assigned to the classical symbol of Mikhlin-Calderon-Zygmund operators on $\mathbb{R}^n$. More precisely, I shall now state the main result of this article.

1.47 Theorem : Let $F, G \in C^\infty(\mathbb{C}\times\mathbb{R}\setminus 0)$ be H-homogeneous of degree -4 with vanishing principal value. Then

$$\hat{\mathfrak{L}}([F *_H G]^\wedge) = \hat{\mathfrak{L}}(\hat{F})\ \hat{\mathfrak{L}}(\hat{G}) \quad . \tag{1.48}$$

The proof is given in sections 2-6. I should point out that Theorem 1.47 works for left-invariant pseudo-differential (convolution) operators with arbitrary homogeneity on H_1 . This remark will be used in section 7, where I apply the Laguerre matrix calculus, i.e. (1.48), to "invert" some well known left-invariant differential operators on H_1 , including the famous Hans Lewy operator. I shall also indicate how, in a rather natural manner, the classical Mikhlin-Calderon-Sygmund calculus for principal value convolution operators on $\mathbb{C} = \mathbb{R}^2$ can be derived from theorem 1.47.

Finally, in section 8, I shall discuss some questions related to Theorem 1.47 which I do not treat in this article.

2. THE TWISTED CONVOLUTION

In section 1 I introduced two representations of $*_H$, one in terms of $F(z,t)$ - see (1.3) and (1.15) - and another as a pseudo-differential operator induced by $\hat{F}(\xi,\tau)$ - see (1.20). In the proof of Theorem 1.47 it turns out to be more convenient to use a third representation, namely the "twisted convolution" given in terms of $\tilde{F}(z,\tau)$, the partial Fourier transform of F with respect to t . Let $\varphi \in C_o^\infty(H_1)$. I define $\tilde{\varphi}(w,\tau)$ by

$$\tilde{\varphi}(w,\tau) = \int_{-\infty}^{\infty} e^{-is\tau}\ \varphi(w,s)\ ds \quad . \tag{2.1}$$

In particular

$$\varphi(w,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{is\tau}\ \tilde{\varphi}(w,\tau)\ d\tau \quad . \tag{2.2}$$

Introducing (2.2) for φ in $F *_H \varphi$ and integrating out the s variable one obtains

(2.3) $$(F *_H \varphi)(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\tau} (\tilde{F} \tilde{*}_\tau \tilde{\varphi})(z,\tau)\, d\tau ,$$

where

(2.4) $$(\tilde{F} \tilde{*}_\tau \tilde{\varphi})(z,\tau) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\tau(z\bar{w} - \bar{z}w)} \tilde{F}(z-w,\tau)\, \tilde{\varphi}(w,\tau)\, dy_1\, dy_2$$

is the "twisted convolution" with parameter $\tau \in (-\infty,\infty)$. Here $w = y_1+iy_2$. I note that

(2.5) $$(F *_H \varphi)^\sim = \tilde{F} \tilde{*}_\tau \tilde{\varphi}$$

and

(2.6) $$\tilde{F}(z,\tau) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i<x,\xi>} \hat{F}(\xi,\tau)\, d\xi .$$

Using (2.6) we find $\tilde{\mathcal{L}}_k^{(p)}$ from $\hat{\mathcal{L}}_k^{(p)}$. More precisely, an elementary calculation yields

<u>2.7 Proposition</u> : Let $z = |z|\, e^{i\theta}$ and $p,k = 0,1,2,\dots$. Then

(2.8) $$\tilde{\mathcal{L}}_k^{(p)}(z,\tau) = \frac{2|\tau|}{\pi} \ell_k^{(p)} (2|\tau|\, |z|^2)\, e^{ip\theta} ,$$

and

(2.9) $$\tilde{\mathcal{L}}_k^{(-p)}(z,\tau) = \frac{2|\tau|}{\pi} (-1)^p \ell_k^{(p)} (2|\tau||z|^2)\, e^{-ip\theta} .$$

To prove Theorem 1.47 we must show that

(2.10) $$\hat{\mathcal{L}}_+ ([F *_H G]^\wedge_+) = \hat{\mathcal{L}}_+(\hat{F}_+)\, \hat{\mathcal{L}}_+(\hat{G}_+) ,$$

and that

(2.11) $$\hat{\mathcal{L}}_-([F *_H G]^\wedge_-) = \hat{\mathcal{L}}_-(\hat{F}_-)\, \hat{\mathcal{L}}_-(\hat{G}_-) .$$

Expanding $\hat{F}$ and $\hat{G}$ in an exponential Laguerre series the proof of (2.10) is reduced to showing that

$$(2.12)\qquad \frac{1}{2\pi}\int_{o}^{\infty} e^{it\tau}(\tilde{\mathcal{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(q)})(z,\tau)\, d\tau = \frac{1}{2\pi}\int_{o}^{\infty} e^{it\tau}\, \tilde{\mathcal{L}}_n^{(r)}(z,\tau)\, d\tau \quad ,$$

where the indices (p,k) , (q,m) and (r,n) are related via the matrix multiplication of Theorem 1.47. Taking partial Fourier transforms in t of both sides of (2.12), this is equivalent to showing that

$$(2.13)\qquad \tilde{\mathcal{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(q)} = \tilde{\mathcal{L}}_n^{(r)} \quad , \quad \tau > 0 \quad .$$

(2.11) is proved by a dual argument.

3. THE ORTHOGONAL PROJECTIONS , $\tilde{\mathcal{L}}_k^{(o)}$, $k=0,1,2,\ldots$

The derivation of (1.48) is based on two important properties of the $\hat{\mathcal{L}}_k^{(p)}$ - s .

(i) The $\hat{\mathcal{L}}_k^{(o)}$ - s are mutually orthogonal projections, and

(ii) the $\hat{\mathcal{L}}_k^{(p)}$ - s can be represented as left-invariant derivatives of the $\hat{\mathcal{L}}_k^{(o)}$ - s.

(i) is derived in section 3 and (ii) in section 4, where, the composition will be represented in the form of the twisted convolution, (2.4).

<u>3.1 Theorem</u> : The operators $\tilde{\mathcal{L}}_k^{(o)} \tilde{*}_\tau$, $k = 0,1,2,\ldots$ are mutually orthogonal projections on $L^2(\mathbb{R}^2)$, i.e.

$$(3.2)\qquad \tilde{\mathcal{L}}_k^{(o)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(o)} = \delta_{km}\, \tilde{\mathcal{L}}_k^{(o)}$$

for each fixed $\tau \in (-\infty,\infty)$, where $\delta_{km} = 0$ or 1 according to $k \neq m$ or $k = m$.

<u>Proof</u> : We may assume that $\tau \neq 0$. Then

$$(3.3)\quad \tilde{\mathcal{L}}_k^{(o)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(o)} = \left(\frac{2\tau}{\pi}\right)^2 \int_{\mathbb{R}^2} e^{\tau(z\bar{w} - \bar{z}w)} e^{-|\tau||z-w|^2 - |\tau||w|^2}$$

$$L_k(2|\tau||z-w|^2)\, L_m(2|\tau||w|^2)\, dy_1\, dy_2 = \frac{2|\tau|}{\pi^2} \int_{\mathbb{R}^2} e^{\pm i(x_2y_1 - x_1y_2)} e^{-\frac{1}{2}|x-y|^2 - \frac{1}{2}|y|^2}$$

$$L_k(|x-y|^2)\, L_m(|y|^2)\, dy_1\, dy_2 \quad ,$$

where $w = y_1 + iy_2$ and we replaced $\sqrt{2|\tau|}\ z$ by $\zeta = x_1 + ix_2$. Using the generating function, (1.24), one has

$$(3.4)\quad \sum_{k,m=o}^{\infty} L_k(x)\, L_m(y)\, s^k\, t^m = \frac{1}{(1-s)(1-t)} \exp\left\{-x\frac{s}{1-s} - y\frac{t}{1-t}\right\} .$$

I shall calculate

$$(3.5)\quad I_{s,t}^{(o)} = \sum_{k,m=o}^{\infty} \tilde{\mathcal{L}}_k^{(o)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(o)} s^k\, t^m = \frac{2|\tau|}{\pi^2} \int_{\mathbb{R}^2} e^{\pm i(x_2y_1 - x_1y_2)} \frac{1}{(1-s)(1-t)}$$

$$\exp\left\{-\frac{1}{2}|x-y|^2 - \frac{1}{2}|y|^2 - \frac{|x-y|^2 s}{1-s} - \frac{|y|^2 t}{1-t}\right\} dy_1\, dy_2 \quad .$$

Now

$$-|x_j - y_j|^2 \left(\frac{1}{2} + \frac{s}{1-s}\right) - |y_j|^2 \left(\frac{1}{2} + \frac{t}{1-t}\right) = -\frac{1}{2}\,\frac{(1+s)(1+t)}{2(1-st)}\, x_j^2 - \frac{1}{2}\,\gamma_j^2 \ ,\ j=1,2 \ ,$$

where I set

$$\gamma_j = \left[\frac{2(1-st)}{(1-s)(1-t)}\right]^{1/2} y_j - \frac{1+s}{1-s}\left[\frac{(1-s)(1-t)}{2(1-st)}\right]^{1/2} x_j \quad .$$

Finally

$$x_2y_1 - x_1y_2 = (x_2\gamma_1 - x_1\gamma_2)\left[\frac{(1-s)(1-t)}{2(1-st)}\right]^{1/2}$$

yields

$$I^{(o)}_{s,t} = \frac{2|\tau|}{\pi^2} \exp\left\{-\frac{1}{2}\,\frac{(1+s)(1+t)}{2(1-st)}\,|x|^2\right\} \frac{1}{2(1-st)}$$

$$\cdot \prod_{j=1}^{2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\,\gamma^2 \pm i \left[\frac{(1-s)(1-t)}{2(1-st)}\right]^{1/2} x_j\,\gamma\right\} d\gamma$$

$$= \frac{2|\tau|}{\pi}\, e^{-\frac{1}{2}|x|^2} \left\{\frac{1}{1-st}\, e^{-|x|^2 \frac{st}{1-st}}\right\} .$$

Replacing $\zeta = x_1+ix_2$ by $\sqrt{2|\tau|}\, z$, we found

$$\text{(3.6)} \qquad \sum_{k,m=o}^{\infty} \tilde{\mathcal{L}}^{(o)}_k \tilde{*}_\tau \tilde{\mathcal{L}}^{(o)}_m\, s^k t^m = \sum_{k=o}^{\infty} \tilde{\mathcal{L}}^{(o)}_k \cdot (st)^k \quad ,$$

which proves Theorem 3.1.

4. CALCULATING $\tilde{\mathcal{L}}^{(p)}_k \tilde{*}_\tau \tilde{\mathcal{L}}^{(q)}_m$ WHEN $\tau > 0$.

H_1 has $Z,\bar{Z}$ and T for a basis of its Lie Algebra of Left-invariant vector fields, where

$$\text{(4.1)} \qquad Z = \frac{\partial}{\partial z} + i\bar{z}\,\frac{\partial}{\partial t} \quad ,$$

$$\text{(4.2)} \qquad \bar{Z} = \frac{\partial}{\partial \bar{z}} - iz\,\frac{\partial}{\partial t} \quad ,$$

$$\text{(4.3)} \qquad T = \frac{\partial}{\partial t} \quad ,$$

with the usual convention

$$\text{(4.4)} \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right) , \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right) \quad .$$

In (z,τ) space I shall need the operators

$$\text{(4.5)} \qquad \tilde{Z}_\tau = \frac{\partial}{\partial z} - \bar{z}\,\tau \quad ,$$

$$\text{(4.6)} \qquad \tilde{\bar{Z}}_\tau = \frac{\partial}{\partial \bar{z}} + z\,\tau \quad .$$

Given $f,g \in L^2(\mathbb{C})$ one has the inner product

(4.7) $$(f,g) = \int_{\mathbb{R}^2} f(x)\,\overline{g(x)}\,dx \quad .$$

The following result is a simple consequence of the definitions of $\tilde{Z}_\tau$ and $\tilde{\mathcal{L}}_k^{(p)}$.

<u>4.8 Proposition</u> : (i) $\tilde{Z}_\tau$ and $-\tilde{\bar{Z}}_\tau$ are mutually adjoint in $L^2(\mathbb{C})$, i.e.

(4.9) $$(\tilde{Z}_\tau f,g) = (f, -\tilde{\bar{Z}}_\tau g) \quad .$$

(ii) $\tilde{\mathcal{L}}_k^{(p)}$ and $\tilde{\mathcal{L}}_k^{(-p)}$ are mutually adjoint $\tilde{*}_\tau$ - convolution operators, i.e.

(4.10) $$(\tilde{\mathcal{L}}_k^{(p)} \tilde{*}_\tau f,g) = (f, \tilde{\mathcal{L}}_k^{(-p)} \tilde{*}_\tau g) \quad .$$

I shall derive the composition of the exponential Laguerre functions, $\tilde{\mathcal{L}}_k^{(p)}$, by representing them as $\tilde{Z}_\tau$ and $-\tilde{\bar{Z}}_\tau$ derivatives of the $\tilde{\mathcal{L}}_k^{(o)}$ - s. Since the $\tilde{\mathcal{L}}_k^{(o)}$ - s induce orthogonal projections, they can be easily composed. For this I need a few simple properties of the generalized Laguerre polynomials, $L_k^{(p)}$ (se [15]).

(4.11) $$L_o^{(p)}(x) = 1 \ , \quad p=0,1,2,\ldots$$

(4.12) $$\frac{d}{dx} L_n^{(p)}(x) = \begin{cases} -L_{n-1}^{(p+1)}(x) & \text{if } n \geq 1 \ , \\ 0 & \text{if } n = 0 \ , \end{cases}$$

and

(4.13) $$L_n^{(p)}(x) + L_{n-1}^{(p+1)}(x) = L_n^{(p+1)}(x) \quad .$$

(4.11), (4.12) and (4.13) can be derived from (1.24).

4.14 Proposition : Let $\tau > 0$. Then

$$(4.15)\qquad -\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\,\tilde{\mathcal{L}}_k^{(p)} = \begin{cases} k^{1/2}\,\tilde{\mathcal{L}}_{k-1}^{(p+1)} & \text{if} \quad n \geq 1\,, \\ 0 & \text{if} \quad k = 0\,, \end{cases}$$

$$(4.16)\qquad \frac{\tilde{Z}_\tau}{\sqrt{2\tau}}\,\tilde{\mathcal{L}}_k^{(-p)} = (k+p+1)^{1/2}\,\tilde{\mathcal{L}}_k^{(-p-1)}\,,$$

where $p,k = 0,1,2,\ldots$

4.17 Corollary : Again, $\tau > 0$ and $p,k = 0,1,2,\ldots$. Then

$$(4.18)\qquad \tilde{\mathcal{L}}_k^{(p)} = \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} \left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right)^p \tilde{\mathcal{L}}_{k+p}^{(o)}\,,$$

and

$$(4.19)\qquad \tilde{\mathcal{L}}_k^{(-p)} = \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} \left(\frac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right)^p \tilde{\mathcal{L}}_k^{(o)}\,.$$

Here $\Gamma(z)$ denotes the Gamma function,

$$(4.20)\qquad \Gamma(z) = \int_o^\infty e^{-t}\,t^{z-1}\,dt$$

if $\operatorname{Re} z > 0$. Note $\Gamma(n+1) = n\,!$

Proof of Proposition 4.14 : Since $\tilde{Z}_\tau$, $\tilde{\bar{Z}}_\tau$ and $\tilde{\mathcal{L}}_k^{(p)} *_\tau$ commute with left translations, it suffices to apply $\tilde{Z}_\tau$ and $-\tilde{\bar{Z}}_\tau$ to the function $\tilde{\mathcal{L}}_k^{(p)}$, i.e. we are permitted to set $w = 0$ in the kernel of the operator $\tilde{\mathcal{L}}_k^{(p)}\,\tilde{*}_\tau$.
Then

$$(4.21)\qquad -\tilde{\bar{Z}}_\tau\,\tilde{\mathcal{L}}_k^{(p)}(z,\tau) = \frac{2\tau}{\pi}\left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} (\sqrt{2\tau}\,z)^p$$

$$\cdot \left(-\frac{\partial}{\partial z} - z\tau\right)\left(e^{-\tau|z|^2} L_k^{(p)}(2\tau|z|^2)\right) = \frac{2\tau}{\pi}\left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} (\sqrt{2\tau}\, z)^p e^{-\tau|z|^2}$$

$$\cdot \quad 2\tau\, z\, L_{k-1}^{(p+1)}(2\tau|z|^2) = \begin{cases} k^{1/2} \sqrt{2\tau}\ \tilde{\mathfrak{L}}_{k-1}^{(p+1)} & \text{if } k \geq 1 , \\ 0 & \text{if } k = 0 , \end{cases}$$

where I used (4.11) and (4.12). This proves (4.15).

Similarly

$$(4.22) \qquad \tilde{Z}_\tau\, \check{\mathfrak{L}}_k^{(-p)} = \frac{2\tau}{\pi}\left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} (\sqrt{2\tau}\, \bar{z})^p$$

$$\cdot \left(\frac{\partial}{\partial z} - \bar{z}\tau\right)\left(e^{-\tau|z|^2} L_k^{(p)}(2\tau|z|^2)\right) = \frac{2\tau}{\pi}\left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} (\sqrt{2\tau}\, \bar{z})^p e^{-\tau|z|^2}$$

$$\cdot \ (-2\tau\, \bar{z})(L_k^{(p)}(2\tau|z|^2) + L_{k-1}^{(p+1)}(2\tau|z|^2)) = (k+p+1)^{1/2} \sqrt{2\tau}\ \check{\mathfrak{L}}_k^{(-p-1)} ,$$

where I used (4.12) and (4.13). This proves (4.16) and we have proved Proposition 4.14.

Now the calculation of

$$(4.23) \quad \tilde{\mathfrak{L}}_k^{(p)} \tilde{*}_\tau \check{\mathfrak{L}}_m^{(q)} = \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} \left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right)^p \check{\mathfrak{L}}_{k+p}^{(o)} \tilde{*}_\tau \left[\frac{\Gamma(m+1)}{\Gamma(m+q+1)}\right]^{1/2} \left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right) \tilde{\mathfrak{L}}_{m+q}^{(o)}$$

is reduced to finding

$$(4.24) \qquad \tilde{\mathfrak{L}}_k^{(o)} \tilde{*}_\tau \left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right) \quad \text{and} \quad \check{\mathfrak{L}}_k^{(o)} \check{*}_\tau \left(\frac{\check{Z}_\tau}{\sqrt{2\tau}}\right) .$$

This can be done via transposition. Namely

$$(4.25) \qquad (\check{\mathfrak{L}}_k^{(o)} \tilde{*}_\tau \check{Z}_\tau f, g) = (f, -\check{\bar{Z}}_\tau \tilde{\mathfrak{L}}_k^{(o)} \check{*}_\tau g)$$

$$= (f, k^{1/2} \sqrt{2\tau}\ \tilde{\mathfrak{L}}_{k-1}^{(1)} \tilde{*}_\tau g)$$

$$= (k^{1/2} \sqrt{2\tau}\ \tilde{\mathfrak{L}}_{k-1}^{(-1)} \tilde{*}_\tau f, g) ,$$

i.e.

$$(4.26)\qquad \tilde{\mathcal{L}}_k^{(o)} \tilde{*}_\tau \frac{\tilde{Z}_\tau}{\sqrt{2\tau}} = \begin{cases} \dfrac{\tilde{Z}_\tau}{\sqrt{2\tau}} \tilde{\mathcal{L}}_{k-1}^{(o)} & \text{if } k \geq 1 \ , \\ 0 & \text{if } k = 0 \ . \end{cases}$$

Similarly

$$(4.27)\qquad \tilde{\mathcal{L}}_k^{(o)} \tilde{*}_\tau \left(-\frac{\overline{\tilde{Z}}_\tau}{\sqrt{2\tau}}\right) = \left(-\frac{\overline{\tilde{Z}}_\tau}{\sqrt{2\tau}}\right) \tilde{\mathcal{L}}_{k+1}^{(o)} \ .$$

Iterating this procedure we have derived

<u>4.28 Proposition</u> : Let $\tau > 0$ and $p,k = 0,1,2,\dots$. Then

$$(4.29)\qquad \tilde{\mathcal{L}}_k^{(o)} \tilde{*}_\tau \left(\frac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right)^p = \begin{cases} \left(\dfrac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right)^p \tilde{\mathcal{L}}_{k-p}^{(o)} & \text{if } k \geq p \ , \\ 0 & \text{if } k < p \ , \end{cases}$$

and

$$(4.30)\qquad \tilde{\mathcal{L}}_k^{(o)} \tilde{*}_\tau \left(-\frac{\overline{\tilde{Z}}_\tau}{\sqrt{2\tau}}\right)^p = \left(-\frac{\overline{\tilde{Z}}_\tau}{\sqrt{2\tau}}\right)^p \tilde{\mathcal{L}}_{k+p}^{(o)} \ .$$

<u>4.31 Theorem</u> : Again, $\tau > 0$ and $p,k,q,m = 0,1,2,\dots$. Then

$$(4.32)\qquad \tilde{\mathcal{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(q)} = \begin{cases} \tilde{\mathcal{L}}_k^{(p+q)} & \text{if } p+k = m \ , \\ 0 & \text{if } p+k \neq m \ , \end{cases}$$

and

$$(4.33)\qquad \tilde{\mathcal{L}}_k^{(-p)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(-q)} = \begin{cases} \mathcal{L}_m^{(-p-q)} & \text{if } q+m = k \ , \\ 0 & \text{if } q+m \neq k \ . \end{cases}$$

<u>Proof</u> : Using (4.23) and (4.30) we have

$$(4.34)\qquad \tilde{\mathfrak{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathfrak{L}}_m^{(q)} = \left[\frac{\Gamma(k+1)\ \Gamma(m+1)}{\Gamma(k+p+1)\ \Gamma(m+q+1)}\right]^{1/2} \cdot \left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right)^{p+q} \tilde{\mathfrak{L}}_{k+p+q}^{(o)} \tilde{*}_\tau \tilde{\mathfrak{L}}_{m+q}^{(o)}$$

$$= \begin{cases} \left[\dfrac{\Gamma(k+1)}{\Gamma(k+p+q+1)}\right]^{1/2} \left(-\dfrac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right)^{p+q} \tilde{\mathfrak{L}}_{k+p+q}^{(o)} & \text{if} \quad m = k+p \ , \\ 0 & \text{if} \quad m \neq k+p \ , \end{cases}$$

which yields (4.32) in view of (4.18). (4.33) follows from (4.32) by transposition. This proves Theorem 4.31.

Next we have the somewhat lengthier calculations of

$$(4.35)\qquad \tilde{\mathfrak{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathfrak{L}}_m^{(-q)} \quad \text{and} \quad \tilde{\mathfrak{L}}_k^{(-p)} \tilde{*}_\tau \tilde{\mathfrak{L}}_m^{(q)} \ ,$$

where $p,q \geq 0$. First of all note that, if $\tau > 0$

$$(4.36)\qquad \tilde{\mathfrak{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathfrak{L}}_m^{(-q)} = \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} \left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right) \tilde{\mathfrak{L}}_{k+p}^{(o)} \tilde{*}_\tau \left[\frac{\Gamma(m+1)}{\Gamma(m+q+1)}\right]^{1/2} \left(\frac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right)^q \tilde{\mathfrak{L}}_m^{(o)}$$

$$= \left[\frac{\Gamma(k+1)\ \Gamma(m+1)}{\Gamma(k+p+1)\ \Gamma(m+q+1)}\right]^{1/2} \cdot \left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right)^p \left(\frac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right)^q \tilde{\mathfrak{L}}_{k+p-q}^{(o)} \tilde{*}_\tau \tilde{\mathfrak{L}}_m^{(o)}$$

$$= \frac{[\Gamma(k+1)\ \Gamma(k+p-q+1)]^{1/2}}{\Gamma(k+p+1)} \cdot \begin{cases} \left(-\dfrac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right)^p \left(\dfrac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right)^q \tilde{\mathfrak{L}}_{k+p-q}^{(o)} & \text{if} \quad k+p = m+q \ , \\ 0 & \text{if} \quad k+p \neq m+q \ . \end{cases}$$

Thus I must calculate $\tilde{\bar{Z}}^p_\tau \; \tilde{Z}^q_\tau$ and $\tilde{Z}^p_\tau \; \tilde{\bar{Z}}^q_\tau$.

<u>4.37 Lemma</u> : Set

(4.38) $$\tilde{\Box}_\tau = -\frac{1}{2}\,(\tilde{Z}_\tau\, \tilde{\bar{Z}}_\tau + \tilde{\bar{Z}}_\tau\, \tilde{Z}_\tau) \quad .$$

Then for $\tau > 0$,

(4.39) $$\tilde{\Box}_\tau\,(\ell_n^{(o)}(2\tau|z|^2)) = 2\tau(n+\frac{1}{2})\;\ell_n^{(o)}(2\tau\;|z|^2) \quad .$$

<u>Proof</u> : Explicitly one has

(4.40) $$\tilde{\Box}_\tau = -\frac{\partial^2}{\partial z\partial\bar{z}} - \tau\left(z\,\frac{\partial}{\partial z} - \bar{z}\,\frac{\partial}{\partial\bar{z}}\right) + |z|^2\,\tau^2 \quad .$$

Hence

(4.41) $$\tilde{\Box}_\tau\, u(|z|^2) = (-\frac{1}{4}\,\Delta + |z|^2\,\tau^2)\; u(|z|^2) \quad ,$$

where

(4.42) $$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \quad .$$

Therefore

(4.43) $$\tilde{\Box}_\tau(\ell_n^{(o)}(2\tau|z|^2)) = -2\tau\; e^{-\tau|z|^2}\;[2\tau|z|^2\; L_n''\;(2\tau\;|z|^2) + (1-2\tau|z|^2)\; L_n'(2\tau|z|^2) - \frac{1}{2}\, L_n\;(2\tau|z|^2)] \quad .$$

Now formula (5.1.2) of [15], namely

(4.44) $$yL_n''(y) + (1-y)\;L_n'(y) + nL_n(y) = 0 \quad ,$$

yields Lemma 4.37.

4.45 Lemma : Let $s = 1,2,\ldots$. Then

$$(4.46)\quad (-\tilde{\bar{\bar{Z}}}_\tau)^s\, \tilde{Z}_\tau^s(\ell_n^{(o)}(2\tau|z|^2)) = (2\tau)^s\,\frac{\Gamma(n+s+1)}{\Gamma(n+1)}\,\ell_n^{(o)}\,(2\tau\,|z|^2)\ ,$$

and

$$(4.47)\quad \tilde{Z}_\tau^s\,(-\tilde{\bar{\bar{Z}}}_\tau)^s\,(\ell_n^{(o)}\,(2\tau|z|^2)) = \begin{cases} (2\tau)^s\,\dfrac{\Gamma(n+1)}{\Gamma(n-s+1)}\,\ell_n^{(o)}(2\tau|z|^2) & \text{if } n \geq s\ , \\ 0 & \text{if } n < s\ , \end{cases}$$

as long as $\tau > 0$.

Proof : Note that

$$(4.48)\qquad -\tilde{\bar{\bar{Z}}}_\tau\,\tilde{Z}_\tau = -\tilde{Z}_\tau\tilde{\bar{\bar{Z}}}_\tau + 2\tau = \tilde{\Box}_\tau + \frac{1}{2}\,(2\tau)\quad .$$

Therefore

$$\begin{aligned} -\tilde{\bar{\bar{Z}}}_\tau\,\tilde{Z}_\tau^s &= (-\tilde{\bar{\bar{Z}}}_\tau\tilde{Z}_\tau)\,\tilde{Z}_\tau^{s-1} \\ &= (-\tilde{Z}_\tau\tilde{\bar{\bar{Z}}}_\tau + 2\tau)\,\tilde{Z}_\tau^{s-1} \\ &= \tilde{Z}_\tau(-\tilde{\bar{\bar{Z}}}_\tau\tilde{Z}_\tau + 2\tau)\,\tilde{Z}_\tau^{s-2} \\ &= \tilde{Z}_\tau(-\tilde{Z}_\tau\tilde{\bar{\bar{Z}}}_\tau + 2(2\tau))\,\tilde{Z}_\tau^{s-2} \\ &\quad \cdots\cdot \\ &= \tilde{Z}_\tau^{s-1}(-\tilde{Z}_\tau\tilde{\bar{\bar{Z}}}_\tau + s(2\tau)) \\ &= \tilde{Z}_\tau^{s-1}(\tilde{\Box}_\tau + (s-\tfrac{1}{2})(2\tau))\quad . \end{aligned}$$

Therefore a simple induction argument yields

$$(4.49) \qquad (-\tilde{\bar{Z}}_\tau)^s \, \tilde{Z}_\tau^s = (-\tilde{\bar{Z}}_\tau)^{s-1} \, \tilde{Z}_\tau^{s-1} \, (\tilde{\Box}_\tau + (s-\tfrac{1}{2})(2\tau))$$

$$= (\tilde{\Box}_\tau + \tfrac{1}{2}(2\tau))$$

$$(\tilde{\Box}_\tau + \tfrac{3}{2}\,(2\tau))$$

$$\cdots\cdot$$

$$(\tilde{\Box}_\tau + (s - \tfrac{1}{2})(2\tau)) \quad .$$

Now (4.49) and Lemma (4.37) imply (4.46). Similarly

$$\tilde{Z}_\tau(-\tilde{\bar{Z}}_\tau)^s = (-\tilde{\bar{Z}}_\tau\tilde{Z}_\tau - 2\tau)(-\tilde{\bar{Z}}_\tau)^{s-1}$$

$$= -\tilde{\bar{Z}}_\tau(-\tilde{\bar{Z}}_\tau\tilde{Z}_\tau - 2(2\tau))(-\tilde{\bar{Z}}_\tau)^{s-2}$$

$$\cdots\cdot$$

$$= (-\tilde{\bar{Z}}_\tau)^{s-1} \, (-\tilde{\bar{Z}}_\tau\tilde{Z}_\tau - s(2\tau))$$

$$= (-\tilde{\bar{Z}}_\tau)^{s-1} \, (\tilde{\Box}_\tau - (s - \tfrac{1}{2}) \, (2\tau)).$$

Since

$$-\tilde{Z}_\tau\tilde{\bar{Z}}_\tau = \tilde{\Box}_\tau - \tfrac{1}{2}(2\tau) \quad ,$$

by induction we have

$$(4.50) \qquad (\tilde{Z}_\tau)^s \, (-\tilde{\bar{Z}}_\tau)^s = (\tilde{\Box}_\tau - \tfrac{1}{2}\,(2\tau))$$

$$(\tilde{\Box}_\tau - \tfrac{3}{2}\,(2\tau))$$

$$\cdots\cdot$$

$$(\tilde{\Box}_\tau - (s - \tfrac{1}{2})(2\tau)) \quad ,$$

which, in view of Lemma 4.37, yields (4.47), and, hence, Lemma 4.45.

4.51 Theorem : Assume $\tau > 0$

(i) If $k+p = m+q$, then

$$(4.52) \qquad \tilde{\mathcal{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(-q)} = \begin{cases} \tilde{\mathcal{L}}_k^{(p-q)} & \text{if } p \geq q , \\ \tilde{\mathcal{L}}_m^{(p-q)} & \text{if } p \leq q . \end{cases}$$

(ii) If $k+p \neq m+q$, then

$$(4.53) \qquad \tilde{\mathcal{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(-q)} = 0 .$$

Proof : (i) If $k+p = m+q$ and $p \geq q$ then (4.36), (4.46) and (4.18) yield

$$\tilde{\mathcal{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(-q)} = \frac{[\Gamma(k+1)\ \Gamma(k+p-q+1)]^{1/2}}{\Gamma(k+p+1)}$$

$$\left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right)^{p-q} \left(\frac{1}{2\tau}\right)^q (-\tilde{\bar{Z}}_\tau)^q\ \tilde{Z}_\tau^q\ \tilde{\mathcal{L}}_{k+p-q}^{(o)}$$

$$= \left[\frac{\Gamma(k+1)}{\Gamma(k+p-q+1)}\right]^{1/2} \left(-\frac{\tilde{\bar{Z}}_\tau}{\sqrt{2\tau}}\right)^{p-q} \tilde{\mathcal{L}}_{k+p-q}^{(o)}$$

$$= \tilde{\mathcal{L}}_k^{(p-q)} .$$

The case $p \leq q$ follows by duality (transposition).

(ii) is a consequence of (4.36).

4.54 Theorem : Again, $\tau > 0$.

(i) Let $m=k$. Then

$$(4.55) \qquad \tilde{\mathcal{L}}_m^{(-q)} \tilde{*}_\tau \tilde{\mathcal{L}}_k^{(p)} = \begin{cases} \tilde{\mathcal{L}}_{m+q}^{(p-q)} & \text{if } q \leq p , \\ \tilde{\mathcal{L}}_{k+p}^{(p-q)} & \text{if } q \geq p . \end{cases}$$

(ii) If $m \neq k$, then

$$\tilde{\mathcal{L}}_m^{(-q)} \tilde{*}_\tau \tilde{\mathcal{L}}_k^{(p)} = 0 \quad . \tag{4.56}$$

<u>Proof</u> : Proceeding as in the proof of Theorem 4.51, one has

$$\tilde{\mathcal{L}}_m^{(-q)} \tilde{*}_\tau \tilde{\mathcal{L}}_k^{(p)} = \left[\frac{\Gamma(m+1)\ \Gamma(k+1)}{\Gamma(m+q+1)\ \Gamma(k+p+1)}\right]^{1/2} \left(\frac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right)^q \left(-\frac{\bar{\tilde{Z}}_\tau}{\sqrt{2\tau}}\right)^p \tilde{\mathcal{L}}_{m+p}^{(o)} \tilde{*}_\tau \tilde{\mathcal{L}}_{k+p}^{(o)} \quad , \tag{4.57}$$

which vanishes if $m \neq k$, proving (4.56). If $m=k$ and $q \geq p$, then

$$\begin{aligned} \tilde{\mathcal{L}}_m^{(-q)} \tilde{*}_\tau \tilde{\mathcal{L}}_k^{(p)} &= \frac{\Gamma(k+1)}{[\Gamma(k+q+1)\ \Gamma(k+p+1)]^{1/2}} \\ &\quad \left(\frac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right)^{q-p} \frac{1}{(2\tau)^p} \tilde{Z}_\tau^p \, (-\bar{\tilde{Z}}_\tau)^p \, \tilde{\mathcal{L}}_{k+p}^{(o)} \\ &= \left[\frac{\Gamma(k+p+1)}{\Gamma(k+q+1)}\right]^{1/2} \left[\frac{\tilde{Z}_\tau}{\sqrt{2\tau}}\right]^{q-p} \tilde{\mathcal{L}}_{k+p}^{(o)} \\ &= \tilde{\mathcal{L}}_{k+p}^{(-[q-p])} \quad , \end{aligned} \tag{4.58}$$

where I used (4.47) and (4.19). The case $q \leq p$ follows by duality, which proves Theorem 4.54.

5. <u>PROOF OF THEOREM 1.47 WHEN $\tau > 0$</u>

According to (2.13) and the discussion leading up to it, all we have to prove is that

$$\tilde{\mathcal{L}}_k^{(p)} \tilde{*}_\tau \tilde{\mathcal{L}}_m^{(q)} = \tilde{\mathcal{L}}_n^{(r)} \quad , \qquad \tau > 0 \quad ,$$

where the indeces (p,k), (q,m) and (r,n) are related via the matrix multiplication of Theorem 1.47. More precisely, (1.43) implies that, for $\tau > 0$, Theorem 1.47 is equivalent to the following result.

5.1 Theorem : Suppose $\tau > 0$. Let (i,j) and (k,ℓ) denote matrix positions. Then

$$(5.2) \qquad \tilde{\mathfrak{L}}^{(j-i)}_{\min(i,j)-1} \;\tilde{*}_{\tau}\; \tilde{\mathfrak{L}}^{(\ell-k)}_{\min(k,\ell)-1} = \begin{cases} \tilde{\mathfrak{L}}^{(\ell-i)}_{\min(i,\ell)-1} & \text{if } j=k\,, \\ 0 & \text{if } j\neq k\,. \end{cases}$$

Proof : Theorem 5.1 is simply the collection of the results of section 4. There are four cases

(i) $i<j$, $k<\ell$. Then (4.32) is equivalent to

$$(5.2) = \tilde{\mathfrak{L}}^{(j-i)}_{i-1} \;\tilde{*}_{\tau}\; \mathfrak{L}^{(\ell-k)}_{k-1}$$

$$= \begin{cases} \tilde{\mathfrak{L}}^{(\ell-i)}_{i-1} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

(ii) $i > j$, $k > \ell$. Then

$$(5.2) = \tilde{\mathfrak{L}}^{(-[i-j])}_{j-1} \;\tilde{*}_{\tau}\; \tilde{\mathfrak{L}}^{(-[k-\ell])}_{\ell-1}$$

$$= \begin{cases} \tilde{\mathfrak{L}}^{(\ell-i)}_{\ell-1} & \text{if } k = j\,, \\ 0 & \text{if } k \neq j\,, \end{cases}$$

is just rephrasing (4.33).

(iii) $i<j$, $k>\ell$. Then

$$(5.2) = \tilde{\mathcal{L}}_{i-1}^{(j-i)} \tilde{*}_\tau \check{\mathcal{L}}_{\ell-1}^{(-[k-\ell])}$$

$$= \begin{cases} \tilde{\mathcal{L}}_{\min(i,\ell)-1}^{(\ell-i)} & \text{if } j = k , \\ 0 & \text{if } j \neq k , \end{cases}$$

is the statement of Theorem 4.51. Finally,

(iv) $i>j$, $k<\ell$. Then

$$(5.2) = \tilde{\mathcal{L}}_{j-1}^{(-[i-j])} \tilde{*}_\tau \check{\mathcal{L}}_{k-1}^{(\ell-k)}$$

$$= \begin{cases} \tilde{\mathcal{L}}_{\min(i,\ell)-1}^{(\ell-i)} & \text{if } j = k , \\ 0 & \text{if } j \neq k , \end{cases}$$

is equivalent to Theorem 4.54. This proves Theorem 5.1 and thus we have proved Theorem 1.47 if $\tau > 0$.

6. THE PROOF OF THEOREM 1.47 WHEN $\tau < 0$

The proof is based on the following observation.

6.1 Lemma : For $\tau \neq 0$ one has

$$\check{\mathcal{L}}_k^{(p)} \tilde{*}_{-\tau} \check{\mathcal{L}}_m^{(q)} = \tilde{\mathcal{L}}_m^{(q)} \tilde{*}_\tau \check{\mathcal{L}}_k^{(p)} .$$

<u>Proof of Lemma 6.1</u> : Substituting $\zeta = z-w$ yields

$$\tilde{\mathcal{L}}_k^{(p)} \mathbin{\tilde{*}}_{-\tau} \tilde{\mathcal{L}}_m^{(q)} = \left(\frac{2\tau}{\pi}\right)^2 \int_{\mathbb{R}^2} e^{-\tau(z\bar{w} - \bar{z}w)}$$

$$\ell_k^{(p)}(2|\tau|\ |z-w|^2)\ \ell_m^{(q)}(2|\tau|\ |w|^2)\ \frac{i}{2}\, dw \wedge d\bar{w}$$

$$= \left(\frac{2\tau}{\pi}\right)^2 \int_{\mathbb{R}^2} e^{\tau(z\bar{\zeta} - \bar{z}\zeta)}$$

$$\ell_m^{(q)}\ (2|\tau|\ |z-\zeta|^2)\ \ell_k^{(p)}(2|\tau||\zeta|^2)\ \frac{i}{2}\, d\zeta \wedge d\bar{\zeta}$$

$$= \tilde{\mathcal{L}}_m^{(q)} \mathbin{\tilde{*}}_{\tau} \tilde{\mathcal{L}}_k^{(p)} ,$$

which proves Lemma 6.1 .

Therefore

$$\hat{F}_{-,k}^{(p)} \tilde{\mathcal{L}}_k^{(p)} \mathbin{\tilde{*}}_{-|\tau|} \hat{G}_{-,m}^{(q)} \tilde{\mathcal{L}}_m^{(q)} = \hat{F}_{-,k}^{(p)} \hat{G}_{-,m}^{(q)} (\tilde{\mathcal{L}}_m^{(q)} \mathbin{\hat{*}}_{|\tau|} \tilde{\mathcal{L}}_k^{(p)}) .$$

In other words,

$$\text{(6.2)} \qquad \hat{\mathcal{L}}_+([F *_H G]^\wedge_-) = \hat{\mathcal{L}}_+(\hat{G}_-)\ \hat{\mathcal{L}}_+(\hat{F}_-) .$$

Transposing both sides yields

$$\hat{\mathcal{L}}_-(\hat{F}_-)\ \hat{\mathcal{L}}_-(\hat{G}_-)$$

$$= [\hat{\mathcal{L}}_+(\hat{G}_-)\ \hat{\mathcal{L}}_+(\hat{F}_-)]^t$$

$$= [\hat{\mathcal{L}}_+([F *_H G]^\wedge_-)]^t$$

$$= \hat{\mathcal{L}}_-([F *_H G]^\wedge_-) ,$$

which is (2.11). Thus we, finally, finished the proof of Theorem 1.47, the main result of this article, and are ready to discuss examples.

7. EXAMPLES AND APPLICATIONS

I shall give three applications of the symbolic calculus, i.e. Theorem 1.47 . The first two are concerned with well known left-invariant differential operators on H_1 , including the Hans Levy operator. Here I discard the restriction that the symbol of the operator is homogeneous of degree zero. For a third example I show that the Mikhlin-Calderon-Zygmund operators on $\mathbb{R}^2$ have a natural extension to H_1 as left invariant principal value convolution operators and the Mikhlin-Calderon-Zygmund (MCZ) calculus is a consequence of Theorem 1.47 .

(1) The operators $\Box_\alpha$ ($\mathcal{L}_\alpha$ in [3]) are defined by

$$\Box_\alpha = -\frac{1}{2}(Z\bar{Z} + \bar{Z}Z) + i\alpha \frac{\partial}{\partial t} \quad , \quad \alpha \in \mathbb{C} \quad , \tag{7.1}$$

where

$$Z = \frac{\partial}{\partial z} + i\bar{z} \frac{\partial}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}\right) + i(x_1 - ix_2) \frac{\partial}{\partial t} \ , \tag{7.2}$$

and $\bar{Z}$ is its complex conjugate - also see section 4. The symbol of $\Box_\alpha$, $\sigma(\Box_\alpha)$, is given by

$$\sigma(\Box_\alpha) = \frac{1}{4} \left[(\xi_1 + 2x_2 \ \tau)^2 + (\xi_2 - 2x_1 \ \tau)^2\right] - \alpha\tau \quad . \tag{7.3}$$

Since I work on the Fourier transform side I introduce the notation

$$F *_H \varphi = \hat{F} \hat{*} \hat{\varphi} \quad , \tag{7.4}$$

where $\hat{F} \hat{*} \hat{\varphi}$ stands for the right hand side of (1.20). Then

$$\Box_\alpha f = |\tau| \left(\frac{|\xi|^2}{4|\tau|} - \alpha(\operatorname{sgn} \tau)\right) \hat{*} \hat{f} \quad . \tag{7.5}$$

Note, that

$$(7.6)\qquad \frac{1}{|\tau|}\hat{\Box}_\alpha = \frac{|\xi|^2}{4|\tau|} - \alpha(\operatorname{sgn}\tau)$$

is homogeneous of degree zero. An easy calculation yields

$$(7.7)\qquad \frac{1}{|\tau|}\hat{\Box}_{\alpha,\pm} = \frac{|\xi|^2}{4|\tau|} \mp \alpha = \sum_{k=o}^{\infty} (2k+1 \mp \alpha)\, \hat{\mathfrak{L}}_k^{(o)} \quad .$$

Consequently

$$(7.8)\qquad \hat{\mathfrak{L}}_\pm\left(\frac{1}{|\tau|}\hat{\Box}_{\alpha,\pm}\right) = \operatorname{diag}\,(1 \mp \alpha,\, 3 \mp \alpha,\, 5 \mp \alpha, \ldots) \ ,$$

i.e.

$$(7.9)\qquad \hat{\mathfrak{L}}\left(\frac{1}{|\tau|}\hat{\Box}_\alpha\right) = \hat{\mathfrak{L}}_+\left(\frac{1}{|\tau|}\hat{\Box}_{\alpha,+}\right) \oplus \hat{\mathfrak{L}}_-\left(\frac{1}{|\tau|}\hat{\Box}_{\alpha,-}\right) \ ,$$

where

$$(7.10)\qquad \hat{\mathfrak{L}}_\pm\left(\frac{1}{|\tau|}\hat{\Box}_{\alpha,\pm}\right) = \begin{bmatrix} 1 \mp \alpha & 0 & 0 & \cdots \\ 0 & 3 \mp \alpha & 0 & \cdots \\ 0 & 0 & 5 \mp \alpha & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

Therefore Theorem 1.47 implies

<u>7.11 Theorem</u> : $\Box_\alpha$ is invertible if and only if $2k+1 \pm \alpha \neq 0$, $k = 0,1,2,\ldots$ In that case the inverse Laguerre matrix is

$$[\hat{\mathcal{L}}(\hat{\Box}_\alpha)]^{-1} = [\hat{\mathcal{L}}_+(\hat{\Box}_{\alpha,+})]^{-1} \oplus [\hat{\mathcal{L}}_-(\hat{\Box}_{\alpha,-})]^{-1} \quad , \tag{7.12}$$

where

$$[\hat{\mathcal{L}}_\pm(\hat{\Box}_{\alpha,\pm})]^{-1} = \frac{1}{|\tau|} \operatorname{diag}\left(\frac{1}{1\mp\alpha}, \frac{1}{3\mp\alpha}, \ldots\right) , \tag{7.13}$$

and, consequently

$$(\Box_\alpha^{-1})^\wedge = \frac{1}{|\tau|} \sum_{k=o}^{\infty} (2k+1-\alpha(\operatorname{sgn}\tau))^{-1} \hat{\mathcal{L}}_k^{(o)} \quad . \tag{7.14}$$

<u>Remark</u> : For other results concerning $\Box_\alpha$ the reader should consult [2], [3], [7] and [8].

(2) A more interesting application is the "inversion" of $-\overline{Z} = -\frac{\partial}{\partial\overline{z}} + iz\frac{\partial}{\partial t}$. $\overline{Z}$ is the Hans Lewy operator ([10]). A convenient way of calculating its Laguerre matrix is to write it as follows

$$\begin{aligned} -\tilde{\overline{Z}}_\tau \tilde{\varphi} &= -\tilde{\overline{Z}}_\tau(1 \,\tilde{*}_\tau\, \tilde{\varphi}) \\ &= -\tilde{\overline{Z}}_\tau\Big(\sum_{k=o}^{\infty} \tilde{\mathcal{L}}_k^{(o)}\Big) \,\tilde{*}_\tau\, \tilde{\varphi} \\ &= \sqrt{2|\tau|} \sum_{k=o}^{\infty} \sqrt{k+1}\, \tilde{\mathcal{L}}_k^{(1)} \,\tilde{*}_\tau\, \tilde{\varphi} \quad , \quad \tau \neq 0 \quad , \end{aligned}$$

where I used (4.15) if $\tau > 0$ and a similar procedure if $\tau < 0$. Therefore

$$\hat{\mathcal{L}}_+(-\hat{\overline{Z}}_+) = \sqrt{2|\tau|} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \sqrt{2} & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \sqrt{3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{bmatrix} , \tag{7.15}$$

(7.16) $$\hat{\mathcal{L}}_-(-\hat{\bar{Z}}_-) = [\hat{\mathcal{L}}_+(-\hat{\bar{Z}}_+)]^t \quad ,$$

and

(7.17) $$\hat{\mathcal{L}}(-\hat{\bar{Z}}) = \hat{\mathcal{L}}_+(-\hat{\bar{Z}}_+) \oplus \hat{\mathcal{L}}_-(-\hat{\bar{Z}}_-) \quad .$$

Setting

(7.18) $$\hat{W}_+ = \frac{1}{\sqrt{2|\tau|}} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{1}{\sqrt{1}} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{bmatrix}$$

and

(7.19) $$\hat{W}_- = \hat{W}_+^t \quad ,$$

one has

(7.20) $$\hat{\mathcal{L}}(-\hat{\bar{Z}})\ (\hat{W}_+ \oplus \hat{W}_-) = \hat{I}_+ \oplus (\hat{I}_- - 1_-^{(1,1)}) \quad ,$$

where I is the identity matrix and $1^{(1,1)}$ is the matrix with 1 in the (1,1) position and zeros everywhere else. Thus $1_-^{(1,1)}$ represents $\mathcal{L}_o^{(o)}$ for $\tau < 0$. Now, Theorem 1.47 yields

<u>7.21 Theorem</u> : Set

(7.22) $$\hat{W} = \frac{1}{\sqrt{2|\tau|}} \sum_{k=o}^{\infty} \frac{1}{\sqrt{k+1}} \hat{\mathcal{L}}_k^{(-1)} \quad .$$

Then

(7.23) $$-\bar{Z}W = I - C \quad ,$$

where

$$\hat{C} = \begin{cases} 2e^{-|\xi|^2/4|\tau|} & \text{if } \tau < 0 \\ 0 & \text{if } \tau > 0 \end{cases} .$$

Remark : A simple calculation yields

(7.24) $$C = \frac{1}{\pi^2(|z|^2+it)^2} ,$$

i.e. the Cauchy - Szegö (singular) projection on to the boundary values of the Hardy space of antiholomorphic functions on the generalized upper half-plane in $\mathbb{C}^2$.

(3) Consider a Mikhlin - Calderon - Zygmund operator on $\mathbb{R}^2 = \mathbb{C}$ induced by a homogeneous function f of degree -2, whose symbol, $\hat{f} = \sigma(f)$, has the following Fourier series expansion

(7.25) $$\sigma(f) = \sum_{k=0}^{\infty} a_k \frac{\zeta^k}{|\zeta|^k} + \sum_{k=1}^{\infty} b_k \frac{\bar{\zeta}^k}{|\zeta|^k} ,$$

where $\zeta = \xi = \xi_1 + i\xi_2$. I shall extend it to a left-invariant principal value convolution operator on H_1 . Since

(7.26) $$\zeta = [-2i\ \hat{\bar{Z}}]_{\tau = 0} ,$$

and

(7.27) $$\bar{\zeta} = [-2i\ \hat{Z}]_{\tau = 0} ,$$

ζ and $\bar{\zeta}$ can be extended to

(7.28) $$E(\zeta) = -2i\ \hat{\bar{Z}} ,$$

and

(7.29) $$E(\bar{\zeta}) = -2i\ \hat{Z} .$$

I also extend ζ^k by

(7.30) $$E(\zeta^k) = [(-2i\,\bar{Z})^k]^\wedge \quad , \quad k = 1,2,3,\ldots \quad ,$$

and similarly for $E(\bar{\zeta}^k)$. It is easy to see that

(7.31) $$\lim_{\tau\to o} E(\zeta^k) = \zeta^k \, .$$

This also holds for $E(\bar{\zeta}^k)$. As for $|\zeta|$, I note that

(7.32) $$|\zeta|^2 = 2|\tau| \left(\frac{|\zeta|^2}{2|\tau|}\right)$$

$$= 4|\tau| \sum_{k=o}^{\infty} (2k+1)\, \hat{\mathfrak{L}}_k^{(o)} \left(\frac{|\zeta|^2}{2|\tau|}\right) \, .$$

Using the fact that $\hat{\mathfrak{L}}_k^{(o)}$ - s are orthogonal projections, a reasonable definition of the extension, $E(|\zeta|^k)$, of $|\zeta|^k$ is given by

(7.33) $$E(|\zeta|^k) = [(4\square_o)^{k/2}]^\wedge \quad ,$$

$k = 0,\pm 1,\pm 2,\ldots$, where

(7.34) $$[(4\square_o)^{k/2}]^\wedge = (2\sqrt{|\tau|})^k \sum_{k=o}^{\infty} (2n+1)^{k/2}\, \hat{\mathfrak{L}}_n^{(o)} \left(\frac{|\zeta|^2}{2|\tau|}\right) \, .$$

$\square_o$ already occurs in example (1) of this section. Recall the formula

(7.35) $$s^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_o^\infty t^\alpha e^{-st} \frac{dt}{t} \qquad , \alpha > 0 \, .$$

For $k = 1,2,\ldots$ I write

(7.36) $$[(4\square_o)^{-k/2}]^\wedge = \sum_{n=o}^{\infty} \frac{1}{\Gamma(\frac{k}{2})} \int_o^\infty s^{k/2-1}\, e^{-4|\tau|(2n+1)s}\, ds\, \hat{\mathfrak{L}}_n^{(o)} \quad ,$$

$$= \frac{1}{\Gamma(\frac{k}{2})} \int_o^\infty s^{k/2-1} \exp\left\{-\frac{|\zeta|^2}{2|\tau|} \tanh\,(4|\tau|s)\right\} \frac{ds}{\cosh(4|\tau|s)}$$

where I used (1.24) to carry out the summation. The last integral in (7.36) permits one to take the limit as $\tau \to 0$ and yields.

<u>7.37 Proposition</u> : Let $k = 1,2,3,\ldots$. Then

$$[(4\square_o)^{-k/2}]^\wedge \in C^\infty(\mathbb{R}^3 \setminus 0) \quad , \tag{7.38}$$

and

$$\lim_{\tau\to o} [(4\square_o)^{-k/2}]^\wedge = |\zeta|^{-k} \quad \text{if } \zeta \neq 0 \quad . \tag{7.39}$$

<u>7.40 Corollary</u> : Set

$$E\left(\frac{\zeta^k}{|\zeta|^k}\right) = [(-2i\ \bar{Z})^k\ (4\square_o)^{-k/2}]^\wedge \quad . \tag{7.41}$$

Then

$$\lim_{\tau\to o} E\left(\frac{\zeta^k}{|\zeta|^k}\right) = e^{ik\theta} \quad , \quad k = 0,1,2,\ldots \quad , \tag{7.42}$$

and, replacing $\bar{Z}$ by Z in (7.41), one has

$$\lim_{\tau\to o} E\left(\frac{\bar{\zeta}^k}{|\zeta|^k}\right) = e^{-ik\theta} \quad , \quad k = 1,2,\ldots \quad , \tag{7.43}$$

where $\zeta = |\zeta|\ e^{i\theta}$.

Since the reader is, by now, probably lost in the technicalities, let me state again, that the purpose of the present discussion is to derive the following formula

$$\lim_{\tau\to o} [E(e^{ip\theta}) \hat{*} E(e^{iq\theta})]^\wedge = e^{i(p+q)\theta} \quad , \tag{7.44}$$

which is the Mikhlin - Calderon - Zygmund calculus on $\mathbb{R}^2$. To derive (7.44) I need a result on the behaviour of the Laguerre series for large argument.

<u>7.45 Proposition</u> : Let $p = 0, \pm 1, \pm 2, \ldots$. Let

(7.46) $$a_n = 1 + \frac{c(p)}{n} + 0\left(\frac{1}{n^2}\right) ,$$

where $c(p)$ is independent of $n = 1,2,3,\ldots$. Then

(7.47) $$\lim_{\tau \to 0} \sum_{n=1}^{\infty} a_n \hat{\mathcal{L}}_n^{(p)} = (-i \operatorname{sgn} p)^p e^{ip\theta} .$$

<u>Proof</u> : Using the integral formula of (7.36) the following limits can be evaluated with no difficulty.

(7.48) $$\lim_{x \to 0} 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{p/2}} e^{-x/2} x^{p/2} L_n^{(p)}(x) = 1 ,$$

and

(7.49) $$\lim_{x \to \infty} 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{p/2+1}} e^{-x/2} x^{p/2} L_n^{(p)}(x) = 0 .$$

This proves (7.47) if $a_n = 1 + \frac{c(p)}{n}$. Note that

(7.50) $$\lim_{x \to \infty} x^{p/2} e^{-x/2} L_n^{(p)}(x) = 0 ,$$

for each individual $n = 0,1,2,\ldots$. Therefore to prove Proposition 7.45 it suffices to prove the following Lemma.

<u>7.51 Lemma</u> : Assume $a_n = 0\left(\frac{1}{n^2}\right)$. Then

(7.52) $$\lim_{x \to \infty} \sum_{n=1}^{\infty} a_n (-1)^n \ell_n^{(p)}(x) = 0 .$$

<u>Proof of Lemma 7.51</u> : Recall the Hankel transform of the Laguerre functions

(7.53) $$(-1)^n \ell_n^{(p)}(x) = \frac{1}{2} \int_0^{\infty} \ell_n^{(p)}(y) \, J_p(\sqrt{xy}) \, dy ,$$

where J_p is the standard Bessel function.
Suppose $p=0$. Using Schwartz's inequality and the boundedness of J_o, one has

(7.54) $$\left|\sum_{n=N}^{\infty} a_n(-1)^n \ell_n^{(o)}(x)\right|^2 \le \sum_{n=N}^{\infty} O(\frac{1}{n^2}) \sum_{n=N}^{\infty} O(\frac{1}{n^2}) \int_o^{\infty} [\ell_n^{(o)}(y)]^2\, dy$$

$$\le \left(\sum_{n=N}^{\infty} O(\frac{1}{n^2})\right)^2 .$$

By choosing N sufficiently large, (7.54) can be made arbitrarily small.
Next choosing x sufficiently large

$$\sum_{n=1}^{N} a_n(-1)^n \ell_n^{(p)}(x)$$

can also be made arbitrarily small. This proves (7.52) when $p = 0$. A similar argument yields Lemma 7.51 for arbitrary p. Thus we proved Lemma 7.51 and, hence, Proposition 7.45.

Now I am ready to derive (7.44). Thus, if $p,q \ge 0$, then

(7.55) $$[E(e^{ip\theta}) \hat{*} E(e^{iq\theta})]^{\sim}_{\tau} = [(-2i\, \tilde{\tilde{Z}}_{\tau})^p (2\sqrt{|\tau|})^{-p} \sum_{n=o}^{\infty} (2n+1)^{-p/2} \tilde{\mathcal{L}}_n^{(o)}]$$

$$\tilde{*}_{\tau} [(-2i\, \tilde{\tilde{Z}}_{\tau})^q (2\sqrt{|\tau|})^{-q} \sum_{\ell=o}^{\infty} (2\ell + 1)^{-q/2} \tilde{\mathcal{L}}_{\ell}^{(o)}] = (-2i\, \tilde{\tilde{Z}}_{\tau})^{p+q} (\sqrt{2}\sqrt{2|\tau|})^{-p-q}$$

$$\cdot \sum_{n=o}^{\infty} (2n+1)^{-p/2} \tilde{\mathcal{L}}_{n+q}^{(o)}$$

$$\tilde{*}_{\tau} \sum_{\ell=o}^{\infty} (2\ell+1)^{-q/2} \tilde{\mathcal{L}}_{\ell}^{(o)}$$

$$= i^{p+q}\, 2^{(1/2)(p+q)} \left(-\frac{\tilde{\tilde{Z}}_{\tau}}{\sqrt{2|\tau|}}\right)^{p+q}$$

$$. \sum_{n=o}^{\infty} (2n+1)^{-p/2} (2[n+q]+1)^{-q/2} \tilde{\mathcal{L}}^{(o)}_{n+q}$$

$$= i^{p+q} 2^{(1+2)(p+q)} \sum_{n=p}^{\infty} (2n+1)^{-p/2} (2[n+q]+1)^{-q/2}$$

$$. [(n+q)(n+q-1) \ldots (n-p+1)]^{1/2} \tilde{\mathcal{L}}^{(p+q)}_{n-p}$$

$$= i^{p+q} \sum_{n=p}^{\infty} \left(1 + \frac{c(p,q)}{n} + O(\frac{1}{n^2})\right) \tilde{\mathcal{L}}^{(p+q)}_{n-p} \quad .$$

Therefore, according to Proposition 7.45, I have derived (7.44). This calculation, of course, works for arbitrary p and q . Extending by linearity, I have obtained the classical Mikhlin - Calderon - Zygmund calculus. To state it, let me recall (7.4), the definition of $\hat{*}$, namely

$$\hat{F} \hat{*} \hat{G} = F *_H G \quad .$$

<u>7.56 Theorem</u> : (M.C.Z.) Let f and g induce principal value convolution operators on $\mathbb{R}^2$. Then

$$\lim_{\tau \to o} [E(\hat{f}) \hat{*} E(\hat{g})]^{\wedge} = \hat{f} \hat{g} \quad . \tag{7.57}$$

To come full circle, Proposition 7.45 implies the following result.

<u>7.58 Theorem</u> : Let F induce a principal value convolution operator on H_1 with Laguerre matrix $\hat{\mathcal{L}}(\hat{F}) = \hat{\mathcal{L}}_+(\hat{F}_+) \oplus \hat{\mathcal{L}}_-(\hat{F}_-)$. Then $\hat{F}(\tau=0) = \hat{F}_+(\tau=0) = \hat{F}_-(\tau=0)$ has the following expansion in a Fourier series

$$\hat{F}(\tau=0) = \sum_{p=o}^{\infty} a_p \frac{(-i\zeta)^p}{|\zeta|^p} + \sum_{q=1}^{\infty} b_q \frac{(i\bar{\zeta})^q}{|\zeta|^q} \quad , \tag{7.59}$$

where

$$a_p = \lim_{k \to \infty} \hat{F}^{(p)}_{+,k} \quad , \; p \geq 0 \quad , \tag{7.60}$$

$$b_q = \lim_{k \to \infty} \hat{F}^{(-q)}_{+,k} \quad , \; q > 0 \quad . \tag{7.61}$$

7.62 Remark : Suppose we consider the "diagonals" of the Laguerre matrix as the analogues of the exponentials in a Fourier series on S^1 - this makes sense in view of Theorem 7.58. Then, by summing the Fourier series one obtains a symbol on $\mathbb{R}^2$, and by summing the Laguerre diagonals one obtains the Laguerre "symbol" (matrix) on H_1. This confirms, heuristically, the idea proposed in this article, that the Laguerre matrix calculus on H_1 replaces the classical calculus of symbols on $\mathbb{R}^2$.

7.63 Remark : I defined the extension operator, E, by

(i) $E(\hat{f}) = \hat{F}$ for some F, which induces a principal value convolution operator on H_1, and

(ii) $\lim_{\tau \to o} E(\hat{f}) = \hat{f}$,

where f induces a principal value convolution operator on $\mathbb{R}^2$. E is not defined uniquely by (i) and (ii) and in example (3) I made a particular choice that I found convenient. Actually f can have arbitrary homogeneity and then $E(\hat{f})$ has the same H-homogeneity. A similar extension operator was used in [4] and [5] to generalize the results of [6] - also note example (2) in section 7 of this article - to some left-invariant homogeneous differential operators on H_n.

8. FINAL COMMENTS

There is a great deal of work to be done to make the "Laguerre calculus" into an effective working tool of analysis. To begin with, (i) Theorem 1.47 should be extended to H_n, i.e. to arbitrary dimensional Heisenberg groups with general positive definite Levi forms (see [1]), (ii) a calculus is needed to invert Laguerre matrices, and (iii) properties of the symbol should be characterized by its exponential Laguerre series and vice-versa. Then all this should be transplanted to manifolds - some of this can already be found in [1].

Next one should include elliptic symbols in the Laguerre calculus. This will allow a more systematic treatment of some non-elliptic boundary value problems, e.g. the $\bar{\partial}$-Neumann problem on strongly pseudo-convex domains, Bergmann kernels, etc. All of this will certainly take a great deal of time and effort. I shall return to these questions in future publications.

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References

[1] Beals, R.W. and Greiner, P.C., "Non-elliptic differential operators of type $\Box_b$", (in preparation).

[2] Folland, G.B., "A fundamental solution for a subelliptic operator", Bull. Amer. Math. Soc. 79 (1973), pp. 373 - 376.

[3] Folland, G.B. and Stein, E.M., "Estimates for the $\bar{\partial}_b$ - complex and analysis on the Heisenberg group", Comm. Pure Appl. Math. 27 (1974), pp. 429 - 522.

[4] Geller, D., "Fourier analysis on the Heisenberg group. I. Schwartz space", J. Func. Analysis 36 (1980), pp. 205-254.

[5] Geller, D., "Local solvability and homogeneous distributions on the Heisenberg group", Comm. PDE, 5 (5) (1980), pp. 475-560.

[6] Greiner, P.C., Kohn, J.J. and Stein, E.M., "Necessary and sufficient conditions for the solvability of the Lewy equation", Proc. Nat. Acad. of Sciences, U.S.A., 72 (1975), pp. 3287-3289.

[7] Greiner, P.C. and Stein, E.M., "Estimates for the $\bar{\partial}$-Neumann problem", Math. Notes Series, n° 19, Princeton, Univ. Press, Princeton, N.J. 1977.

[8] Greiner, P.C., and Stein, E.M., "On the solvability of some differential operators of type $\Box_b$", Proc. of the Seminar on Several Complex Variables, Cortona, Italy, 1976-1977, pp. 106-165.

[9] Koranyi, A. and Vagi, S., "Singular integrals in homogeneous spaces and some problems of classical analysis", Ann. Scuola Norm. Sup. Pisa 25 (1971), pp. 575-648.

[10] Lewy, H., "An example of a smooth linear partial differential equation without solution", Ann. of Math., 66 (1957), pp. 155-158.

[11] Mauceri, G., "The Weyl transform and bounded operators on $L^p(\mathbb{R}^n)$", Report n° 54 of the Math. Inst. of the Univ. of Genova, 1980.

[12] Mikhlin, S.G., "Multidimensional singular integrals and integral equations", Pergamon Press, 1965.

[13] Nagel, A. and Stein, E.M., "Lectures on pseudo-differential operators", Math. Notes Series, n° 24, Princeton Univ. Press, Princeton, N.J. 1979.

[14] Seeley, R.T., "Elliptic Singular Integral Equations", Amer. Math. Soc. Proc. Symp. Pure Math. 10 (1967), pp. 308-315.

[15] Szegö, G., "Orthogonal polynomials", Amer. Math. Soc. Colloquium Publ., V. 23, Amer. Math. Soc., Providence, R.I., 1939.