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SINGULAR SPECTRUM OF PRODUCTS OF DISTRIBUTIONS AT u = 0 POINTS

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§ 1. INTRODUCTION

Let us consider a tempered distribution f defined on the n-dimensional real vector space $\mathbb{R}^n_{(x)}$ of the variables $x = x_1, \dots, x_n$ (The index x in $\mathbb{R}^n_{(x)}$ serves uniquely to indicate the name of the variables in the space $\mathbb{R}^n$ considered).

The essential support [1] $ES_X(f)$ of f at any given point X of $\mathbb{R}^n_{(x)}$ - which coincides, as we recall below, with its singular spectrum [2] - is the closed cone with apex at the origin in the dual space $\mathbb{R}^n_{(u)}$ of $\mathbb{R}^n_{(x)}$ composed of the "singular directions" along which the localized Fourier transform of f at X does not decrease exponentially in a well specified sense. Namely, the localized Fourier transform of f is defined for every $\gamma > 0$ by :

(1) $$\hat{F}_\gamma(u;X) = \int f(x)\, e^{-i\langle u,x\rangle - \gamma|u|(x-X)^2} dx$$

where $\langle u,x\rangle$ is the scalar product, usually defined as $u.x = \sum_{i=1}^{n} u_i x_i$. Then a point $U \neq 0$ in $\mathbb{R}^n_{(u)}$ (as also all points λU, $\lambda > 0$) is by definition outside $ES_X(f)$ if there exists an open cone $\mathcal{V}$ with apex at the origin in $\mathbb{R}^n_{(u)}$ containing U, and constants $\alpha > 0$, $\gamma_0 > 0$ (as also a polynomial $\mathcal{P}$ and $q \geq 0$) such that the bounds

(2) $$|\hat{F}_\gamma(u;X)| < [\mathcal{P}(|u|)(\gamma|u|)^{-q}]\, e^{-\alpha\gamma|u|}$$

be satisfied in the region $u \in \mathcal{V}$, for all γ satisfying $0 < \gamma < \gamma_0$.

In the bounds (2), the important factor is the exponential fall-off factor $e^{-\alpha\gamma|u|}$, bounds of the form $|\hat{F}_\gamma(u;X)| < [\mathcal{P}(|u|)(\gamma|u|)^{-q}]$ being always satisfied. The bounds (2), when they are satisfied, express the fact that the localized Fourier transform of f at X decreases exponentially for all $\gamma > 0$ sufficiently small, with a rate of decrease at least proportional to γ. The notion of essential support thus defined characterizes, as explained in detail in [1], the micro-local analytic structure of f, namely the various possible decompositions of f into sums of boundary values of analytic functions (in the sense of distributions). The directions from which these boundary values are obtained may depend on X. This characterization coincides with that associated in [2] with the singular spectrum, apart from the fact that all boundary values are distributions in [1], whereas they can be hyperfunctions in [2], even when f itself is a distribution. It is, however, proved in [3] that the two notions do coincide for distributions.

Let us now consider a product $f_1 f_2$ of two distributions. A standard theorem [2,1] says that this product can be well defined in a neighborhood of a point X, and it gives information on $ES_X(f_1 f_2)$, if the singular spectra of f_1 and f_2 at X do not contain opposite directions. Namely one has :

<u>Definition 1</u> : A point X is called a $u=0$ point with respect to the product $f_1 f_2$ if there exist $U_1 \in ES_X(f_1)$ and $U_2 \in ES_X(f_2)$, $U_1, U_2 \neq 0$, such that $U_1 + U_2 = 0$.

<u>Theorem 1</u> : If X is not a $u=0$ point, $f_1 f_2$ can be well defined as a distribution in the neighborhood of X and

$$ES_X(f_1 f_2) \subset ES_X(f_1) + ES_X(f_2) \tag{3}$$

In (3), $ES_X(f_1) + ES_X(f_2)$ is the set $\{u = u_1 + u_2,\ u_1 \in ES_X(f_1),\ u_2 \in ES_X(f_2)\}$.

Theorem 1 is an extension of a previous well-known result on the product of two distributions f_1, f_2 that are boundary values of analytic functions $\underline{f}_1$, $\underline{f}_2$ from common directions : $f_1 f_2$ is then the boundary value of $\underline{f}_1 \underline{f}_2$ from these directions. It can be proved either by using this latter result together with local decomposition theorems of distributions into sums of boundary values of analytic functions, or directly in terms of localized Fourier transforms (see [1]).

Let us now consider a $u=0$ point X. Theorem 1 then gives no information :

(i) it does not provide a definition of the product $f_1 f_2$ and

(ii) even in cases when this product is well defined from the outset by standard procedures (product of functions, etc...), it gives no information on $ES_X(f_1 f_2)$.

There have been a number of works devoted to problem (i), namely to define $f_1 f_2$ in various situations where it is not a priori well defined. We do not treat it here, but are interested instead in problem (ii). In the applications to relativistic quantum theory that we have in mind (see Appendix 1) one is in fact interested essentially in products of scattering operators, which are bounded operators, and related quantities, and the product is then always a well defined bounded operator. The link with products of distributions is as follows. Being given two bounded operators A', A" from $L^2(\mathbb{R}^n_{(x)})$ to $L^2(\mathbb{R}^p_{(t)})$ and from $L^2(\mathbb{R}^p_{(t)})$ to $L^2(\mathbb{R}^n_{(y)})$ respectively, the kernel a(x,y) of the product A = A"A' can be written (formally) as :

$$a(x,y) = \int [a'(x,t)\,a''(t,y)]\,dt \tag{4}$$

where a', a" are the kernels of A', A". We recall on the other hand that kernels of bounded operators are always well defined (tempered) distributions in view of the Schwartz nuclear theorem.

In cases when it applies, Theorem 1 allows one to determine the singular spectrum of the product a'(x,t)a"(t,y) and the standard theorems [2,1] on integrals of distributions then allow one (under some conditions on the supports of a', a" that are satisfied in the physical application) to get information on the singular spectrum of a. Namely, if we denote by u, v, w the dual variables of x, y, t respectively and adopt, for convenience, the definition $\langle(u,v),(x,y)\rangle = u.x - v.y$, and similar ones for $\langle(u,w),(x,t)\rangle$ and $\langle(w,v),(t,y)\rangle$, one has :

<u>Definition 2</u> : (X,Y) is a (u,v) = 0 point with respect to the product A"A' if $\exists\ T \in \mathbb{R}^p_{(t)}$ and $W \in \mathbb{R}^p_{(w)}$, $W \neq 0$, such that $(0,W) \in ES_{X,T}(a')$ and $(W,0) \in ES_{T,Y}(a'')$.

<u>Theorem 2</u> : If (X,Y) is not a (u,v) = 0 point, then $ES_{X,Y}(a) \subset \{(u,v);\ \exists\ T \in \mathbb{R}^p_{(t)},\ W \in \mathbb{R}^p_{(w)}$ such that $(u,W) \in ES_{X,T}(a')$ and $(W,v) \in ES_{T,Y}(a'')\}$.

This theorem can alternatively be proved in a more direct way (i.e. without using successively theorems on products and integrals) in terms of localized Fourier transforms : see [4]. The absence of information on $ES_{X,Y}(a)$ at (u,v) = 0 points turns out, however, to be a crucial problem in the application : all points are (u,v) = 0 points for some of the simplest cases encountered, for instance for the product $\equiv\!\oplus\!\equiv\!\odot\!\equiv$ where $\equiv\!\oplus\!\equiv$ and $\equiv\!\odot\!\equiv$ denote respectively the connected parts of the scattering operator $S_{3,3}$ and of $(S^{-1})_{3,3}$ between three initial and three final particles. This problem is at the origin, in recent years, of mathematical works [5,4], which have in turn been applied to the physical situation in [6] and [4] respectively, in a way which is briefly outlined in Appendix 1. The results of [5] are briefly mentioned in Appendix 2, the main part of the present text being devoted to present the mathematical results of [4] and related ones. We shall as a matter of fact be interested here, from a mathematical viewpoint, in the two following problems :

(i) essential support of a product of functions, and more particularly of locally square integrable functions and

(ii) essential support of the kernel of a product of bounded operators.

Let us first make some preliminary remarks. We consider here for instance case (i), but similar remarks apply to case (ii). First, if X is a u = 0 point, then the set $ES(f_1) + ES(f_2) = \bigcup_x (x, ES_x(f_1) + ES_x(f_2))$ is not always a closed subset of $\mathbb{R}^n_{(x)} \times \mathbb{R}^n_{(u)}$. Since the essential support $ES(f) = \bigcup_x (x, ES_x(f))$ of a distribution is always closed (see [1,2]), the best result that might a priori be expected is to show that $ES_X(f_1 f_2)$ is contained in $\overline{ES(f_1) + ES(f_2)}_{|X}$, which is the fiber at X of the closure of $ES(f_1) + ES(f_2)$. However, such a result cannot be expected in general. There do exist examples for which $ES_X(f_1 f_2)$ is strictly larger than $\overline{ES(f_1) + ES(f_2)}_{|X}$, and examples for which $ES_X(f_1 f_2)$ is certainly believed to be all $\mathbb{R}^n_{(u)}$, even though $\overline{ES(f_1) + ES(f_2)}_{|X}$ is much smaller : see Appendix 3. Some conditions on f_1, f_2 are therefore needed if one wishes to get information on $ES_X(f_1 f_2)$.

In the various works carried out on the problem, the results obtained, under various conditions, do not yield $\overline{ES(f_1) + ES(f_2)}_{|X}$ as a bound on $ES_X(f_1 f_2)$. Besides the necessity already mentioned of considering the closure of $ES(f_1) + ES(f_2)$, it appears necessary either in [5] or in the results described here to consider moreover certain limiting procedures that may enlarge the essential support. The present results apply under a general regularity property R on individual terms that is presented in Section 2. They are described in Section 3.

As will appear below, property R is not a property that can be expected to be automatically satisfied in "simple cases". Examples of simple functions that do not satisfy it are given in Section 2. It is a general property that is satisfied by a class of distributions that include "simple" cases, and others. As will appear in Section 3 , the results presented here, which rely on this property, are the best possible ones "in general", in a certain sense. On the other hand, other types of results can be obtained under other conditions on individual terms : this is the case for the results of [5], which may give for instance information, in certain cases, on products of functions that do not satisfy property R. These latter results are in fact of a different nature and apply to different situations. The limiting procedures obtained in both cases are correspondingly different : see Appendix 2.

§ 2. REGULARITY PROPERTY R

The regularity property R is essentially a condition on the way rates of exponential fall-off of localized Fourier transforms tend to zero in certain situations, when one approaches the essential support. Its content in terms of analyticity properties can be easily understood in simple cases as will be explained at the end of this section.

Let us first make some preliminary remarks. Being given a (tempered) distribution f and any direction $\hat{u}$ that does not belong to $ES_X(f)$, there exist by definition (see Section 1) an open cone $\mathcal{V}$ around $\hat{u}$ and constants $\alpha > 0$, $\gamma_o > 0$ such that the localized Fourier transform of f at X decreases exponentially like $e^{-\alpha\gamma|u|}$ in the direction of $\hat{u}$ for all $\gamma > 0$ smaller than γ_o. Let $\alpha(\hat{u})$ be the upper limit of all possible values of α for the given direction $\hat{u}$, all possible neighboring cones $\mathcal{V}$ and all possible γ_o. Information on $ES_X(f)$ alone yields no information on $\alpha(\hat{u})$: one only knows that $\alpha(\hat{u})$ is strictly positive, and it will in general tend to zero when $\hat{u}$ approaches a direction that belongs to $ES_X(f)$.

Information on $\alpha(\hat{u})$ may be derived, however, from the knowledge of the essential support of f at points x in the neighborhood of X. In fact, it follows from general results of essential support theory (see Theorems 9, 8 and 4 of [1]) that :

$$(5) \qquad \alpha(\hat{u}) = \sup_{\alpha'>o} \{\alpha' ; \hat{u} \notin ES_x(f) \ , \ \forall x \text{ such that } (x-X)^2 < \alpha'\}$$

Let us then consider a cone $\mathcal{C}$ with apex at the origin in $\mathbb{R}^n_{(u)}$ composed of a continuous set of directions that all lie outside $ES_X(f)$ and lie also outside $ES_x(f)$ for all points x such that $(x-X)^2 < \alpha_o$, $\alpha_o > 0$. The closure of $\mathcal{C}$ may, however, contain directions that do belong to $ES_X(f)$. It follows from the above analysis that, given any $\alpha < \alpha_o$, $\alpha > 0$, the localized Fourier transform of f at X falls off like $e^{-\alpha\gamma|u|}$, with the same uniform constant α, for all directions $\hat{u}$ in $\mathcal{C}$ and all sufficiently small γ. The maximal value $\gamma_o(\hat{u})$ of γ may, on the other hand, depend on $\hat{u}$ and in fact it must necessarily tend to zero when $\hat{u}$ approaches a direction that belongs to $ES_X(f)$.

The crucial content of property R is the requirement that $\gamma_o(\hat{u})$ does not tend to zero faster than linearly with respect to the angle $\widehat{(\hat{u};\partial\mathcal{C})}$ of $\hat{u}$ with the boundary of $\mathcal{C}$.

Applied to the case of a square integrable function, property R is more precisely the following requirement.

Regularity property R for square integrable functions.
A square integrable function f satisfies by definition the regularity property R at a point X if, being given any neighborhood $\mathcal{N}$ of X and any cone $\mathcal{C}$ with apex at the origin in $\mathbb{R}^n_{(u)}$, composed of a continuous set of directions $\hat{u}$ such that $\hat{u} \notin ES_x(f)$, $\forall x \in \mathcal{N}$, there exist $\alpha > 0$, a constant $\chi > 0$ and a function d of the variables u, γ, which is square integrable with respect to u and whose norm $[\int (d(u;\gamma))^2 du]^{1/2}$ is independent of γ, such that :

(6) $$|\hat{F}_\gamma(u;X)| < d(u;\gamma)e^{-\alpha\gamma|u|}$$

for all points u in $\mathcal{C}$ and all γ satisfying :

(7) $$0 < \gamma < \chi\widehat{(\hat{u},\partial\mathcal{C})}.$$

Condition (7) can be equivalently replaced by :

(8) $$0 < \gamma|u| < \chi' d(u;\partial\mathcal{C})$$

for some constant $\chi' > 0$, where $d(u,\partial\mathcal{C})$ is the distance of u to $\partial\mathcal{C}$.

The existence of $\alpha > 0$ is a direct consequence of the previous analysis, since a neighborhood of X always contains a subneighborhood of the form $(x-X)^2 < \alpha_o$, $\alpha_o > 0$, and the essential content of property R is, as already mentioned, condition (7), or (8). The bound (6) contains moreover a uniform function d. This is a slight extension of the two following results, which are valid [4] for any square integrable function f :

(i) Bounds of the form $|\hat{F}_\gamma(u;X)| < d(u;\gamma,X)$, where d is a square integrable function of u, whose norm is independent of γ and X, always hold for all $\gamma > 0$.

(ii) Bounds of the form (6) always hold if the closure of $\mathcal{C}$ does not intersect $ES_X(f)$ (apart from the origin) —

For a locally square integrable function, property R at X is defined similarly by considering the localized Fourier transform of χf, where χ has a compact support, is bounded and is equal to one in the neighborhood of X (the statement does not depend on the choice of χ).

In the case of the kernel a(x,y) of a bounded operator, property R is a requirement similar to above, which includes a function d of u, v, γ which is square integrable with respect to u, or alternatively v, and whose corresponding norm is independent of the other variables. The crucial content of the property is, however, again the condition analogous to (7), or (8). The precise statement is given in [4] and is therefore omitted here. Note that property R, as stated in [4], includes also certain uniformity conditions with respect to the point (X,Y) considered : bounds analogous to (6) are required to hold uniformly on $\hat{F}_\gamma(u,v;x,y)$ when (x,y) varies in a sufficiently small neighborhood of (X,Y).

Remarks :

1) Property R is automatically satisfied if, being given any neighborhood $\mathcal{N}$ of X, $ES_X(f)$ is contained (apart from the origin) in the interior of the set $\bigcup_{x\in\mathcal{N}} ES_x(f)$. This directly follows from the definition of the essential support and, for instance in the case of a square integrable function f, of result (ii) mentioned above.

2) Property R may depend on local coordinate systems, as will appear below in the discussion of analyticity properties. For instance, we shall see that the function $f(x_1,x_2) = (x_1 - x_2^3 + io)^\lambda$ of Example 3 does not satisfy property R at the origin. If, however, one uses the new variables $x_1' = x_1 - x_2^3$, $x_2' = x_2$, the function $(x_1' + i0)^\lambda$ satisfies it.

Regularity property R and analyticity

The content of property R in terms of analyticity properties is most easily understood in the following simple case.

Let us consider a distribution f whose essential support $ES_x(f)$ at all points x in a neighborhood of a point X is a closed convex salient cone C, independent of x. Then f is, in the neighborhood of X, the boundary value of an analytic function $\underline{f}$ from the directions of the open dual cone Γ of C. More precisely, being given any open cone Γ' with apex at the origin whose closure is contained (apart from the origin) in Γ, there exists $\varepsilon > 0$ such that $\underline{f}$ is analytic in $\omega \times \{\mathrm{Im} z \in \Gamma', |\mathrm{Im} z| < \varepsilon\}$ where z is the complexified variable of x and ω is a real neighborhood of X. However, ε may tend to zero when the cone Γ' expands to Γ, in which case $\underline{f}$ is analytic in a domain of the form $\omega \times \{\mathrm{Im} z \in B\}$, where B is of the form shown in Fig. 1 a). In view of the last part of Theorem 4 of [1], the basic content of property R, namely the fact that $\gamma_0(\hat{u})$ does not decrease faster than linearly with respect to the angle of $\hat{u}$ with C, is then equivalent, when it is satisfied, to the fact that ε does not tend to zero i.e. that $\underline{f}$ is analytic is a domain of the form $\omega \times \{\mathrm{Im} z \in \Gamma, |\mathrm{Im} z| < \varepsilon,\ \varepsilon > 0\}$ $= \omega \times \{\mathrm{Im} z \in B\}$, when B is of the form shown in Figure 1b .

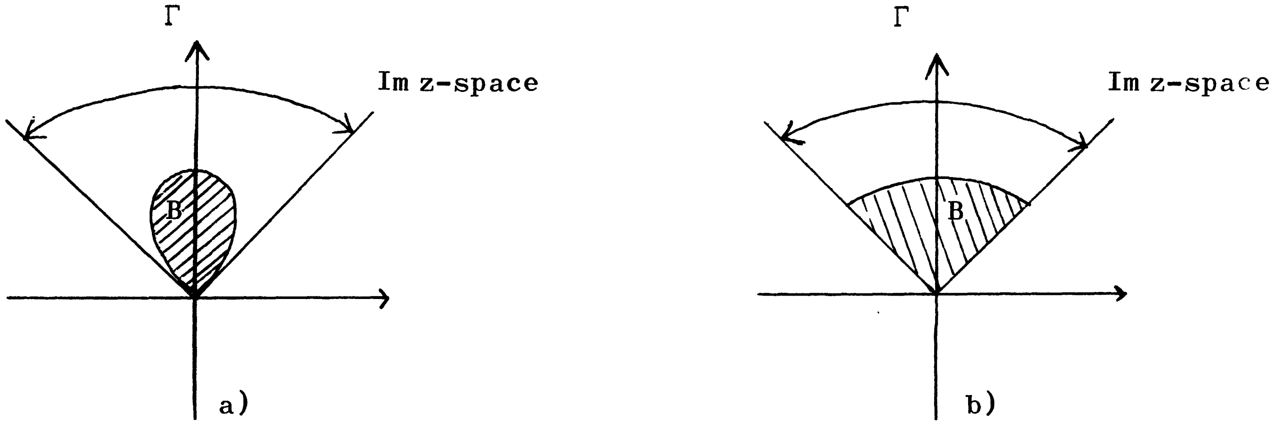


Figure 1

As stated above, for instance in the case of a square integrable function f, property R is essentially equivalent to this analyticity property, together with some type of bound on $\underline{f}$ in its analyticity domain. For instance, if $\underline{f}$ is uniformly bounded in $\omega \times \{\mathrm{Im} z \in B\}$, then property R is satisfied.

Property R is of interest, in particular in applications to physics, in much more general cases than that considered above. Its general expression in terms of analyticity properties is however more difficult, in particular because a distribution f is not in general the boundary value of an analytic function, but can be expressed in many ways as a sum of boundary values of analytic functions. We shall not discuss this here, and we conclude this section with some examples.

Examples

The examples given below of functions that satisfy (examples 1,4), or do not satisfy (examples 2,3) property R, have voluntarily been chosen simple. In these examples, λ denotes a positive non-integer rational number and f is of the form $(a(x) + io)^{\lambda}$ where a is a real analytic function, which is real valued in the examples 1, 3, 4. This form is that considered in [5] . Similar considerations would apply if $(a(x) + io)^{\lambda}$ is replaced by $\delta(a(x))$. For simplicity, we only consider below points X that are not critical points of a.

Example 1 : $f(x_1,x_2) = (x_1 - x_2^2 + io)^\lambda$, $X = (0,0)$. $ES_X(f)$ is the direction (1,0). At neighboring points (x_1,x_2), $ES_x(f)$ is the direction $(1,-2x_2)$. For any neighborhood $\mathcal{N}$ of X, $\bigcup_{x\in\mathcal{N}} ES_x(f)$ contains the direction (1,0) in its interior and hence property R is automatically satisfied (see Remark 1 above).

The following fact is on the other hand interesting in the case of that function. Let $\alpha_o > 0$ be given and let $C_{\alpha_o} = \bigcup_{x;(x-X)^2\le\alpha_o} ES_x(f)$. Then a factor of fall-off of the form $e^{-\alpha_o\gamma|u|}$ is obtained in a region of the form $0<\gamma|u| < \chi\, d(u,C_{\alpha_o})$, where $\chi > 0$ is independent of α_o. I.e. the constant α in this factor can be chosen equal to α_o and needs not be strictly less than α_o. This is due to the fact that the analyticity domain $\mathrm{Im}(z_1 - z_2^2)>0$ of the function $z_1 - z_2^2$ contains , for any $\alpha_o > 0$, a region of the form shown in Figure 1b, with Γ being the dual cone of C_{α_o} , for any point $x = (x_1,x_2)$ such that $(x - X)^2 \le \alpha_o$, and that the "width" of the region B_{α_o} can be chosen independent of α_o .

Example 2 : $f(x_1,x_2) = (x_1 + ix_2^2 + io)^\lambda$, $X = (0,0)$. This function is analytic at all real points, apart from the origin. Its essential support away from the origin is thus empty, its essential support at the origin being the direction (1,0).

However, the analyticity domain $\mathrm{Im}(z_1 + iz_2^2) > 0$ of the function $(z_1 + iz_2^2 + io)^\lambda$ does not include the intersection of any complex neighborhood of the origin with the region $\mathrm{Im}\, z_1 > 0$. Correspondingly, f does not satisfy property R (see previous discussion and Fig.1).

Example 3 : $f(x_1,x_2) = (x_1 - x_2^3 + io)^\lambda$, $X = (0,0)$, $ES_X(f)$ is the direction (1,0). When $x = (x_1,x_2)$ lies in a neighborhood $\mathcal{N}$ of the origin $Es_x(f)$ which is the direction $(1, - 3x_2^2)$ lies always on the same side of the direction (1,0) : see Figure 2 a) where the shaded area represents $\bigcup_{x\in\mathcal{N}} ES_x(f)$. To check whether property R is satisfied or not, it is then necessary (and sufficient in this case) to consider a set of directions that lie in the region $u_2 > 0$ and approach the direction (1,0) : see Fig. 2a)

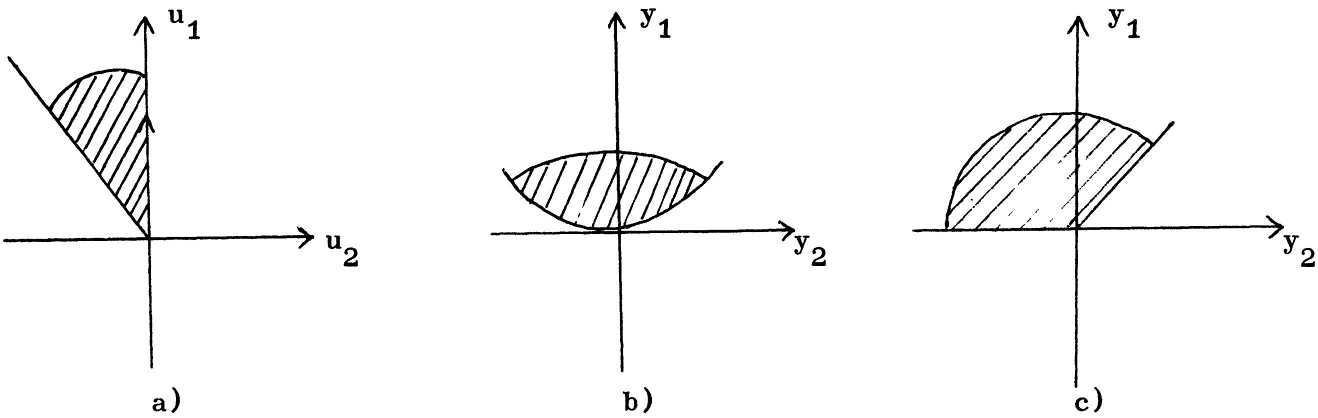


Figure 2

In view of Theorem 4 of [1] , property R would be satisfied at X if the function $(z_1 - z_2^3 + io)^\lambda$ was analytic in a domain of the form $\omega \times \{\mathrm{Im}\, z \in B\}$ with B of the form shown in Figure 2c (the important fact is here the existence of a segment of finite length $-b < y_2 < 0$, $b > 0$ in the closure of B, at $y_1 = 0$, $y_2 < 0$). This is not the case here, the region $\mathrm{Im}\,(z_1 - z_2^3) > 0$ being of the form shown in Figure 2b at $x_1 = x_2 = 0$. Correspondingly, the function f does not satisfy property R.

Example 4 :

$$f(x_1,x_2,x_3) = [(\vec{x}_1^2+1)^{1/2} + (\vec{x}_2^2+1)^{1/2} - (\vec{x}_3^2+1)^{1/2} - ((\vec{x}_1+\vec{x}_2-\vec{x}_3)^2+1)^{1/2} + io]^\lambda$$

$$X = (\vec{X}_1,\vec{X}_2,\vec{X}_3),\ \vec{X}_1 = \vec{X}_3,\ \vec{X}_2 \neq \vec{X}_1 ,$$

where $\vec{x}_i$, $i = 1,2,3$, is a multidimensional variable. $ES_X(f)$ is the direction $(\vec{A},\vec{0},-\vec{A})$ where $\vec{A} = \vec{V}_1 - \vec{V}_2$, $\vec{V}_i = (\vec{X}_i^2 + 1)^{-1/2}\vec{X}_i$. On the other hand, one checks, for instance, that all other directions of the space $\vec{u}_2 = 0$ are outside $\bigcup_{x \in \mathcal{N}} ES_x(f)$ where x varies in a neighborhood $\mathcal{N}$ of X, although $ES_x(f)$ varies with x (and is different from $ES_X(f)$). Hence property R has to be checked in particular for sets of directions of that space that approach the direction $(\vec{A},0,-\vec{A})$: see Figure 3 .

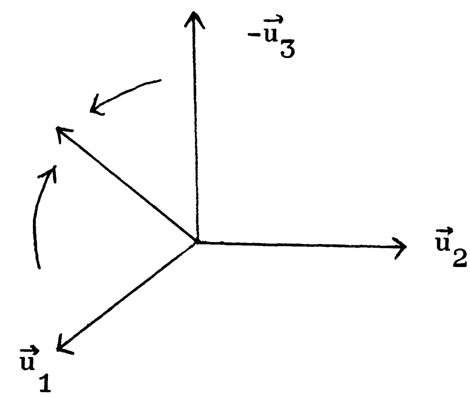


Figure 3

The detailed study of the analyticity properties of f allows one to conclude that property R is satisfied. The function f, when replaced by the corresponding δ-function, is very close to the momentum-energy conservation δ-functions encountered in the physical application.

§ 3. $u = 0$ THEOREMS

Theorem 3 : Let f_1, f_2 be two locally square integrable functions that satisfy property R at a $u = 0$ point X. Then :

$$ES_X(f_1 f_2) \subset \{u \,;\forall\, m \in \mathbb{N},\ \exists x'_{(m)},\ x''_{(m)} \in \mathbb{R}^n_{(x)},\ u'_{(m)},\ u''_{(m)} \in \mathbb{R}^n_{(u)}$$

(9) $$u'_{(m)} \in ES_{x'_{(m)}}(f_1),\ u''_{(m)} \in ES_{x''_{(m)}}(f_2),$$

$$x'_{(m)} \to X,\ x''_{(m)} \to X,\ u'_{(m)} + u''_{(m)} - u \to 0 \text{ when } m \to \infty\}$$

Proof : We prove Theorem 3 below for square integrable functions. For locally square integrable functions, it is sufficient to replace f_1, f_2 by $\chi_1 f_1$, $\chi_2 f_2$ with appropriate functions χ_1, χ_2 (see Section 2).

Let $f = f_1 f_2$. It will be convenient to use the function F defined for every $v_o > 0$ by :

(10) $$F(u;v_o,X) = \int f(x) e^{-i\langle u,x\rangle - v_o(x-X)^2} dx$$

which is related to $\hat{F}_\gamma$ through the relation $\hat{F}_\gamma(u;X) = F(u;\gamma|u|,X)$. Functions F_1, F_2 associated with f_1, f_2 are defined similarly, and the following formula is easily derived :

(11) $$F(u;v_o,X) = \int F_1(u';v_o,X)\, F_2(u-u';v_o,X) du'$$

Let U be a given point, such that for instance $|U| = 1$, that does not belong to the cone C defined in the r.h.s. of (9). To show that $U \notin ES_X(f)$, it is clearly sufficient to show that there exists an open cone $\mathcal{V}$ with apex at the origin, containing U, $\alpha > 0$, $\gamma_o > 0$ and a constant C such that :

(12) $$|F(u;v_o,X)| < C e^{-\alpha v_o}$$

in the region $u \in \mathcal{V}$, $0 < v_o < \gamma_o |u|$. This is achieved as follows.

Since $U \notin C$, there exists $\varepsilon > 0$ such that $|u' + u'' - U| > \varepsilon$ whenever $u' \in ES_{x'}(f_1)$, $u'' \in ES_{x''}(f_2)$, $|x' - X| < \varepsilon$, $|x'' - X| < \varepsilon$. The cone $\mathcal{V}$ will be chosen to be the set of points of the form λu, $\lambda > 0$, $|u - U| < \varepsilon'$, $0 < \varepsilon' \ll \varepsilon$. To prove the bounds (12), it is then sufficient to use the bound :

(13) $$|F(u;v_o,X)| < \int |F_1(u';v_o,X)| \times |F_2(u-u';v_o,X)| du'$$

which follows from (11), and to check bounds of the form (12) for the integrals over the two following regions :

(i) the set of points u' whose distance to $\bigcup_{x;|x-X|<\varepsilon} ES_x(f_1)$ is larger than $\frac{1}{3}\varepsilon|u|$ and

(ii) the set of points u' whose distance to $u - \bigcup_{x;|x-X|<\varepsilon} ES_x(f_2)$ is larger than $\frac{1}{3}\varepsilon|u|$.

One checks in fact easily that the union of these two sets is all $\mathbb{R}^n_{(u')}$. The needed bounds for each one of these integrals are derived from the exponential fall-off properties associated with F_1 and F_2 respectively,(and from the fact that F_2, F_1 always satisfy, as already mentioned in Section 2, trivial bounds of the form $|F_i(u';v_o,X)| < d_i(u';v_o)$

for all $v_o > 0$, where d_i is square integrable with respect to u' and has a norm independent of v_o).

For instance, property R applied to F_1 ensures that :

(14) $$|F_1(u';v_o,X)| < d_1(u';v_o,X)e^{-\alpha v_o}$$

for some $\alpha > 0$, in a region of the form $0 < v_o < \chi'$ dist.$(u';C_{1,\varepsilon})$, $\chi' > 0$, where $C_{1,\varepsilon} = \bigcup_{x;|x-X|<\varepsilon} ES_x(f_1)$, and hence in particular in the region $0 < v_o < (\frac{1}{3}\chi'\varepsilon)|u|$, since dist.$(u';(1,\varepsilon) \geq \frac{1}{3}\varepsilon|u|$. (The cone $\mathcal{C}$ of Section 2 is here chosen to be the complement of $C_{1,\varepsilon}$). Q.E.D.

A similar theorem, which extends Theorem 2 to (u,v) = 0 points, applies to products of bounded operators that satisfy property R : see [4]. A direct proof is now based on the formula :

(15) $$F(u,v;v_o,X,Y) = \int(2v_o)^{1/2}dT\,dW\,F_1(u,W;v_o,X,T)F_2(W,v;v_o,T,Y)$$

We conclude this section with some remarks, for instance in the case of a product of functions.

1) Formulae (11) (13) are valid for arbitray square integrable functions. One can see, however, that Theorem 3 is the best possible type of result that can be obtained on the basis of Formula (13) : if property R is not satisfied by f_1 and f_2 at X, at least for appropriate sets of directions, one cannot expect to extract (as is required to prove results on $ES_X(f)$) a uniform exponential fall-off factor $e^{-\alpha v_o}$ on F in a uniform region $0 < v_o < \gamma_o|u|$ (independently of u').

The use of the bound (13) seems to be the best possible method "in general", although this method may not be the best under special conditions on f_1 and f_2.

2) Consider a product f_1f_2 of functions which are, in a neighborhood of X, boundary values of analytic functions $\underline{f}_1$, $\underline{f}_2$ from the directions of open cones Γ_1, Γ_2. Theorem 3 gives information when the closures of Γ_1 and Γ_2 have an intersection that is not reduced to the origin, although $\Gamma_1 \cap \Gamma_2$ itself is empty, and when property R is satisfied. This result can be understood in terms of analyticity properties as follows : the intersection of the closures of the analyticity domains of $\underline{f}_1$, $\underline{f}_2$ in Im z -space is non empty if property R is satisfied and contains in fact

a region of the form $\text{Im}\, z \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2$, $|\text{Im}\, z| < b$, $b > 0$. This fact, together with appropriate bounds on $\underline{f}_1$, $\underline{f}_2$ in the closures of their analyticity domains can be used for an alternative proof of Theorem 3 in this case. Such analyticity properties appear again to be necessary "in general" in such cases, to get information on $ES_X(f)$.

Example : Let $ES_x(f_1)$ and $ES_x(f_2)$ be the closed cones C_1, C_2 shown in Figure 4 a), when x varies on a neighborhood of X. The dual cones Γ_1, Γ_2 are shown in Figure 4b . If property R is satisfied, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ contains a segment $0 < y_2 < b$, $b > 0$, $y_1 = 0$. Theorem 3 says that $ES_X(f_1 f_2)$ is contained in the region $u_2 \geq 0$, which is the dual cone of the direction $y_2 > 0$, $y_1 = 0$

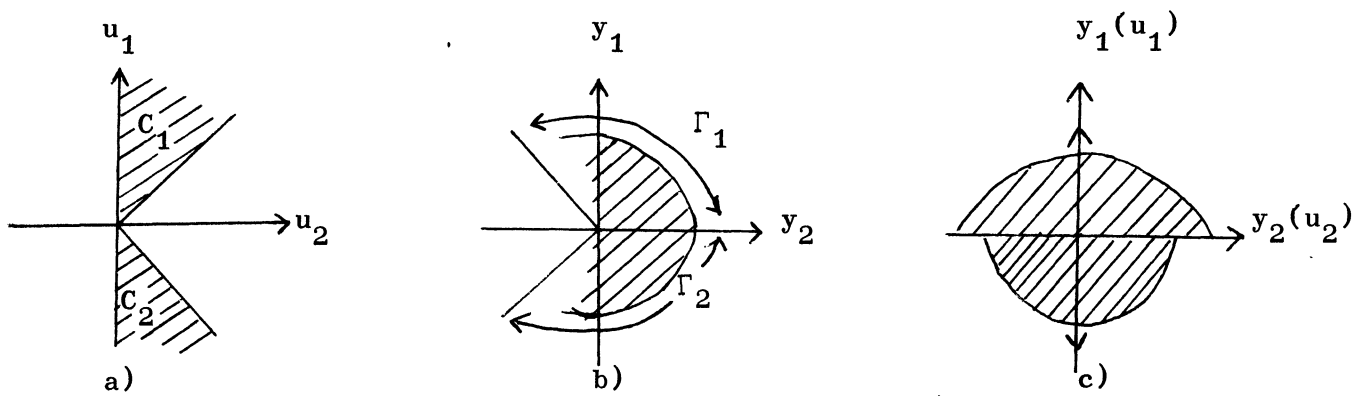
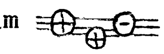


Figure 4

If C_1, C_2 are the lines $u_2 = 0$, $u_1 > 0$ and $u_2 = 0$, $u_1 < 0$ respectively, the situation is represented in Figure 4c: $ES_X(f_1 f_2)$ is then (contained in) the full line $u_2 = 0$.

Appendix 1

In the applications to physics, one is interested in determining the analytic structure of bubble diagram functions, a problem which is crucial in the derivation of discontinuity formulae for multiparticle scattering functions and related results. For a general presentation of the S-matrix formalism, more details and references, see [7]. Bubble diagrams are certain topological graphs whose vertices are replaced by + or - bubbles that represent connected scattering operators. They define corresponding "bubble diagram operators", which are always well defined bounded operators, and whose kernels are the so-called bubble diagram functions. If B is for instance the bubble diagram , then the corresponding kernel F_B can be written as :

$$F_B(\vec{p}_1,\vec{p}_2,\vec{p}_3;\vec{p}_4,\vec{p}_5,\vec{p}_6) = \int S^c_{3,3}(\vec{p}_1,\vec{p}_2,\vec{p}_3;\vec{k}_1,\vec{k}_2,\vec{k}_3) \times S^c_{2,2}(\vec{k}_2,\vec{k}_3;\vec{k}_4,\vec{k}_5) \times (S^{-1})^c_{3,3}(\vec{k}_1,\vec{k}_4,\vec{k}_5;\vec{p}_4,\vec{p}_5,\vec{p}_6) \prod_{\ell=1}^{5} \frac{d^3\vec{k}_\ell}{2(\vec{k}_\ell^2+\mu_\ell^2)^{1/2}} \tag{16}$$

where $\vec{p}_1,\dots,\vec{p}_6$, resp. $\vec{k}_1,\dots,\vec{k}_5$ are three-momenta variables associated with the external, resp. internal, lines and μ_ℓ is the mass of particle ℓ. (Each line is associated with a given particle).

The S-matrix kernels, and hence F_B, satisfy energy-momentum conservation (the energy of a particle is $(\vec{p}^2+\mu^2)^{1/2}$).

The essential supports of the individual "bubbles" are known from the physical principles of macroscopic causality, and unitary (= conservation of probabilities). They involve the so-called Landau surfaces. Corresponding results on the essential support of F_B follow at non $u=0$ points [8]. The $u=0$ problem is, however, crucial, as already mentioned in Section 1.

The results of [9] give partial results in certain situations, under certain assumptions on the nature of the singularities of the S-matrix kernels. The results of [5,6] provide on the other hand a general solution of the $u=0$ problem for phase-space integrals:in the latter, the individual scattering kernels associated with each bubble are replaced by more elementary quantities, namely constants (or regular functions) times energy-momentum conservation δ-functions. These results have led [6] to propose a "conjecture" on the singular spectrum of bubble diagram functions. The results of [4] do give information, on the other hand, on the essential support of bubble diagram functions, if the regularity property R is assumed to hold for the individual scattering operators. Property R is expected to hold in a number of cases (see [4]) where it can be considered to correspond to a refinement of the macrocausality principle, based on the same physical ideas. Work is in progress [10] to check whether property R can be reasonably postulated in general.

The result of [4] and the conjecture of [6] both say that the rules at non $u=$ points are still valid at $u=0$ points, apart from the introduction of limiting procedures. The limiting procedures of [6] and [4] are different so far, as a consequence of the difference in the limiting procedures obtained in the respective mathematical works.

APPENDIX 2

The results of [5] apply to products of the form $\prod_{j=1}^{d} \delta(\varphi_j) \prod_{\ell=1}^{N} (a_\ell(x)+io)^{\lambda_\ell}$ where φ_j and a_ℓ are real-valued real analytic function of x and $\mathrm{Re}\,\lambda_\ell >> 0$. For simplicity, we restrict below our attention to a product $f = f_1 f_2$ of two functions, $f_1 = (a_1(x)+io)^{\lambda_1}$, $f_2 = (a_2(x)+io)^{\lambda_2}$. The result of [5] is then :

$$
(18) \quad ES_X(f) \subset \{u; \forall m \in \mathbb{N}, \exists z_{(m)} \in \mathbb{C}^n, \alpha_1^{(m)}, \alpha_2^{(m)} \in \mathbb{C} \text{ such that } z_{(m)} \to X,
$$
$$
\alpha_1^{(m)} \nabla a_1(z_{(m)}) + \alpha_2^{(m)} \nabla a_2(z_{(m)}) - u \to 0
$$
$$
\alpha_1^{(m)} a_1(z_{(m)}) \to 0, \ \alpha_2^{(m)} a_2(z_{(m)}) \to 0
$$
$$
\text{when } m \to \infty \}
$$

It is on the other hand conjectured in [5] that $\alpha_1^{(m)}, \alpha_2^{(m)}$ can as a matter of fact be restricted to $\mathbb{R}^+$. We note that the result given by Theorem 3, if property R holds, is in this case (if X is not a critical point of a_1 or a_2) :

$$
(19) \quad ES_X(f) \subset \{u; \exists x'_{(m)}, x''_{(m)} \in \mathbb{R}^n_{(x)}, \alpha_1^{(m)}, \alpha_2^{(m)} \in \mathbb{R}^+ \text{ such that}
$$
$$
a_1(x'_{(m)}) = a_2(x''_{(m)}) = 0, \ x'_{(m)} \to X, \ x''_{(m)} \to X, \ \alpha_1^{(m)} \nabla a_1(x'_{(m)}) + \alpha_2^{(m)} \nabla a_2(x''_{(m)}) - u \to 0\}
$$

APPENDIX 3

Consider the products $(x_1 - x_2^2 + io)^{\lambda} \times (x_1 + x_2^2 - io)^{\mu}$, or $(x_1 + ix_2^2 + io)^{\lambda} \times (x_1 - ix_2^2 - io)^{\mu}$. In both cases, the respective singular spectra of f_1, f_2 at the origin are the opposite direction (1,0) and (-1,0) : see Section 2. The set $\overline{ES(f_1) + ES(f_2)}_{|X}$, where X is the origin, coincides with $ES_X(f_1) + ES_X(f_2)$ and is the line $u_2 = 0$. It is however believed that $ES_X(f_1 f_2)$ is all $\mathbb{R}^2_{(u_1,u_2)}$. (In the first case, either Theorem 3 or the results of [5] - see (18), (19)- give no information. In the second one, property R is not satisfied : see Section 2, Example 2).

A case for which $ES_X(f_1 f_2)$ has been proved to be strictly larger than $\overline{ES(f_1) + ES(f_2)}_{|X}$ is the following : $f(x_1,x_2,x_3) = (x_1^3 - x_2)_+^{\lambda} \times (x_2 - x_3^3)_+^{\mu}$ where $a(t)_+^{\lambda} = a(t)^{\lambda}$ if $t > 0$, $= 0$ if $t < 0$. The essential support $ES_x(f_1)$, resp. $ES_x(f_2)$, at any point x of the surface $x_2 = x_1^3$, resp. $x_2 = x_3^3$, is the line $\alpha_1(3x_1^2,-1,0)$, α_1 real, resp. the line $\alpha_2(0,1,-3x_3^2)$, α_2 real. $ES_X(f_1)$ and $ES_X(f_2)$ at the origin ($X = 0$) are the line $u_1 = u_3 = 0$. One checks on the other hand that the only directions of the plane $u_2 = 0$ that lie in $\overline{ES(f_1) + ES(f_2)}_{|X}$ are the two opposite directions of the line $u_1 = -u_3$. However, it is proved in [5] that the essential support at the origin of the function $(x_1^3 - x_3^3)^{\lambda+\mu-1}$, which is equal to $\int f(x_1,x_2,x_3)\, dx_2$ up to constant factors, is all $\mathbb{R}^2_{(u_1,u_3)}$. Correspondingly, $ES_X(f)$ must contain the whole plane $u_2 = 0$.

The limiting procedures of Theorem 3, if Theorem 3 was applicable, would entail that the only possible directions of the plane $u_2 = 0$ in $ES_X(f_1 f_2)$ would be those lying in the region $u_1 u_3 < 0$, a result which is not correct, as we have just seen. The reason is that Theorem 3 does not apply, because the functions f_1, f_2 do not satisfy property R : see Example 3 in Section 2. (The result (18) of [5], with complex values of $z_{(m)}$, is in agreement with the fact that $ES_X(f) \cap \{u_2 = 0\}$, is all $\mathbb{R}^2_{(u_1,u_3)}$).

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