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DIFFRACTION BY CONVEX BODIES

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In the spectral theory of the laplacian in exterior domains "distorted plane waves" are fundamental. For the exterior domain $\mathbb{R}^n \setminus K$, where K is a compact set with smooth boundary ∂K, one defines the distorted plane wave $\Phi(x,\omega,k)$ for the Dirichlet problem as follows :

i) $(\Delta + k^2)\Phi = 0$ on $\mathbb{R}^n \setminus K$,

ii) $\Phi = 0$ on ∂K (Dirichlet condition),

iii) $\Phi = e^{-ikx\cdot\omega} + v$, where as $|x| \to \infty$

$$v = |x|^{\frac{1-n}{2}} e^{-ik|x|}\left(f\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right)\right) \text{ (Sommerfeld condition).}$$

For a proof of the existence and uniqueness of Φ satisfying i) - iii) one may consult [11].

This seminar deals with an approximate construction of $\Phi(x,\omega,k)$ in the case that K is strictly convex -- in the sense that the normal curvatures of ∂K are everywhere strictly positive. The construction is asymptotic to order k^{-N} for any given N as k tends to ∞, and it permits the explicit asymptotic expansion of two quantities of interest in scattering theory, the scattering phase s(k) and the forward diffraction peak $a(\theta,\theta,k)$. These can be expressed in terms of $\Phi(x,\omega,k)$ as follows :

$$\frac{ds}{dk} = \frac{1}{8\pi^2}\left(\frac{k}{2\pi}\right)^{n-3} \int_{|\omega|=1} d\omega \int_{\partial K} \left|\frac{\partial\Phi}{\partial\nu}\right|^2 (x\cdot\nu)\, dS$$

$$a(\theta,\omega,k) = \frac{i}{4\pi}\left(\frac{k}{2\pi}\right)^{n-2} \int_{\partial K} e^{ik\theta\cdot x} \frac{\partial\Phi}{\partial\nu}\, dS ,$$

where ν is the unit normal to $\partial K_\wedge$ pointing into $\mathbb{R}^n \setminus K$. By substituting the approximations for Φ into these formulas one can conclude that ds/dk and $a(\theta,\theta,k)$ have complete asymptotic expansions of the form

$$\sim \Sigma\, k^{n_i}(a_i + b_i \log k),\ n_i \searrow -\infty ,$$

and compute the first few terms :

$$(1)\quad \frac{1}{2\pi}\frac{ds}{dk} = \frac{n(4\pi)^{-\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} k^{n-1} V(K) - \frac{(n-1)(4\pi)^{-\frac{n-1}{2}}}{\Gamma(1+\frac{n-1}{2})} k^{n-2} A(\partial K) + O(k^{n-3})$$

(2) $$a(\theta,\theta,k) = (\frac{k}{2\pi})^{n-1} A(\theta) + c_n k^{n-\frac{5}{3}} \int_\Gamma K^{-\frac{1}{3}}(\theta)dS + \dots$$

Here $V(K)$ is the volume of K, $A(\partial K)$ is volume of ∂K, $A(\theta)$ is the volume of the projection of K onto $x\cdot\theta = 0$, Γ is the boundary of this projection, dS is the volume form on Γ, and $K(\theta)$ is the normal curvature in direction θ on the pre-image of Γ in K. The constant c_n is the finite part of a definite integral of Airy functions and depends only on n.

The constructions given here follows those of Ludwig [3] very closely but make use of improvements made possible by Melrose's proof of the symplectic equivalence of glancing hypersurfaces [7]. For a discussion of (1) one may see [5]. The expansion (2) was derived when K is a sphere by Rubinow and Wu [10], and conjectured for convex bodies by Keller and Rubinow [2]. The leading term was derived rigorously by Majda and Taylor [6]. The complete asymptotic expansion is due to R. Melrose [9]. The method of [9] is different from that used here and appears to be more powerful as it yields the same results for the Neumann problem. Still more refined results on $a(\theta,\omega,k)$-which permit a uniform expansion near $\theta = \omega$ -have been obtained by Melrose and M. E. Taylor. The construction given here seems sufficiently intuitive -at least to the author - that it may serve as a prologue to the results of Melrose and Melrose-Taylor.

Localization

Using the standard construction of geometric optics one can decompose $e^{-ikx\cdot\omega}$ into a sum of terms u_e , where

$$u_e = e^{-ikx\cdot\omega}(a_o + \frac{a_1}{k} + \dots + \frac{a_M}{k^M})$$

such that

i) $(\Delta + k^2)u_e = O(k^{-N})$

ii) the projections of the supports of u_e onto $x\cdot\omega = 0$ can be made subordinate to any given cover of $x\cdot\omega = 0$.

The strategy here will be, given u_e to construct a u_s satisfying

i) $(\Delta + k^2)u_s = O(k^{-N})$

ii) $u_s = -u_e$ on ∂K

iii) u_s satisfies the Sommerfeld condition.

Actually one has only to construct u_s on a neighborhood of ∂K in $\mathbf{R}^n \backslash K$ satisfying i) and ii) with wave fronts -or more precisely "frequency set" (see [1]) - over points near ∂K but strictly inside $\mathbf{R}^n \backslash K$ directed toward ∂K. Then u_s can be extended to satisfy the Sommerfeld condition by the outgoing Green's function for the laplacian on $\mathbf{R}^n$ (see [4], pp.521-3).

If the projection of the support of u_e on $x \cdot \omega$ does not intersect Γ, the construction of u_s is a standard application of geometric optics. Hence from here on we consider only u_e whose support projects onto a neighborhood -which we may take as small as we wish - of a point on Γ.

The Ludwig-Melrose construction

The idea here is to find a representation of u_e in the form

(3) $$u_e = \int_{\mathbf{R}^{n-1}} e^{ik\theta}(a\,Ai(-k^{2/3}\rho) + b\,Ai'(-k^{2/3}\rho))\,d\xi + O(k^{-N})$$

where the integrand is an asymptotic solution to $(\Delta + k^2)w = 0$ uniformly in ξ, and one has additionally

(4) i) $\rho = \xi_1$ and $\frac{\partial\rho}{\partial\nu} > 0$ on ∂K ,

(5) ii) $b = 0$ on ∂K.

The function Ai is the standard Airy function

$$Ai(s) = \int_{-\infty}^{\infty} e^{i(\beta s + \frac{\beta^3}{3})}\,d\beta ,$$

and a and b have the form

$$a = \sum_{i=0}^{R} a_i(x,\xi)k^{-i+\frac{n}{2}-\frac{1}{3}}, \quad b = \sum_{i=0}^{R} b_i(x,\xi)k^{-i+\frac{n}{2}-\frac{2}{3}}.$$

Once we have (3) - (5) the function u_s will be given by

$$(6) \quad u_s = -\int_{\mathbb{R}^{n-1}} e^{ik\theta}(aA(-k^{2/3}\rho) + bA'(-k^{2/3}\rho)) \frac{A_i(-k^{2/3}\xi_1)}{A(-k^{2/3}\xi_1)} d\xi ,$$

where $A(s) = Ai(e^{-\frac{2\pi i}{3}} s)$. Note that , since A satisfies Airy's differential equation, the integrand is automatically an asymptotic solution to $(\Delta + k^n)w = 0$ in $\mathbb{R}^n \backslash K$ -since $A(s)$ is exponentially increasing as $s \to +\infty$, we use the fact $\rho > \xi$ in $\mathbb{R}^n \backslash K$ here. The choice of A is made so that the frequency set of u_s is directed toward ∂K from $\mathbb{R}^n \backslash K$.

As we mentioned earlier the constructions here are strictly local. We assume that we are given $x_o \in \partial K$ with $\omega \cdot \nu(x_o) = 0$ and, writing $\xi = (\xi_1, \xi')$, a ξ'_o such that $\nabla\theta(x_o, 0, \xi'_o) = -\omega$. All the assertions (of existence etc...) in the constructions that follow are to be qualified by "for (x,ξ) in a neighborhood of $(x_o, 0, \xi'_o)$" -even though this will always be omitted. Just how small the support of u_e must be is only determined at the end of the construction.

The representation (3) with conditions (4), (5) is the delicate part of the construction. One first determines θ and ρ and then a and b . In order that the integrand in (3) be an asymptotic solution to $(\Delta + k^2)w = 0$, θ and ρ must satisfy the "eichonal" equations :

$$(7) \quad \begin{aligned} &a) \quad |\nabla_x \theta|^2 + \rho|\nabla_x \rho|^2 = 1 \\ &b) \quad \nabla_x\rho \cdot \nabla_x \theta = 0 \end{aligned}$$

on $\rho \geq 0$. These equations are solved by choosing a smooth family of strictly convex surfaces S_ξ with $S_\xi = \partial K$ when $\xi_1 = 0$, and defining $\rho(x,\xi) = 0$ on S_ξ . Note that, since we want $\nabla_x\rho \neq 0$, this implies $|\nabla_x\theta(x,\xi)|^2 = 1$ on S_ξ and $\nabla_x\theta(x,\xi)$ is tangent to S_ξ . Thus we must choose θ on S_ξ to be a solution of the surface eichonal on S_ξ. With these choices 7a) and 7b) determine θ and ρ uniquely for x outside S_ξ, i.e. in the region where we will have $\rho \geq 0$. The condition (4) implies and, modulo a change of variables in ξ, is equivalent to the following geometric condition on S_ξ and $\theta \restriction S_\xi$: if the straight line through $x_o \in S_{\xi_o}$ with direction $\nabla_x \theta(x_o,\xi_o)$ hits ∂K at x', then the reflection of this line in ∂K is, for some $x_1 \in S_{\xi_o}$, the line through x_1 with direction $\nabla_x \xi(x_1,\xi_o)$. In [3] the surfaces S_ξ were only chosen so that (4) held up to an error which was $O(\xi_1^N)$ for all N. However, it is a direct consequence of [7] (the derivation is given in [8] that S_ξ and $\theta \restriction S_\xi$ can be chosen so that (4) holds exactly. Then one completes the construction by extending θ and ρ as C^∞ functions in the complement of $\rho \geq 0$, maintaining (4).

If we replace Ai and Ai' by their integral representations, (3) becomes

$$u_e = \int e^{ik(\theta - \beta\rho + \frac{\beta^3}{3})} (a + ik^{1/3}\beta b)\, d\xi\, d\beta \ .$$

Note that, writing $\xi = (\xi_1,\xi')$, if $\det \dfrac{\partial^2\theta}{\partial\xi'\partial\xi'} \neq 0$, then this integral can be expanded by the method of stationary phase. If the result of this expansion agrees with u_e, then we must have

$$(8) \qquad -x\cdot\omega = \Phi(x) \equiv (\theta - \beta\rho + \frac{\beta^3}{3}) \restriction \xi = \xi(x), \beta = \beta(x),$$

where $\xi(x)$ and $\beta(x)$ are defined by

$$\theta_\xi - \beta\rho_\xi = 0 \quad \text{and} \quad -\rho + \beta^2 = 0$$

However, since Φ is automatically a solution of the standard eichonal $(|\nabla\Phi|^2 = 1)$ it suffices to have (8) hold for x on a surface transverse to ω. The eichonal equations (7) and condition (4) remain valid if we replace θ by $\theta + \psi(\xi)$, and we must exploit this freedom to obtain $\det \partial^2\theta/\partial\xi'\partial\xi' \neq 0$ and (8) .

Introducing local coordinates (z,y) where z = 0 on ∂K, $\partial f/\partial z = \partial f/\partial\nu$ on ∂K and $y_1 = x\cdot\omega$ on ∂K, we can assume S_ξ is given by $z = \alpha(y,\xi)$. Writing $y = (y_1,y')$, it is a consequence of the constructions in [7] that S_ξ and $\theta \restriction S_\xi$ can be chosen so that $\det \partial^2\theta/\partial y\partial\xi \neq 0$ (this is used in [8]) and $\det \partial^2\theta/\partial y'\partial y' \neq 0$. We let β denote $x\cdot\omega$ written as a function of (z,y).

To achieve 8) we begin by solving $(\theta_z,\theta_y) = (\beta_z,\beta_y)$ on $z=\alpha(y,\xi)$ for $y=y(\xi)$. This is over determined, but since $|\nabla_x\theta| = |\nabla_x(x\cdot\omega)| = 1$ when $z=\alpha(y,\xi)$ it suffices to solve $(\theta_z,\theta_{y'}) = (\beta_z,\beta_{y'})$ on $z=\alpha(y,\xi)$. To check the hypothesis of the implicit function theorem, we set $\xi_1 = 0$ (so that $\alpha=\theta_z = 0$) and compute

$$\begin{pmatrix} \dfrac{\partial^2(\theta-\beta)}{\partial z\,\partial y_1} & \dfrac{\partial^2(\theta-\beta)}{\partial z\partial y'} \\[2ex] \dfrac{\partial^2(\theta-\beta)}{\partial y'\partial y_1} & \dfrac{\partial^2(\theta-\beta)}{\partial y'\partial y'} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial^2\beta}{\partial z\,\partial y_1} & -\dfrac{\partial^2\beta}{\partial z\,\partial y'} \\[2ex] \dfrac{\partial^2\theta}{\partial y'\partial y_1} & \dfrac{\partial^2\theta}{\partial y'\partial y'} \end{pmatrix}$$

Since $\partial^2\beta/\partial z\,\partial y_1$ is nonzero by the strict convexity of ∂K, and we may assume $\partial^2\theta/\partial y'\partial y_1$ vanishes at the base point, we conclude that $(\theta_z\,,\,\theta_y) = (\beta_z\,,\,\beta_y)$ can be solved for $y(\xi)$ on $z=\alpha(y,\xi)$.

Now we defined ψ by the requirement

$$\theta_\xi(\alpha(y(\xi),\xi),y(\xi),\xi) + \psi_\xi = 0 \quad . \tag{9}$$

To check that

$$\theta_{\xi\xi} + \theta_{\xi z}\alpha_y y_\xi + \theta_{\xi y}\, y_\xi$$

is symmetric, we note that

$$\theta_{z\xi} + \theta_{zz}\,\alpha_y\, Y_\xi + \theta_{zy}\, y_\xi = \beta_{zz}\,\alpha_y\, y_\xi + \beta_{zy}\, y_\xi$$

$$\theta_{y\xi} + \theta_{yz}\,\alpha_y\, y_\xi + \theta_{yy}\, y_\xi = \beta_{yz}\,\alpha_y\, y_\xi + \beta_{yy}\, y_\xi \; .$$

Since (9) determines ψ up to an additive constant, we complete the construction of ψ by choosing this constant so that $\theta+\psi = -x\cdot\omega$ at the base point. Further work along exactly, the same lines shows that $\det\partial^2(\theta+\psi)/\partial\xi'\,\partial\xi' \neq 0$ (This uses $\det \dfrac{\partial^2\theta}{\partial y\,\partial\xi} \neq 0$) and that

$z=\alpha(y(\xi),\xi))$, $y=y(\xi)$ defines a surface transverse to ω. Then it follows that (8) holds when θ is replaced by $\theta+\psi$.

We will not discuss the construction of the amplitudes $a(x,\xi,k)$ and $b(x,\xi,k)$ here. In [3] a and b are constructed so that , given the preceding construction of θ and ρ, (3) holds and in place of (5) one has $b = O(\xi_1^N)$ for any N on ∂K. The modifications needed to improve this to (5), i.e. $b = 0$ on ∂K are substantially simpler than those that were used in obtaining (4) - no use of [7] is involved. Actually, imposing (5) for all $x \in \partial K$ (or even the weaker condition $b = O(\xi_1^N)$)) would make it impossible to keep the intersection of the support of a and b with ∂K strictly inside the set where θ and ρ are defined. This is a turn would prevent us from making the integrand in (3) an asymptotic solution to $(\Delta + k^2)w = 0$ on a neighborhood of ∂K in $\mathbb{R}^n \setminus K$. However, we only impose (5) for (x,ξ) in a small neighborhood of the base point. Provided the projection of the support of u_e is made sufficiently small, one still has $u_s(x,k) + u_e(x,k) = O(k^{-N})$ for $x \in \partial K$ in this case.

<u>The representation of</u> $\partial\Phi/\partial\nu$

Away from the intersection of ∂K with the pre-image of Γ the expansion of $\partial\Phi/\partial\nu$ is easy to compute from geometric optics ; the leading term is

$$\frac{\partial\Phi}{\partial\nu} = \begin{cases} -2ik\omega\cdot\nu e^{-ikx\cdot\omega} & \text{if } \omega\cdot\nu < 0 \\ 0 & \text{if } \omega\cdot\nu > 0 \end{cases}$$

(the "Kirchhoff approximation"). The next term is O(1) and it does not contribute to the second term in (1) and (2).

In a neighborhood of a point $x_0 \in \partial K$ where $\omega\cdot\nu(x_0) = 0$, i.e. a point that projects to Γ, one can combine (3)-(6) to get

$$\frac{\partial\Phi}{\partial\nu} = \int_{\mathbb{R}^{n-1}} e^{ik\theta}(-k^{2/3}\frac{\partial\rho}{\partial\nu}a + \frac{\partial b}{\partial\nu})F(-k^{2/3}\xi_1)\,d\xi + O(k^{-N})$$

where $F(x) = Ai'(x) - \frac{A'(x)}{A(x)}Ai(x)$. Expanding by stationary phase in the variable ξ', this can be further simplified to a representation in the form

$$(10)\quad \frac{\partial\Phi}{\partial\nu}(x,k) = \int_{\mathbb{R}} e^{ik\check{\theta}(x,\xi_1)} G(x,\xi_1,k)F(-k^{2/3}\xi_1)\,d\xi_1 + O(k^{-N})$$

where G has the form

$$G = \sum_{i=o}^{M} G_i(x,\xi_1)k^{\frac{1}{6}+i}$$

Substituting (10) and the analogous expression (derived from (3)),

$$e^{-ikx\cdot\omega} = \int_{\mathbb{R}} e^{ik\tilde{\theta}(x,\xi_1)} H(x,\xi_1,k) Ai(-k^{2/3}\xi_1)d\xi_1 + O(k^N)$$

into the integral formulas for ds/dk and $a(\omega,\omega,k)$ one derives (1) and (2). The crucial advantage here of (10) over the formulas that could be obtained from [3] is that one can eliminate the oscillatory $e^{ik\tilde{\theta}}$ factors by an integration in an x-variable without disturbing the Airy functions. At the final stage in the derivation of (1) and (2) one must expand integrals of the form

$$\int_{\mathbb{R}} H(s)G(-k^{2/3}s)ds$$

where G is a polynomial in A'/A, $\bar{A}'/\bar{A}$, Ai and their derivatives; it is here that the $k^r \log k$ terms seem to appear in the asymptotic expansions. It is known that there are no logarithms (and only integral powers of k) in the expansion of ds/dk (see [5]), but unknown whether logarithms actually appear in the expansion of $a(\theta,\theta,k)$.

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