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FLAT SOLUTIONS AND SINGULAR SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS

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§ 0. INTRODUCTION

The results presented here have been obtained in collaboration with M. S. Baouendi and F. Treves. We deal with a linear partial differential operator $P = P(x,D)$ of order $m \geq 1$ with complex-valued coefficients defined and analytic in an open subset Ω of $\mathbb{R}^n$. If M is an analytic submanifold of Ω, we are interested in the question of whether one can find solutions of the homogeneous equation

$$(0.1) \qquad P(x,D)u = 0$$

which are flat or, alternatively, singular precisely on M. (We say that a C^∞ function is flat on M if all its derivatives vanish there).

This is of course a topic on which many results are known, beginning with the classical Holmgren's theorem which concerns non characteristic hypersurfaces. Let us introduce some terminology and notation : by $T^*\Omega\backslash 0$ we denote the cotangent bundle over Ω from which the zero section has been excised ; by $N^*(M)$ we mean the conormal bundle over M, which is the subset of $T^*\Omega\backslash 0$ consisting of the points (x,ξ) with x in M and ξ orthogonal to all tangent vectors to M at x ; by Char P the characteristic set of P, i.e. the subset of $T^*\Omega\backslash 0$ where the principal symbol $p_m(x,\xi)$ of P vanishes. We shall say that M is noncharacteristic (with respect to P) if $N^*(M)$ and Char P do not intersect, that M is totally characteristic if $N^*(M) \subset \mathrm{Char}\,P$, and finally that a hypersurface S is simply characteristic at one of its points, x^0, if $p_m(x^0,\xi^0) = 0$ and $\mathrm{grad}_\xi p_m(x^0,\xi^0) \neq 0$ where ξ^0 is the normal to S at x^0.

Holmgren's theorem implies that if codim M = 1 and M is non characteristic, any C^∞ solutions of (0.1) which is flat on M must vanish identically in a full neighborhood of M. The work [2] extends this result to the case of codim M > 1.

On the other hand, when M is totally characteristic there are cases in which one can prove the existence (in a neighborhood of a point of M) of solutions of (0.1) which are flat on M and non vanishing (in fact are analytic) in the complement of M. When M is a hypersurface such a result is an immediate consequence of the work of Mizohata [8] in the simply characteristic case, and has recently been obtained by Komatsu [7] in the constant multiplicity case. Theorems 1 and 1F below extend Mizohata's result in a different direction by allowing codim M to exceed 1.

There are cases which fall between those two categories (non characteristic and totally characteristic) in which it is still possible to prove the existence of solutions which are flat on M and vanish nowhere else. The model for such behavior is provided by the Mizohata operator

$$L = \partial_x + ix\partial_y \qquad (\text{in } \mathbb{R}^2). \tag{0.2}$$

Here, our manifold M is the origin in $\mathbb{R}^2$, which is neither noncharacteristic nor totally characteristic. The function (C^∞ in $\mathbb{R}^2$ and analytic in $\mathbb{R}^2\setminus\{0\}$)

$$u(x,y) = \exp[-(\frac{x^2}{2} + iy)^{-1/2}] \tag{0.3}$$

(where $z^{1/2} > 0$ for $z > 0$) satisfies $Lu = 0$ in $\mathbb{R}^2$, is flat at (0,0) and does not vanish anywhere else. Theorems 2, 2', 3 and 3' stated below generalize this kind of result.

The construction of flat solutions can be slightly modified to yield (under the same hypotheses) solutions of (0.1) which are analytic in the complement of M and have singular support exactly equal to M. For instance, in the above example of the Mizohata operator, the function

$$u(x,y) = (\frac{x^2}{2} + iy)^{3/2} \tag{0.4}$$

is such a solution of $Lu = 0$.

We present in section 1 the statements of our results and in section 2 an outline of the proofs. The details can be found in [1] (See also [12] for the first order, completely characteristic case).

§ 1. STATEMENT OF RESULTS

We consider first the case in which the manifold M is totally characteristic. For the statement of the theorem we need the following definitions (see [4] and [6]).

Let f be a complex valued C^∞ function defined in some open subset $\mathcal{O}$ of $T^*\Omega\setminus 0$. The Hamiltonian field of f is the complex vector field over $\mathcal{O}$,

$$H_f = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j}\right)$$

A C^1 curve $\Gamma : I \to \mathcal{O}$ (I an interval in $\mathbb{R}$) is called a <u>bicharacteristic</u> of f if, for every $t \in I$,

$$f(\Gamma(t)) = 0$$
$$0 \neq \Gamma'(t) = z(t) H_f(\Gamma(t)), \text{ for some } z(t) \in \mathbb{C} .$$

Note that at each point of a bicharacteristic the real part and the imaginary part of H_f must be parallel. A <u>semibicharacteristic</u> of f is a bicharacteristic of $\mathrm{Re}(qf)$ for some C^∞ function q in $\mathcal{O}$, $q \neq 0$. Note that if f is real-valued, any semibicharacteristic of f on which f vanishes is a bicharacteristic of f. Moreover, if g is any nonvanishing complex valued C^∞ function in $\mathcal{O}$, a (semi)bicharacteristic of f is also a (semi)bicharacteristic of gf.

<u>Theorem 1</u> : Let M be an analytic submanifold of Ω, totally characteristic with respect to P. Suppose that there exists an analytic hypersurface S containing M and having the following properties :

(1.3) S is simply characteristic along M.

(1.4) The only semibicharacteristics of p_m leaving $\Sigma = N^*(S)|_M$ are characteristics of p_m .

Then the following conclusions hold :

(a) every point $x^o \in M$ has an open neighborhood $U(x^o) \subset \Omega$ in which there is a C^∞ solution of (0.1), flat on M, and analytic and nowhere zero in $U(x^o)\backslash M$.

(b) Given any integer $p \geq m$ and any point $x^o \in M$, there is a C^p solution u of (0.1) in an open neighborhood $V_p(x^o)$ such that

(1.5) u is analytic in $V_p(x^o) \backslash M$;

(1.6) u is not C^{p+1} in the neighborhood of any point of $M \cap V_p(x^o)$.

The conjunction of (1.5) and (1.6) implies that the C^∞ singular support of u [in $V_p(x^o)$] is equal to its analytic singular support and is exactly equal to $M \cap V_p(x^o)$.

It should be remarked that the hypothesis that M is totally characteristic and contained in a hypersurface which is simply characteristic along M implies that

$$\dim M \geq 1 \tag{1.7}$$

Indeed if M were a point x^o, then $p_m(x^o,\xi) = 0$ for all $\xi \subset \mathbb{R}^n$. Hence $\mathrm{grad}_\xi p_m(x^o,\xi) \equiv 0$, and no hypersurface through x^o could be simply characteristic at x^o.

Let us examine the hypothesis (1.4) more closely. It can be shown, by examining the Hamilton Jacobi equations for the bicharacteristics, that if M is any totally characteristic manifold, any semibicharacteristic of p_m intersecting $N^*(M)$ must be completely contained in $N^*(M)$. Condition (1.4) states that if a semibicharacteristic of p_m contains points of $\Sigma = N^*(M) \cap N^*(S)$ and points of $N^*(M)\setminus\Sigma$, then it must be a bicharacteristic of p_m. Now, if S were a simply characteristic hypersurface, no semibicharacteristic of p_m can leave $N^*(S)$. Since it cannot leave $N^*(M)$, it cannot leave Σ, and we have proved :

Corollary 1 : Suppose that the analytic manifold M is totally characteristic and contained in a simply characteristic hypersurface S. Then the conclusions (a) and (b) of Theorem 1 are valid.

Another corollary follows easily if p_m is real. In this case any semibicharacteristic Γ of p_m intersecting $N^*(M)$ is necessarily a bicharacteristic of p_m. Indeed Γ stays in $N^*(M)$ on which p_m vanishes and therefore Γ must be a bicharacteristic of p_m (by the remark following the definition of semibicharacteristics).

Corollary 2 : Suppose that the principal symbol of P is real and let M be an analytic submanifold of Ω, totally characteristic and contained in an analytic hypersurface S which is simply characteristic along M. Then conclusions (a) and (b) of Theorem 1 are valid.

For first order operators we prove in [12] :

Theorem 1F : Let P be first order and nondegenerate (i.e. its principal part $p_1(x,D)$ does not vanish identically at any point of Ω) and let the analytic submanifold M of Ω be totally characteristic with respect to P. Then conclusion (a) and (b) of Theorem 1 are valid, and moreover,

(c) Every point $x^o \in M$ has an open neighborhood $U(x^o) \subset \Omega$ in which there is a (distribution) solution u of (0.1) such that $\mathrm{supp}\, u = \mathrm{sing\,supp}\, u = U(X^o) \cap M$.

Actually conclusion (a) of this theorem can be proved as a corollary of Theorem 1, as is done in [1], but a simple direct proof, which takes advantage of the fact that P is first order, is given in [12] . A slight modification of this proof yields conclusion (b). Conclusion (c) was first proved in [13] for the case in which M is a leaf of the natural foliation of Ω defined by the Lie algebra generated by the real and imaginary parts of the principal part of P. The proof in [13] does not make full use of the leaf structure of M but only uses the fact that M is totally characteristic with respect to P. In connection with this we should mention the following easily proved proposition.

Proposition 1 : Suppose that P is first order and let $A+iB$ denote its principal part with A and B real (and real analytic) vector fields in Ω. An analytic submanifold M of Ω is totally characteristic with respect to P if and only if every vector field in the Lie algebra generated by A and B is tangent to M.

The connected analytic manifolds which are totally characteristic with respect to $A+iB$ and are maximal with respect to connectedness and minimal with respect to dimension are called the leaves of the natural foliation of Ω defined by the Lie algebra generated by A and B. According to a Theorem of Nagano [9], through every point of Ω passes one and only one leaf. In general, the dimension of a leaf might be any integer ≥ 0 and $\leq n$, but since we are assuming here that P is nondegenerate, the dimension of a leaf must be ≥ 1. Of course when the dimension of a leaf is equal to n, its relevance to our results becomes nil.

We turn now to the case in which the submanifold M is neither noncharacteristic nor totally characteristic.

Theorem 2 : Let M be an analytic submanifold of Ω and suppose that there is an analytic hypersurface S containing M, an odd integer $k \geq 1$, and $z \in \mathbb{C}$ such that if

(1.8) $$A = \mathrm{Re}(zp_m), \quad B = \mathrm{Im}(zp_m) ,$$

then the following is true

(1.9) For all $\gamma \in \Sigma = N^*(S)\big|_M$, the base projection of $H_A(\gamma)$ is not tangent to M .

(1.10) $$A\big|_\Sigma = H_A^j B\big|_\Sigma = 0 \quad \text{for } 0 \leq j < k, \text{ and } H_A^k B\big|_\Sigma \neq 0 .$$

(1.11) For all $\gamma \in \Sigma$ and $0 \leq j < \frac{k-1}{2}$, $H_{H_A^j B}(\gamma)$ is in the span of $H_A(\gamma)$ and of the tangent space to Σ at γ .

Then conclusions (a) and (b) of Theorem 1 are valid.

It should be noted that the hypothesis that A and B vanish on $\Sigma = N^*(S)|_M$ means that the hypersurface S is characteristic along M. Condition (1.9) implies that S is simply characteristic along M. Incidentally (1.9) is automatically satisfied when M is a point and P is of principal type.

Condition (1.10) can be rephrased by saying that at every point $\gamma \in \Sigma$ the restriction of B to the bicharacteristic of A through γ vanishes exactly of order k. Note that (1.11) is void when $k = 1$.

The hypothesis of Theorem 2 can also be expressed in terms of two integers related to those introduced by Hörmander in his study of subellipticity [5]. Although this alternate form of the hypothesis is more difficult to verify computationally, it has the advantage that it is clearly invariant under multiplication of p_m by a nonvanishing factor. This fact is used in the proof of Theorem 2, in conjunction with the implicit function theorem, to locally reduce p_m to a first order symbol.

Let A and B be real valued C^∞ functions defined in a neighborhood of a point $\gamma \in T^*\Omega$. With the pair (A,B) we associate the integer

$$(1.12)\qquad k = k(\gamma) = \sup\{j \in \mathbb{N};\ H_{C_1} \cdots H_{C_{\ell-1}} C_\ell(\gamma) = 0 \text{ for } 1 \le \ell \le j \text{ and } C_i = A \text{ or } B\}$$

Thus, $k = 0$ means that (at γ) A or B is $\neq 0$ while $k = 1$ means that $A = B = 0$ but $H_A B \neq 0$. Rememberring that $H_A B$ is the Poisson bracket $\{A,B\}$, we can say, alternatively, that k is the largest integer for which all repeated Poisson brackets of length $\le k$ vanish at γ.

For $j \in \mathbb{N}$, let V_j be the span in $T_\gamma(T^*\Omega)$ of all commutators

$$[H_{C_1},[H_{C_2},\ldots,[H_{C_{\ell-1}},H_{C_\ell}]\ldots]](\gamma)$$

with $1 \le \ell \le j$ and $C_i = A$ or B. If Σ is a smooth manifold in $T^*\Omega$ passing through γ, let $V_{j,\Sigma}$ be the canonical image of V_j in $T_\gamma(T^*\Omega)/T_\gamma\Sigma$. Now we define a new integer related to γ, (A,B) and Σ :

$$(1.13)\qquad s_\Sigma = s_\Sigma(\gamma) = \sup\{j \in \mathbb{N};\ j \le k(\gamma), \dim V_{j,\Sigma} \le 1\}\ .$$

If Σ is the single point γ, then s_Σ is the number s introduced in [5]. If Σ is $T^*\Omega$ or a hypersurface then $s_\Sigma = k$. If $k \neq 0$, then $s_\Sigma = 0$ means that $H_A(\gamma)$ and $H_B(\gamma)$ span a two dimensional vector space in $T_\gamma(T^*\Omega)$ transversal to $T_\gamma\Sigma$.

The definitions of k and s_Σ are invariant under canonical transformations (Σ being replaced by its image under the canonical transformation). Moreover k and s_Σ are also invariant under multiplication of the 2-vector (A,B) by a non singular 2×2 matrix of C^∞ real functions. In particular, if A and B are the real and imaginary parts of the principal symbol p_m, then k and s_Σ remain invariant under multiplication of p_m by a non vanishing complex-valued C^∞ function.

Theorem 2' stated below is equivalent to Theorem 2.

<u>Theorem 2'</u> : Let M be an analytic submanifold of Ω and suppose that there is an analytic hypersurface S containing M such that

(1.14) At every $\gamma \in \Sigma = N^*(S)|_M$ the base projection of $H_{\mathrm{Re}\,p_m}(\gamma)$ or of $H_{\mathrm{Im}\,p_m}(\gamma)$ is not tangent to M,

(1.15) $\qquad k$ is constant and odd on Σ ,

(1.16) $$s_\Sigma \geq \frac{k-1}{2} \quad \text{on } \Sigma ,$$

where k and s_Σ are defined by (1.12) and (1.13) with $A = \mathrm{Re}\,p_m$ and $B = \mathrm{Im}\,p_m$. Then conclusions (a) and (b) of Theorem 1 are valid.

The hypotheses of Theorems 2 and 2' imply that the hypersurface S is simply characteristic along M. However, S cannot be totally characteristic, for then the bicharacteristic of A through any point of Σ would be entirely contained in $N^*(S)$ and B would vanish identically on it, contadicting condition (1.10). (Also, if S were totally characteristic, then $k = +\infty$ on $N^*(S)$, contradicting condition (1.15)). As a consequence, we must have

(1.17) $$\operatorname{codim} M \geq 2 ,$$

For if codim M were equal to one, we would then have $M = S$ and S would be totally characteristic.

Actually in the top dimensional case (codim $M = 2$) we can show that condition (1.11) in Theorem 2 and condition (1.16) in Theorem 2' are redundant. We have

Theorem 3 : If codim $M = 2$ in Theorem 2, then condition (1.11) can be disregarded.

Theorem 3' : If codim $M = 2$ in Theorem 2', then condition (1.16) can be disregarded.

In particular, when M is a point in $\mathbb{R}^2$ we have

Corollary 3 : Let Ω be an open set in $\mathbb{R}^2$ and $P(x,D)$ be a differential operator with analytic coefficients in Ω. Assume that there is $(x^o,\xi^o) \in T^*\Omega$ satisfying

$$p_m(x^o,\xi^o) = 0, \quad \operatorname{grad}_\xi p_m(x^o,\xi^o) \neq 0,$$

and, for some $z \in \mathbb{C}$, the restriction of $\operatorname{Im}(zp_m)$ to the bicharacteristic of $\operatorname{Re}(zp_m)$ passing through (x^o,ξ^o) changes sign. Then conclusions (a) and (b) of Theorem 1 are valid with $M = \{x^o\}$.

An obvious consequence of the above results is

Corollary 4 : Under the assumptions of any one of the above theorems or corollaries, the operator P is not hypo-elliptic near any point of M.

Actually, corollary 4 is a particular case of Theorem II in [11].

§ 2. OUTLINE OF THE PROOFS

In the proofs of all the results, the first and crucial step is the determination of a complex phase function i.e. of a solution φ of the characteristic equation

$$p_m(x,\varphi_x) = 0 \tag{2.1}$$

in an open neighborhood V of x^o, such that

(i) $\varphi(x) = 0$ if and only if $x \in M \cap V$,

(ii) the values of φ in $\mathbb{C}$ avoid the negative imaginary half-axis,

and, when $m > 1$

(iii) $\operatorname{grad} \varphi \neq 0$.

Such a phase function φ is determined by solving an appropriate Cauchy problem for (2.1). Assuming for the moment that φ has been determined, the desired solutions of (0.1) are obtained in the form of asymptotic expansions similar to those used in Mizohata [8],

(2.2) $$u(x) = \sum_{j=o}^{\infty} E_j(\varphi(x))u_j(x) .$$

The functions E_j are chosen to be holomorphic in the complex plane cut along the negative imaginary half-axis and to satisfy

(2.3) $$\frac{d}{dz} E_j(z) = E_{j-1}(z), \quad j = 1,2,...$$

The analytic functions u_j are obtained by successively solving a sequence of Cauchy problems for first order equations with appropriately chosen initial conditions so that the series (2.2) converges and satisfies (0.1). For the construction of solutions which are flat on M we take

(2.4) $$\begin{cases} E_o(z) = \exp(-\frac{1}{z^{1/5}}) \\ E_j(z) = \int_o^z E_{j-1}(z)dz \qquad j = 1,2,... \end{cases}$$

where $z^{1/5} = r^{1/5} e^{i\theta/5}$ if $z = re^{i\theta}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, and for solutions which are C^p and not C^{p+1} (where p is any integer $\geq$ m) we take

(2.5) $$\begin{cases} E_o(z) = z^{p+1/2} \\ E_j(z) = \frac{z^{p+j+1/2}}{(p+j+1/2)...(p+3/2)} \qquad j = 1,2,... \end{cases}$$

Let us turn now to the determination of the phase functions φ having the desired properties. Here the first step is to factor the principal symbol in the form

(2.6) $$p_m(x,\xi) = q(x,\xi)[\xi_1 - \lambda(x,\xi')], \quad \xi' = (\xi_2,...,\xi_n),$$

near a certain point in $T^*\Omega\backslash 0$, where q and λ are analytic and homogeneous in ξ of degree m-1 and 1 respectively and $q \neq 0$. This is done using the implicit function theorem and the assumption "simply characteristic". The factoring (2.6) allows us to replace the characteristic equation (2.1) by

(2.7) $$\varphi_{x_1} - \lambda(x, \varphi_{x'}) = 0$$

The next step is to transfer the assumptions of the theorems from p_m to its first degree factor $\xi_1 - \lambda(x,\xi')$. This requires showing that the assumptions are invariant under multiplication of p_m by a non vanishing function. While this is obvious for Theorem 1 and easy to show for Theorem 2', it is accomplished for Theorem 2 by showing that the assumptions of Theorems 2 and 2' are equivalent. Next we express the assumptions of the theorems in terms of properties of the function λ. For Theorem 1 we do this using the theory of first order (non linear) equations while for Theorem 2 we first make an analytic change of variables to locally straighten out the bicharacteristics of $\xi_1 - \mathrm{Re}\,\lambda$. The final steps consist of assigning appropriate Cauchy data on suitable hypersurfaces, solving the resulting Cauchy problems (using the Cauchy-Kowalewsky theorem) and then showing that the solutions φ have the desired properties. For Theorem 2 the necessary estimates are found by refining those obtained by Hörmander [3]. (M a point, $k = 1$) and by Nirenberg and Treves [10] (M a point, $k > 1$ and under a condition more restrictive than (1.11)).

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