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STRUCTURE THEORY OF FUNCTION SPACES

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In this lecture are considered two kinds of function spaces : 1. Function spaces of Lebesgue - Besov type which are Banach spaces, and 2. Nuclear function spaces. We shall give a survey about some isomorphic properties for spaces of such a type. We do not give any proofs. All proofs are given in the technical reports [6], which are the first version of a planed book "interpolation theory, function spaces, differential operators". If the results are published elsewhere, we shall give special references.

§ 1. STRUCTURE THEORY OF FUNCTION SPACES OF LEBESGUE-BESOV-TYPE WITHOUT WEIGHTS

1.1 Definitions : R_n denotes the real n-dimensional euclidean space. $S'(R_n)$ denotes the set of tempered distributions ; F is the Fourier-transformation in $S'(R_n)$, and F^{-1} is the inverse Fourier-transformation. Let $-\infty < s < \infty$ and $1 < p < \infty$. Then

$$H_p^s(R_n) = \{f \mid f \in S'(R_n) \;;\; F^{-1}(1+|\xi|^2)^{s/2} Ff \in \mathcal{L}_p(R_n)\},$$

$$\|f\|_{H_p^s(R_n)} = \|F^{-1}(1+|\xi|^2)^{s/2} Ff\|_{\mathcal{L}_p(R_n)} ,$$

are the Lebesgue-spaces. ($\mathcal{L}_p(R_n)$ has the usual meaning). Let $\Omega \subset R_n$ be a bounded domain. Then $H_p^s(\Omega)$ denotes the restriction of $H_p^s(R_n)$ on Ω,

$$\|f\|_{H_p^s(\Omega)} = \inf_{\substack{g(x) = f(x) \\ \text{for } x \in \Omega \,(a.e)}} \|g\|_{H_p^s(R_n)}$$

where $g \in H_p^s(R_n)$. Let $1 < p < \infty$; $1 \leq q \leq \infty$ and $-\infty < s < \infty$. If $s_o < s < s_1$ and $s = (1-\theta)s_o + \theta s_1$, then

$$B_{p,q}^s(\Omega) = (H_p^{s_o}(\Omega),\, H_p^{s_1}(\Omega))_{\theta,q} ,$$

where $(\ ,\ )_{\theta,q}$ denotes the real interpolation method of Lions-Peetre. The definition is independent of the choice of s_o and s_1. For $s > 0$ one obtains the usual Besov spaces. At least for $s > 0$ there are many equivalent norms ; see [2] and the references given there. Let

$$W_p^s(\Omega) = \begin{cases} H_p^s(\Omega) & \text{if } s = 0,1,2,\dots \\ B_{p,p}^s(\Omega) & \text{if } 0 < s \neq \text{integer.} \end{cases}$$

$W_p^s(\Omega)$ are the Sobolev-Slobodeckij spaces. Let Ω be either $\Omega = R_n$ or a bounded domain of the class C^∞. Then

$$W_p^s(\Omega) = \{f \mid f \in D'(\Omega),\ \|f\|_{W_p^s}^p = \sum_{|\alpha| \le s} \|D^\alpha f\|_{\mathcal{L}_p(\Omega)}^p < \infty\}$$

if $s = 0,1,2,\dots$, or

$$W_p^s(\Omega) = \{f \mid f \in D'(\Omega),\ \|f\|_{W_p^s}^p = \|f\|_{W_p^{[s]}}^p + \sum_{|\alpha| = [s]} \int_{\Omega\times\Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}} dxdy < \infty\}$$

if $0 < s = [s] + \{s\}$, $[s]$ integer ; $0 < \{s\} < 1$.

1.2 The case $\Omega = R_n$: Let A be a Banach space, and let $1 \le q \le \infty$. Then

$$l_q(A) = \{a \mid a = (a_j)_{j=1}^\infty,\ a_j \in A,\ \|a\|_{l_q(A)} = (\sum_{j=1}^\infty \|a_j\|_A^q)^{1/q} < \infty\}$$

(with the usual modification for the case $q = \infty$) denotes the vector valued l_q-space. "$\simeq$" means isomorphic property.

Theorem : $H_p^s(R_n) \simeq \mathcal{L}_p(0,1)$; $-\infty < s < \infty$; $1 < p < \infty$;

$B_{p,q}^s(R_n) \simeq l_q(l_p)$; $-\infty < s < \infty$; $1 < p < \infty$; $1 \le q \le \infty$.

Remarks : (1) The proof is given in [3]. There are further results of such a type.

(2) The spaces $\mathcal{L}_p(0,1)$ and $l_q(l_p)$ have unconditional Schauder bases ; $1 < p < \infty$; $1 \le q < \infty$. The Theorems shows that this is true for the spaces $H_p^s(R_n)$ and $B_{p,q}^s(R_n)$ also ; $-\infty < s < \infty$; $1 < p < \infty$; $1 \le q < \infty$.

(3) As a special case of the theorem one obtains

$$W_p^s(R_n) \simeq \begin{cases} \mathcal{L}_p(0,1) & \text{for } s = 0,1,2,\dots \text{ (Sobolev spaces)} \\ l_p & \text{for } 0 < s \neq \text{integer (Slobodeckij spaces)} \end{cases}$$

This shows that the spaces $W^s_p(R_n)$ for integers s on the one side and for non-integers s on the other side belong to different isomorphic classes.

(4) The spaces $B^s_{p,\infty}(R_n)$ are not separable.

1.3 The case $\Omega \in C^\infty$: Let Ω be a bounded domain in R_n with smooth boundary, $\Omega \in C^\infty$.

Theorem : (a) $H^s_p(\Omega) \simeq \mathcal{L}_p(0,1)$; $-\infty < s < \infty$; $1 < p < \infty$;

(b) $B^s_{p,p}(\Omega) \simeq l_p$; $-\infty < s < \infty$; $1 < p < \infty$

(c) $B^s_{p,q}(\Omega)$ has a Schauder basis ; $-\infty < s < \infty$; $1 < p < \infty$; $1 \le q < \infty$.

Remarks : (1) There are similar results for the spaces $\overset{o}{H}{}^s_p(\Omega)$ and $\overset{o}{B}{}^s_{p,q}(\Omega)$, which are defined as the completion of $C^\infty_o(\Omega)$ in the norm of $H^s_p(\Omega)$, resp. $B^s_{p,q}(\Omega)$.

(2) The proof is given in [3] ; there are further results in this direction.

(3) One can carry over the remarks (2) and (3) from section 1.2 for $p = q$.

(4) One can show that all the spaces $B^s_{p,q}(\Omega)$; $1 < p < \infty$; $1 \le q < \infty$; $0 < s < \frac{1}{p}$ have a common Schauder basis : a system of functions of Haar-type (see [3]).

(5) Problem : What is the structure of $B^s_{p,q}(\Omega)$; $p \neq q$?

§ 2. STRUCTURE THEORY OF FUNCTION SPACES OF SOBOLEV-SLOBODECKIJ TYPE WITH WEIGHTS

We shall consider two kinds of Sobolev-Slobodeckij spaces with weights.

2.1 The spaces $W^s_p(\Omega;\rho^\mu;\rho^\nu)$: Let $\Omega \subset R_n$ be an arbitrary domain (bounded or unbounded, no smoothness assumptions are needed). Let $0 < \rho(x) \in C^\infty(\Omega)$; $\exists\, c > 0$, $|\nabla\rho(x)| \le c\,\rho^2(x)$ for all $x \in \Omega$; $\rho(x) \to \infty$ for $x \to \partial\Omega$ or $|x| \to \infty$.

Let $1 < p < \infty$; μ and ν are real numbers ; $s > 0$, and $\nu \geq \mu + sp$. Then $W_p^s(\Omega;\ \rho^{\mu},\rho^{\nu})$ denotes the completion of $C_o^{\infty}(\Omega)$ in the norm

$$\|f\|_{W_p^s(\Omega;\rho^{\mu};\rho^{\nu})} = \Big(\int_{\Omega} \Big(\sum_{|\alpha|=s} \rho^{\mu}(x)\,|D^{\alpha}f(x)|^p + \rho^{\nu}(x)\,|f(x)|^p \Big)\,dx \Big)^{1/p} \quad \text{if}$$

$s = 0,1,2,3,\ldots.$ (for $s = 0$ we assume $\mu = \nu$), and

$$\|f\|_{W_p^s(\Omega;\rho^{\mu};\rho^{\nu})} = \Big(\int_{\Omega\times\Omega} \sum_{|\alpha|=[s]} \frac{|\rho^{\mu/p}(x)\,D^{\alpha}f(x) - \rho^{\mu/p}(y)\,D^{\alpha}f(y)|^p}{|x-y|^{n+\{s\}p}}\,dxdy$$

$$+ \int_{\Omega} \rho^{\nu}(x)\,|f(x)|^p dx \Big)^{1/p}$$

if $0 < s = [s] + \{s\}$; $[s]$ integer ; $0 < \{s\} < 1$. A more detailled consideration of these spaces is given in [5] .

<u>Theorem</u> :

$$W_p^s(\Omega;\rho^{\mu};\rho^{\nu}) \simeq \begin{cases} \mathcal{L}_p(0,1) & \text{if } s = 0,1,2,\ldots \\ l_p & \text{if } 0 < s \neq \text{integer} \end{cases}$$

2.2 <u>The spaces</u> $W_{p,\mu}^s(R_n)$ (<u>spaces of Kudrjavzev-type</u>) : Let $\sigma(x) = (1 + |x|^2)^{1/2}$, $x \in R_n$, a weight function. Let $1 < p < \infty$ and $-\infty < \mu < \infty$. Then

$$W_{p,\mu}^s(R_n) = \{f \mid f \in D'(R_n),$$

$$\|f\|_{W_{p,\mu}^s} = \Big(\int_{R_n} \sum_{|\alpha|\leq s} \sigma^{\mu-(s-|\alpha|)p}(x)\ |D^{\alpha}f(x)|^p dx \Big)^{1/p} < \infty \}$$

for $s = 0,1,2,\ldots$, and

$$W^s_{p,\mu}(R_n) = \{f \mid f \in D'(R_n), \|f\|_{W^s_{p,\mu}} =$$

$$\|f\|_{W^{[s]}_{p,\mu-\{s\}p}} + \Big(\int_{R_n \times R_n} \sum_{|\alpha|=[s]} \frac{|\sigma^{\mu/p}(x) D^\alpha f(x) - \sigma^{\mu/p}(y) D^\alpha f(y)|^p dxdy \Big)^{1/p}}{|x-y|^{n+\{s\}p}} < \infty \}$$

for $0 < s = [s] + \{s\}$; $[s]$ integer ; $0 < \{s\} < 1$. Spaces of such a type where s is an integer, are considered by Kudrjavzev.

Theorem :

$$W^s_{p,\mu}(R_n) \simeq \begin{cases} \mathcal{L}_p(0,1) & \text{if } s = 0,1,2,\ldots \\ l_p & \text{if } 0 < s \neq \text{integer} \end{cases}$$

Remark : An investigation of these spaces from the point of view of interpolation theory, a proof of the theorem, and also a proof of the theorem of the last section is given in [6] (and will be given in the announced book).

§ 3. STRUCTURE THEORY OF NUCLEAR FUNCTION SPACES

3.1 Nuclear (F)-spaces : An (F)-space is a complete locally convex space, the topology is generated by a countable set of semi-norms. Let be (without loss of generality)

$$\|a\|_1 \leq \|a\|_2 \leq \ldots \leq \|a\|_j \leq \ldots$$

the countable set of semi-norms. For sake of simplicity we assume that $\|a\|_j$ are norms. Let F_j be the completion of the (F)-space F in the norm $\|a\|_j$. Then holds $F_k \subset F_j$ for $k > j$. The imbedding operator is denoted by $I_{j,k}$. By definition the (F)-space F is nuclear, if for each $j = 1,2,$.there exists such a number $k = k(j)$ that the imbedding operator $I_{j,k}$ is a nuclear operator. [An operator $S \in \mathcal{L}(A,B)$, where A and B are Banach spaces, is nuclear, if it has a representation of the type

$$Sa = \sum_{j=1}^{\infty} f_j(a) b_j \ ; \ b_j \in B \ ; \ f_j \in A' \ ; \ \sum_{j=1}^{\infty} \|f_j\|_{A'} \|b_j\|_B < \infty \ ] .$$

An important nuclear (F)-space is the space

$$s = \{\xi \mid \xi = (\xi_j)_{j=1}^{\infty} \; ; \; \xi_j \text{ complex} \; ; \; \|\xi\|_l = \sup j^l |\xi_j| < \infty \text{ for } l = 0,1,2,\ldots\}$$

of rapidly decreasing sequences. Grothendieck conjectured, and T. and Y. Komura proved that each nuclear (F)-space is isomorphic to a subspace of the topological product $(s)^{\{1,2,3,\ldots\}}$.

3.2 The spaces $D(A^\infty)$: An (F)-space is (by definition) a Montel space if each bounded set is a pre-compact set.

Let H be a separable (complex) Hilbert space. Let A be a self adjoint operator, acting in H. The domain of definition is denoted by D(A). Then

$$D(A^\infty) = \bigcap_{j=1}^{\infty} D(A^j) \; , \; \|a\|_j = \|A^j a\|_H + \|a\|_H, \; (j = 1,2,\ldots),$$

is an (F)-space.

Theorem : (a) $D(A^\infty)$ is a Montel space iff A is an operator with pure point spectrum.

(b) $D(A^\infty)$ is a nuclear space iff A is an operator with pure point spectrum and there exist $c > 0$ and $\tau > 0$ such that holds

$$N(\lambda) \leq c\,\lambda^\tau \; . \qquad (\lambda > 1)$$

(The eigenvalues of A denoted by λ_j; $j = 1,2,\ldots$; $N(\lambda) = \sum_{|\lambda_j| \leq \lambda} 1$ is the function of the eigenvalue distribution).

(c) $D(A^\infty)$ is isomorphic to s iff A is an operator with pure point spectrum and there exist $c_1 > 0$, $c_2 > 0$, $\tau_1 > 0$, and $\tau_2 > 0$, such that holds

$$c_1 \lambda^{\tau_1} \leq N(\lambda) \leq c_2 \lambda^{\tau_2} \qquad (\lambda > 1)$$

Remark : The theory of the spaces $D(A^\infty)$ are developped by Mitjagin, Pietsch, and the author. Detailed references are given in [1,6].

3.3 General structure theorem for nuclear function spaces

Theorem : Let $\Omega \subset R_n$ be an arbitrary domain ; let

$$\overset{o}{A}u = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u \ ; \ a_\alpha(x) \in C^\infty(\Omega) \ ; \ D(\overset{o}{A}) = C_o^\infty(\Omega) \ ;$$

a symmetric operator ; let A be a self adjoint extension of $\overset{o}{A}$ in $\mathcal{L}_2(\Omega)$, and let $D(A^\infty)$ be a nuclear (F)-space. Then $D(A^\infty)$ is isomorphic to s.

Remark : In the formalism of the theorem no assumptions for the type of $\overset{o}{A}$ are needed. But one can expect "good" spaces $D(A^\infty)$ only in the cases if $\overset{o}{A}$ is elliptic, degenerate-elliptic, semi-elliptic..., but not hyperbolic,

3.4 Special function spaces

(a) Let $\Omega \subset R_n$ be an arbitrary domain; let $0 < \rho(x) \in C^\infty(\Omega)$,

$$\rho(x) \to \infty \quad \text{for} \quad x \to \partial\Omega \text{ and for } |x| \to \infty \ ;$$

$$|D^\gamma \rho(x)| \leq c_\gamma \, \rho^{1+|\gamma|}(x) \text{ for suitable numbers } c_\gamma > 0 \ ;$$

$\exists$ a > 0 such that $\rho^{-a}(x) \in \mathcal{L}_1(\Omega)$. Then

$$S_{\rho(x)}(\Omega) = \{f | f \in C^\infty(\Omega), \ \sup_{x\in\Omega} \rho^l(x) \ |D^\alpha f(x)| < \infty \text{ for all } l = 0,1,2,.. \text{and all } \alpha\}$$

is an (F)-space.

Theorem 1 : $S_{\rho(x)}(\Omega) \simeq s$

Remark :a) Each operator $\overset{o}{A}u = -\Delta u + \rho^\nu(x) u$; $\nu > 2$, $\mathcal{D}(\overset{o}{A}) = C_o^\infty(\Omega)$ fullfils the assumptions of the theorem in sect.3.3. It is possible to show that holds $D(A^\infty) = S_{\rho(x)}(\Omega)$. We refer to [4].

b) Let Ω be a bounded domain. Then

$$\tilde{C}_o^\infty(\Omega) = \{f | f \in C_o^\infty(R_n), \ \text{supp } f \subset \bar{\Omega}\} \ ;$$

equipped with the seminorms $\sup_{x\in\Omega} |\mathcal{D}^\alpha f(x)|$, $0 \leq |\alpha| < \infty$; is an (F)-space.

Theorem 2 : $\overline{C}_o^\infty(\Omega) \simeq s$.

Remarks : (1) If $\Omega \in C^\infty$, then $\overline{C}_o^\infty(\Omega)$ is the completion of $C_o^\infty(\Omega)$ in the topology of $\overline{C}_o^\infty(\Omega)$.

(2) Let $\rho^{-1}(x) \sim d(x)$ = distance of a point $x \in \Omega$ from the boundary. Then holds $S_{\rho(x)}(\Omega) = \overline{C}_o^\infty(\Omega)$ (in the sense of topological equality). This shows that theorem 2 is a special case of theorem 1.

(c) Let Ω be a bounded domain, $\Omega \in C^\infty$. Let ν_z be the inner normal for $z \in \partial\Omega$, and let $j = 1,2,\dots,$ and $l = 0,\dots,j$. Then

$$\overline{C}_{j,l}^\infty(\Omega) = \{f \mid f \in \overline{C}^\infty(\Omega),\ \frac{\partial^r f}{\partial \nu_z^r}\Big|_{\partial\Omega} = 0 \text{ for } r = n_1 j + n_2\ ;$$

$$n_1 = 0,1,2,\dots,n_2 = 0,1,\dots,l-1\}$$

is an (F)-space. (The topology is given by $\sup_{x\in\Omega} |D^\alpha f(x)|; 0 \le |\alpha| < \infty$; special cases are

$$\overline{C}_{j,o}^\infty(\Omega) = \overline{C}^\infty(\Omega) = \{f \mid \exists D^\alpha f \text{ in } \overline{\Omega} \text{ for all } \alpha\}$$

$$\overline{C}_{j,j}^\infty(\Omega) = \overline{C}_o^\infty(\Omega).$$

Theorem 3 : $\overline{C}_{j,l}^\infty(\Omega) \simeq s$.

Remark : The proof uses techniques of degenerate elliptic differential operators. A description of the principles is given in [1]. A full proof is given in [6].

REFERENCES

In the list of references are included only papers of the author obtaining proofs of the described theorems (or proofs of special cases of the theorems). In these papers (especially in [6], and in the announced book) are given detailed references to the important papers of Aronszajn, Besov, Calderón, Il'in, Kudrjavzev, Lions, Magenes, Muramatu, Nikol'skij, Peetre, Slobodeckij, Sobolev, Stein, Taibleson,... in the theory of Sobolev-Besov spaces, and Grothendieck, Komura, Mitjagin, Pietsch,... in the theory of nuclear spaces.

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