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THE SOBOLEV-BESOV IMBEDDING THEOREM FROM THE VIEWPOINT OF SEMI-GROUPS OF OPERATORS

by H. KOMATSU

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INTRODUCTION

In 1938 Sobolev [20] proved his famous imbedding theorem.

Let $1 \leqq p \leqq \infty$, $m = 0, 1, 2, \cdots$ and Ω be a domain in $\mathbb{R}^n$ satisfying a certain cone condition. We define the Sobolev space $W_p^m(\Omega)$ by

(0.1) $$W_p^m(\Omega) = \{x \in L^p(\Omega);\ D^\alpha x \in L^p(\Omega),\quad |\alpha| \leqq m\},$$

where $D^\alpha x$ is the derivative in the sense of distribution. Then his imbedding theorem is formulated as follows:

<u>Theorem</u>. (i) <u>Let</u> $1 \leqq p < p' < \infty$. <u>Then</u>

(0.2) $$W_p^m(\Omega) \subset W_{p'}^{m'}(\Omega) \quad \underline{\text{if}} \quad 0 \leqq m' \leqq m - (\frac{1}{p} - \frac{1}{p'})n .$$

(ii)

(0.3) $$W_p^m(\Omega) \subset L_p^{m'+\alpha}(\Omega) \quad \underline{\text{if}} \quad 0 < m'+\alpha \leqq m - \frac{n}{p}, \quad 0 < \alpha < 1.$$

Originally Sobolev [20] proved part (i) under the restriction that $p > 1$. Part (ii) is attributed to Morrey, Kondrashov, Nikol'skii, Gagliardo, Nirenberg etc.

If we consider only integer m' in part (i), it is clear that we lose some information corresponding to the remainder $m - (p^{-1} - p'^{-1})n - m'$. On the other hand, it is known that the imbedding $W_p^m(\Omega) \subset C^{m'}(\Omega)$ does not hold even if $m' = m - n/p$ is an integer. However, we may expect something better than the fact that $W_p^m(\Omega) \subset C^{m'-1+\alpha}(\Omega)$ for any $\alpha < 1$ or that $W_p^m(\Omega) \subset W_{p'}^{m'}(\Omega)$ for any $p' < \infty$.

A natural idea is to introduce Sobolev spaces of fractional orders and complete the statements. A large amount of works have been done in this direction. We mention in particular the works by Nikol'skii, Uspenskii, Besov and Il'in (see Nikol'skii [16]). Finally in 1961 Besov [2] obtained a satisfactory result in the case where $\Omega = \mathbb{R}^n$.

Russian school employs the theory of approximation by entire functions. Shortly after, Besov's results were reproved by Taibleson [21], Lions-Peetre [13], Grisvard [4] and Peetre [19] by completely different methods.

We remark that in one-dimensional case the imbedding theorem had been obtained by Hardy-Littlewood [5], [6]. Sobolev [20] does not mention their works but the proof itself is very similar. Interesting is the fact that Hardy-Littlewood already considered Besov spaces.

Taibleson's proof may be regarded as a direct succession of Hardy-Littlewood. In the latter three papers the imbedding theorem is proved as an application of the theory of interpolation of Banach spaces but some results from the potential theory etc. are employed as well.

In this report we try to give a proof minimizing the potential theory. Our proof is a generalization of Yoshikawa's treatment [22], [23], [24] of the Hardy-Littlewood theorem. Since we use semi-groups, our proof applies to domains Ω with some cone condition, while the papers mentioned above discuss only the case $\Omega = \mathbb{R}^n$.

§1. NON-NEGATIVE OPERATORS.

A closed linear operator A in a Banach space X is said to be <u>non-negative</u> if the negative real axis $(-\infty, 0)$ is contained in the resolvent set $\rho(A)$ of A and if

(1.1) $$M = \sup_{0<\lambda<\infty} \|\lambda(\lambda+A)^{-1}\| < \infty .$$

In this case we have also

(1.2) $$L = \sup_{0<\lambda<\infty} \|A(\lambda+A)^{-1}\| \leq M+1 < \infty .$$

A non-negative operator A is said to be <u>of type</u> ω, $0 \leq \omega < \pi$, if the domain $D(A)$ is dense and if the resolvent $(\zeta+A)^{-1}$ exists on the sector $\{\zeta \neq 0;\ |\arg \zeta| < \pi - \omega\}$ and $\zeta(\zeta+A)^{-1}$ is uniformly bounded on the subsector $\{\zeta \neq 0;$ $|\arg \zeta| \leq \pi - \omega - \varepsilon\}$ for any $\varepsilon > 0$.

If $-A$ is the infinitesimal generator of a bounded continuous semi-group $T(t)$ of operators, then A is a non-negative operator of type $\pi/2$. (The converse does not hold in general. Ōuchi [17] has shown that if A is a non-negative operator of type $\pi/2$, then $-A$ generates a hyperfunction semi-group.)

In case $\omega < \pi/2$, A is a non-negative operator of type ω if and only if $-A$ generates a bounded analytic semi-group $T(t)$ of type $\pi/2 - \omega$ in the sense that $T(t)$ is a continuous semi-group of operators which has an analytic continuation to the sector $\{t;\ |\arg t| < \pi/2 - \omega\}$ such that $T(t)$ is uniformly bounded on the subsector $\{t;\ |\arg t| \leq \pi/2 - \omega - \varepsilon\}$ for any $\varepsilon > 0$ (Kato, Komatsu [7]).

Examples. Let $X = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, or X be the space $BUC(\mathbb{R}^n)$ of all bounded uniformly continuous functions on $\mathbb{R}^n$.
(i) The translation semi-group $T(t)$ is clearly bounded continuous and $A = -\partial/\partial s_j$ is a non-negative operator of type $\pi/2$.
(ii) The Gauss-Weierstrass integral $T(t)$ is a bounded analytic semi-group of type $\pi/2$ and $A = -\Delta$ is a non-negative operator of type 0. (iii) The Poisson integral $T(t)$ is also a bounded analytic semi-group of type $\pi/2$ and $A = \sqrt{-\Delta}$ is a non-negative operator of type 0.

§2. REAL INTERPOLATION SPACES OF $(X, D(A^m))$ AND $(X, R(A^m))$.

We assume that A is a non-negative operator in a Banach space X.

Let $0 < \sigma < \infty$ and $1 \leq r \leq \infty$ or $r = \infty-$. Choose an integer $m > \sigma$ and define the spaces $D_r^\sigma(A)$ and $R_r^\sigma(A)$ by

(2.1) $$D_r^\sigma(A) = \{x \in X;\ \lambda^\sigma (A(\lambda + A)^{-1})^m x \in L_*^r(X)\},$$

(2.2) $$R_r^\sigma(A) = \{x \in X;\ \lambda^{-\sigma} (\lambda(\lambda + A)^{-1})^m x \in L_*^r(X)\},$$

where $L_*^r(X)$ is the X-valued L^p space on $(0, \infty)$ with respect to the measure $d\lambda/\lambda$ and $L_*^{\infty-}(X)$ is the subspace of $L_*^\infty(X)$

composed of all $f(\lambda)$ such that

$$f(\lambda) \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow 0 \quad \text{or} \quad \infty .$$

$D_r^\sigma(A)$ and $R_r^\sigma(A)$ are independent of the choice of integer $m > \sigma$ and form Banach spaces under the norms

$$(2.3) \qquad \|x\|_{D_r^\sigma(A)} = \|x\|_X + \|\lambda^\sigma (A(\lambda + A)^{-1})^m x\|_{L_*^r(X)} ,$$

$$(2.4) \qquad \|x\|_{R_r^\sigma(A)} = \|x\|_X + \|\lambda^{-\sigma} (\lambda(\lambda + A)^{-1})^m x\|_{L_*^r(X)} .$$

In case $-A$ generates a bounded continuous semi-group $T(t)$, we have

$$(2.5) \qquad D_r^\sigma(A) = \{x \in X; \quad t^{-\sigma}(1 - T(t))^m x \in L_*^r(X)\},$$

$$(2.6) \qquad R_r^\sigma(A) = \{x \in X; \quad t^\sigma (t^{-1} I(t))^m x \in L_*^r(X)\},$$

where

$$(2.7) \qquad I(t)x = \int_0^t T(s)x \, ds \quad .$$

In case $-A$ generates a bounded analytic semi-group $T(t)$, we have

$$(2.8) \qquad D_r^\sigma(A) = \{x \in X; \quad t^{-\sigma}(AT(t))^m x \in L_*^r(X)\} ,$$

$$(2.9) \qquad R_r^\sigma(A) = \{x \in X; \quad t^\sigma T(t)x \in L_*^r(X)\}.$$

It is shown that the spaces $D_r^\sigma(A)$ and $R_r^\sigma(A)$ coincide with the real interpolation spaces of Lions and Peetre [13], [18]:

$$(2.10) \qquad D_r^\sigma(A) = (X, D(A^m))_{\sigma/m,r} ,$$

$$(2.11) \qquad R_r^\sigma(A) = (X, R(A^m))_{\sigma/m,r} ,$$

where the domain $D(A^m)$ and the range $R(A^m)$ are regarded as Banach spaces under the norms

$$\|x\|_{D(A^m)} = \|x\|_X + \|A^m x\|_X , \tag{2.12}$$

$$\|x\|_{R(A^m)} = \|x\|_X + \inf_{A^m y=x} \|y\|_X \tag{2.13}$$

(Lions, Lions-Peetre [13] for (2.5); Berens [1], Komatsu [9] for (2.8); Grisvard [4], Komatsu [9] for (2.1); Komatsu [10] for (2.2), (2.6) and (2.9)).

We define the operators A^{α}_{σ}, $A^{\alpha}_{-\rho}$ and $A^{\alpha}_{\sigma,-\rho}$ with the domains $D^{\sigma}_r(A)$, $R^{\rho}_r(A)$ and $D^{\sigma}_r(A) \cap R^{\rho}_r$, by

$$A^{\alpha}_{\sigma}x = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{\infty} \lambda^{\alpha}(A(\lambda+A)^{-1})^m x \frac{d\lambda}{\lambda} , \quad 0 < \mathrm{Re}\,\alpha < \sigma , \tag{2.14}$$

$$A^{\alpha}_{-\rho}x = \frac{\Gamma(n)}{\Gamma(-\alpha)\Gamma(n+\alpha)} \int_0^{\infty} \lambda^{\alpha}(\lambda(\lambda+A)^{-1})^n x \frac{d\lambda}{\lambda} , \quad -\rho < \mathrm{Re}\,\alpha < 0, \tag{2.15}$$

$$A^{\alpha}_{\sigma,-\rho}x = \frac{\Gamma(m+n)}{\Gamma(m-\alpha)\Gamma(n+\alpha)} \int_0^{\infty} \lambda^{\alpha}(A(\lambda+A)^{-1})^m(\lambda(\lambda+A)^{-1})^n x \frac{d\lambda}{\lambda} , \quad -\rho < \mathrm{Re}\,\alpha < \sigma . \tag{2.16}$$

Actually the integrals depend only on x and α . We define <u>fractional powers</u> A^{α}_{+}, A^{α}_{-} and $A^{\alpha}_{\pm}$ to be the smallest closed extensions of A^{α}_{σ}, $A^{\alpha}_{-\rho}$ and $A^{\alpha}_{\sigma,-\rho}$ respectively. A^{α}_{+} etc. are sufficiently large restrictions of A^{α} when α is an integer, and they satisfy all properties that the powers of A should do. We have

$$D_1^{\mathrm{Re}\,\alpha}(A) \subset D(A^{\alpha}_{+}) \subset D_{\infty-}^{\mathrm{Re}\,\alpha}(A) , \quad \mathrm{Re}\,\alpha > 0 , \tag{2.17}$$

$$R_1^{-\mathrm{Re}\,\alpha}(A) \subset D(A_-^{\alpha}) \subset R_{\infty-}^{-\mathrm{Re}\,\alpha}(A), \quad \mathrm{Re}\,\alpha < 0 . \tag{2.18}$$

Moreover, let $0 < \mathrm{Re}\,\alpha < \sigma$. Then x belongs to $D_r^{\sigma}(A)$ if and only if x belongs to $D(A_+^{\alpha})$ and $A_+^{\alpha}x$ belongs to $D_r^{\sigma-\mathrm{Re}\,\alpha}(A)$.

In particular, write $\sigma = \nu + \tau$ with ν an integer and $0 < \tau \leqq 1$. Then we have

$$\begin{aligned} x \in D_r^{\sigma}(A) \iff x \in D(A^{\nu}) \quad & \text{and} \\ & \lambda^{\tau}(A(\lambda + A)^{-1})A^{\nu}x \in L_*^r(X) \\ \text{or} \quad & t^{-\tau}(1 - T(t))A^{\nu}x \in L_*^r(X) \\ \text{or} \quad & t^{-\tau}(tA)T(t)A^{\nu}x \in L_*^r(X) , \end{aligned} \tag{2.19}$$

where ( ) should be replaced by $(\quad)^2$ when $\tau = 1$. Thus in case $-A$ generates a bounded continuous semi-group $T(t)$, x belongs to $D_r^{\sigma}(A)$ if and only if $T(t)\,x$ is ν times continuously differentiable and the ν-th derivative is Hölder continuous of exponent τ in the sense of L_*^r $(0 < \tau < 1)$ or smooth in the sense of Zygmund and L_*^r $(\tau = 1)$.

If A is a non-negative operator of type ω and if $0 < \alpha < \pi/\omega$, then A_+^{α} is a non-negative operator of type $\alpha\omega$ and we have

$$D_r^{\sigma}(A_+^{\alpha}) = D_r^{\alpha\sigma}(A) , \tag{2.20}$$

$$R_r^{\sigma}(A_+^{\alpha}) = R_r^{\alpha\sigma}(A) . \tag{2.21}$$

Most of the equivalence of (semi-)norms proved in Taibleson [21] is a special case of the equivalence of (2.1), (2.5), (2.8), (2.19) and (2.20).

§3. INTERPOLATION OF NON-NEGATIVE OPERATORS.

Let (X_0, X_1) be an interpolation pair of Banach spaces or a pair of Banach spaces continuously imbedded in a Hausdorff space. We write the real and complex interpolation spaces

(3.1) $$X_{\theta,q} = (X_0, X_1)_{\theta,q}, \quad 0 < \theta < 1, \quad 1 \leqq q \leqq \infty \text{ or } q = \infty-,$$

(3.2) $$X_\theta = [X_0, X_1]_\theta, \quad 0 < \theta < 1$$

(see Peetre [18] and Calderón [3]). In general $\theta*$ denotes either θ, q or θ.

We assume that A_0 and A_1 are non-negative operators in X_0 and X_1 respectively satisfying the following compatibility conditions:

(3.3) $$A_0x = A_1x, \quad x \in D(A_0) \cap D(A_1),$$

(3.4) $$(\lambda + A_0)^{-1}x = (\lambda + A_1)^{-1}x, \quad x \in X_0 \cap X_1, \quad 0 < \lambda < \infty.$$

Then there is a unique non-negative operator A in $X = X_0 + X_1$ defined by

(3.5) $$D(A) = D(A_0) + D(A_1),$$

(3.6) $$Ax = A_0x_0 + A_1x_1, \quad x = x_0 + x_1 \in D(A_0) + D(A_1).$$

We define the <u>interpolation</u> $A_{\theta*}$ of A_0 and A_1 to be the restriction of A to the domain

(3.7) $$D(A_{\theta*}) = \{x \in D(A) \cap X_{\theta*};\ Ax \in X_{\theta*}\}.$$

We have

(3.8) $$(\lambda + A_{\theta *})^{-1} = (\lambda + A)^{-1} | X_{\theta *} , \quad 0 < \lambda < \infty ,$$

or the interpolation of $(\lambda + A_i)^{-1}$. Hence it follows that $A_{\theta *}$ is a non-negative operator in $X_{\theta *}$.

If $-A_i$, $i = 0, 1$, generate bounded continuous (analytic) semi-group $T_i(t)$, $-A_{\theta *}$ generates a bounded continuous (analytic) semi-group $T_{\theta *}(t)$ which is the interpolation of $T_i(t)$ unless $\theta * = \theta, \infty$.

§4. IMBEDDING THEOREM OF HARDY-LITTLEWOOD-YOSHIKAWA.

Let $\rho > 0$. A compatible pair (A_0, A_1) of non-negative operators in an interpolation pair (X_0, X_1) is said to be <u>of class</u> $R^{\rho}(X_0, X_1)$ if

(4.1) $$\|(\lambda(\lambda + A)^{-1})^m x\|_{X_0} \leqq K \lambda^{\rho} \|x\|_{X_1} , \quad x \in X_1, \quad 0 < \lambda < \infty ,$$

for some (all) integer $m > \rho$ and a constant K (Yoshikawa [22]).

The notation comes from the fact that every element $x \in X_1$ behaves as if it belongs to $R^{\rho}_{\infty}(A_0)$ except for the fact that $x \in X_0$. We can prove the following in the same way as the equivalence of (2.2), (2.6) and (2.9):

In case $-A_i$ generate bounded continuous semi-groups $T_i(t)$, (A_0, A_1) is of class $R^{\rho}(X_0, X_1)$ if and only if for every $x \in X_1$, $I(t)^m x$ is a strongly measurable function with values in X_0 and

(4.2) $\|(t^{-1}I(t))^m x\|_{X_0} \leq K t^{-\rho} \|x\|_{X_1}$, $x \in X_1$, $0 < t < \infty$.

In case $-A_i$ generate bounded analytic semi-groups $T_i(t)$, (A_0, A_1) is of class $R^\rho(X_0, X_1)$ if and only if

(4.3) $\|T(t)x\|_{X_0} \leq K t^{-\rho} \|x\|_{X_1}$, $x \in X_1$, $0 < t < \infty$.

Theorem (Yoshikawa [22], [23], [24]). Suppose that (A_0, A_1) is of class $R^\rho(X_0, X_1)$.

(i) If $\sigma > \rho$, then

(4.4) $$D_r^\sigma(A_1) \subset D_r^{\sigma-\rho}(A_0).$$

(ii) If $0 < \sigma < \rho$, then

(4.5) $$D_r^\sigma(A_1) \subset X_{1-\sigma/\rho, r}.$$

(iii) If $\sigma = \rho$ and $r = 1$, then

(4.6) $$D_r^\sigma(A_1) \subset X_0.$$

(iv) If $\sigma = \rho$ and $r > 1$, then

(4.7) $D_r^\sigma(A_1) \subset X_{\theta,q}$ for any $0 < \theta < 1$, $1 \leqq q \leqq \infty$

and

(4.8) $$\|x\|_{X^K_{\theta,q}} \leqq C\,\theta^{-1-1/q+1/r} \|x\|_{D_r^\sigma(A_1)} ,$$

where the left hand side of (4.8) denotes Peetre's K norm in $X_{\theta,q}$ (Peetre [18]).

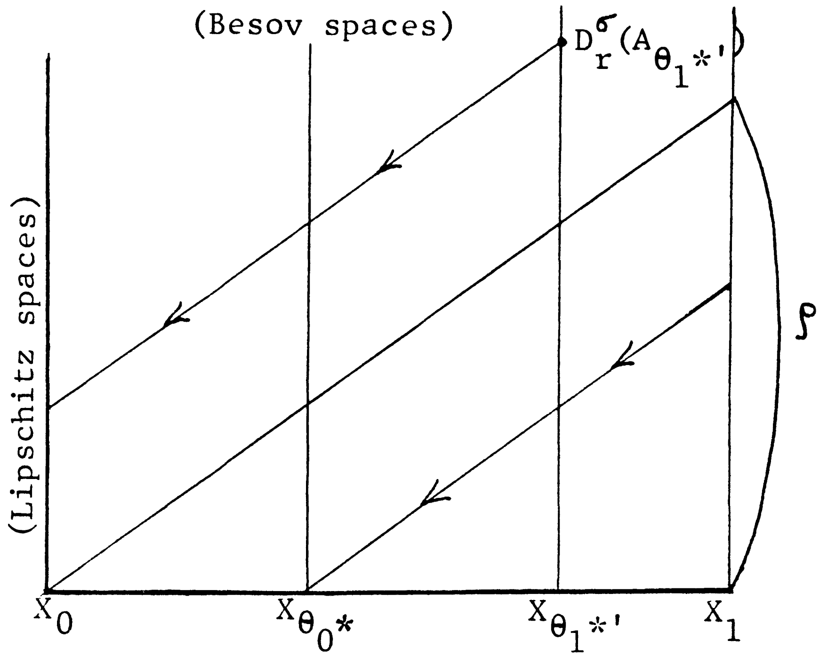
Proof. Since $D_r^\sigma(A_1) \subset D_r^\sigma(A) \cap R_\infty^\rho(A)$, we have by (2.16) the identity

$$(4.9) \qquad x = c\int_0^\infty (A(\lambda + A)^{-1})^m(\lambda(\lambda + A)^{-1})^n x \, d\lambda/\lambda, \quad x \in D_r^\sigma(A_1),$$

in $X = X_0 + X_1$. We divide the integral into the one over $(0, \tau)$ and the one over (τ, ∞) and estimate each integral suitably (see [12]).

If (A_0, A_1) is of class $R^\rho(X_0, X_1)$ and if $0 \leqq \theta_0 < \theta_1 \leqq 1$ then $(A_{\theta_0*}, A_{\theta_1*'})$ is of class $R^{(\theta_1 - \theta_0)\rho}(X_{\theta_0*}, X_{\theta_1*})$ (Yoshikawa [22]).

Hence we have the imbedding relations as shown in the following figure:



Let $X_0 = BUC(\mathbb{R}^n)$ and $X_1 = L^1(\mathbb{R}^n)$. Then X_θ is the Lebesgue space $L^{1/\theta}(\mathbb{R}^n)$ and $X_{\theta,q}$ is the Lorentz space $L^{(1/\theta,q)}(\mathbb{R}$

If $n = 1$ and $T(t)$ is the translation semi-group, we can easily prove by (4.2) that (A_0, A_1) is of class $R^1(X_0, X_1)$.

Hence we obtain the imbedding theorem of Hardy-Littlewood.

If $A = -\Delta$ or $\sqrt{-\Delta}$, the estimate of the Gauss-Weierstrass kernel or the Poisson kernel shows by (4.3) that (A_0, A_1) is of class $R^{n/2}(X_0, X_1)$ or $R^n(X_0, X_1)$. Thus we obtain a proof of the imbedding theorem of Sobolev-Besov in $\mathbb{R}^n$. This proof is essentially the same as that of Taibleson [21]. We note that the case $p = 1$ is not exceptional in this proof.

§5. COMMUTATIVE FAMILIES OF NON-NEGATIVE OPERATORS.

According to Muramatu [14] two non-negative operators A and B are said to be commutative if their resolvents are commutative

Let

$$\mathbb{A} = \{ A^{(1)}, \cdots, A^{(n)} \} \tag{5.1}$$

be a family of non-negative operators commutative with each other. We write

$$D_r^{\sigma}(\mathbb{A}) = \bigcap_{j=1}^{n} D_r^{\sigma_j}(A^{(j)}) \tag{5.2}$$

when $\sigma = (\sigma_1, \cdots, \sigma_n)$ is an n-tuple of positive numbers and $1 \leqq r \leqq \infty$ or $r = \infty-$.

Suppose that

$$\alpha^{(k)} = (\alpha_1^{(k)}, \cdots, \alpha_n^{(k)}), \quad k = 1, \cdots, N,$$

satisfy the following properties:

(i) Either $\alpha_j^{(k)} = 0$ or $\mathrm{Re}\, \alpha_j^{(k)} > 0$;

(ii) $\sum_{j=1}^{n} \frac{\mathrm{Re}\,\alpha_j^{(k)}}{\sigma_j} = 1, \quad k = 1, \cdots, N$;

(iii) For each $j = 1, \cdots, n$, there is an $\alpha^{(k)}$ such that $\mathrm{Re}\,\alpha_j^{(k)} = \sigma_j$.

Then we have

$$(5.3) \qquad (X, \bigcap_{k=1}^{N} D(A^{(1)\alpha_1^{(k)}} \cdots A^{(n)\alpha_n^{(k)}}))_{\theta,r} = D_r^{\theta\sigma}(\mathbb{A})$$

(Muramatu [14], Komatsu [12], see also Grisvard [4]).

We assume that (X_0, X_1) is an interpolation pair of Banach spaces and that

$$(5.4) \qquad \mathbb{A}_0 = \{A_0^{(1)}, \cdots, A_0^{(n)}\},$$

$$(5.5) \qquad \mathbb{A}_1 = \{A_1^{(1)}, \cdots, A_1^{(n)}\}$$

are commutative families of non-negative operators in X_0 and X_1 such that $A_0^{(j)}$ and $A_1^{(j)}$ are compatible for every j.

Then, the commutative families $\mathbb{A}$, $\mathbb{A}_{\theta,q}$ and $\mathbb{A}_\theta$ of non-negative operators in X, $X_{\theta,q}$ and X_θ are defined in the same way as in §4.

Let $\rho = (\rho_1, \cdots, \rho_n)$, $\rho_j > 0$. The pair $(\mathbb{A}_0, \mathbb{A}_1)$ is said to be of class $\mathbb{R}^\rho(X_0, X_1)$ if

$$(5.6) \qquad \|(\lambda_1(\lambda_1 + A^{(1)})^{-1})^{m_1} \cdots (\lambda_n(\lambda_n + A^{(n)})^{-1})^{m_n} x\|_{X_0} \leq K\lambda_1^{\rho_1} \cdots \lambda_n^{\rho_n} \|x\|_{X_1}, \quad x \in X_1, \quad 0 < \lambda_j < \infty,$$

for some (all) integers $m_j > \rho_j$ and a constant K.

In case $-A_i^{(j)}$, $i = 0, 1$, $j = 1, \cdots, n$, generate bounded continuous semi-groups $T_i^{(j)}(t)$, (5.6) holds if and only if for every $x \in X_1$ $I^{(1)}(t_1)^{m_1} \cdots I^{(n)}(t_n)^{m_n} x$ is a strongly measurable functions with values in X_1 and

$$\|(t_1^{-1}I^{(1)}(t_1))^{m_1} \cdots (t_n^{-1}I^{(n)}(t_n))^{m_n} x\|_{X_0} \leq K\, t_1^{-\rho_1} \cdots t_n^{-\rho_n} \|x\|_{X_1}, \quad x \in X_1, \quad 0 < t_j < \infty . \tag{5.7}$$

In case $-A_i^{(j)}$ generate bounded analytic semi-groups $T_i^{(j)}(t)$, (5.6) holds if and only if

$$\|T^{(1)}(t_1) \cdots T^{(n)}(t_n) x\|_{X_0} \leq K t_1^{-\rho_1} \cdots t_n^{-\rho_n} \|x\|_{X_1}, \quad x \in X_1, \quad 0 < t_j < \infty . \tag{5.8}$$

<u>Theorem</u>. <u>Suppose that</u> $(\mathbb{A}_0, \mathbb{A}_1)$ <u>is of class</u> $\mathbb{R}^{\rho}(X_0, X_1)$. <u>Let</u> $\sigma = (\sigma_1, \cdots, \sigma_n)$ <u>with</u> $\sigma_j > 0$ <u>and</u>

$$\kappa = \sum_{j=1}^{n} \frac{\rho_j}{\sigma_j} . \tag{5.9}$$

(i) <u>If</u> $\kappa < 1$, <u>then</u>

$$D_r^{\sigma}(\mathbb{A}_1) \subset D_r^{(1-\kappa)\sigma}(\mathbb{A}_0) . \tag{5.10}$$

(ii) <u>If</u> $\kappa > 1$, <u>then</u>

$$D_r^{\sigma}(\mathbb{A}_1) \subset X_{1-1/\kappa, r} . \tag{5.11}$$

(iii) <u>If</u> $\kappa = 1$ <u>and</u> $r = 1$, <u>then</u>

$$D_r^{\sigma}(\mathbb{A}_1) \subset X_0 . \tag{5.12}$$

(iv) <u>If</u> $\kappa = 1$ <u>and</u> $r > 1$, <u>then</u>

(5.13) $$D_r^{\sigma}(\mathbb{A}_1) \subset X_{\theta,q}, \quad 0 < \theta < 1, \quad 1 \leq q \leq \infty$$

and

(5.14) $$\|x\|_{X^K_{\theta,q}} \leq C\,\theta^{-1-1/q+1/r}\|x\|_{D_r^{\sigma}(\mathbb{A}_1)} .$$

In the proof we employ the identity

(5.15) $$x = c\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{n}\left\{(A^{(j)}(\lambda_j+A^{(j)})^{-1})^{m_j}(\lambda_j(\lambda_j+A^{(j)})^{-1})^{m_j}\right\}x\,\frac{d\lambda_1}{\lambda_1}\cdots\frac{d\lambda}{\lambda_n}$$

The domain of integral is divided into several parts according to each problem. The proof is obtained by estimating the integral on each part by an elementary computation (see [12]).

If $(\mathbb{A}_0, \mathbb{A}_1)$ is of class $\mathbb{R}^{\rho}(X_0, X_1)$ and if $0 \leq \theta_0 < \theta_1 \leq 1$, then $(\mathbb{A}_{\theta_0*}, \mathbb{A}_{\theta_1*'})$ is of class $\mathbb{R}^{(\theta_1-\theta_0)\rho}(X_{\theta_0*}, X_{\theta_1*'})$. Hence we have imbedding theorems between spaces $D_r^{\sigma}(\mathbb{A}_{\theta*})$.

§6. IMBEDDING THEOREM OF SOBOLEV-BESOV.

Let Ω be a domain in $\mathbb{R}^n$ satisfying the following cone condition: There is an open convex cone Γ with summit at the origin such that for every $s \in \Omega$ $s+\Gamma$ is contained in Ω .

We may assume that Γ contains the first octant:

(6.1) $$\Gamma \supset \{s;\ s_j > 0,\ j = 1, \cdots, n\}.$$

Let $X_0 = L^{\infty}(\Omega)$ and $X_1 = L^1(\Omega)$. Then we have for $1 < p < \infty$ and $1 \leq q \leq \infty$ or $q = \infty-$

(6.2) $$X_{1/p} = X_{1/p,p} = L^p(\Omega) : \text{ Lebesgue spaces,}$$

$$(6.3) \qquad X_{1/p,q} = L^{(p,q)}(\Omega) \ : \ \text{Lorentz spaces.}$$

We define for $x \in X = L^{\infty}(\Omega) + L^{1}(\Omega)$ and $j = 1, \cdots, n$

$$(6.4) \qquad T^{(j)}(t)x(s) = x(s_1, \cdots, s_j + t, \cdots, s_n).$$

The restrictions of $T^{(j)}(t)$ to X_i, $i = 0, 1$, form commutative families of bounded semi-groups of operators. In $X_0 = L^{\infty}(\Omega)$ $T^{(j)}(t)$ do not possess the strong continuity in t but a similar treatment is possible because they are the duals of strongly continuous semi-groups $S^{(j)}(t)$ in $L^{1}(\Omega)$. We may also restrict ourselves to $BUC(\Omega)$. The interpolation spaces $D_r^{\sigma}(\mathbb{A}_0)$ and $X_{\theta *}$ remain the same if we replace $L^{\infty}(\Omega)$ by $BUC(\Omega)$ (see [11]).

Corresponding non-negative operators are the maximal restrictions to respective spaces of the differential operators

$$(6.5) \qquad A^{(j)} = - \partial / \partial s_j$$

in the sense of distribution.

$(\mathbb{A}_0, \mathbb{A}_1)$ is of class $\mathbb{R}^{(1,1,\cdots,1)}(X_0, X_1)$ because we have

$$(6.6) \qquad \| I^{(1)}(t_1) \cdots I^{(n)}(t_n)x \|_{L^{\infty}(\Omega)} \leq \| x \|_{L^{1}(\Omega)} .$$

We define <u>Besov spaces</u> on Ω by

$$(6.7) \qquad B^{\sigma}_{p,r}(\Omega) = D^{\sigma}_r(\mathbb{A}_{1/p}),$$

$$(6.8) \qquad B^{\sigma}_{(p,q),r}(\Omega) = D^{\sigma}_r(\mathbb{A}_{1/p,q}).$$

By (2.19) we have

$$\|x\|_{B^{\sigma}_{p,r}(\Omega)} = \|x\|_{L^p(\Omega)} + \sum_{j=1}^{n} \|t_j^{-\sigma_j}(1 - T^{(j)}_{1/p}(t_j))^{m_j} x\|_{L^r_*(L^p(\Omega))}$$

(6.9)
$$= \|x\|_{L^p(\Omega)} + \sum_{j=1}^{n}\left\{\int_0^{\infty} t^{-r\sigma_j - 1} \Big\| \sum_{k=0}^{m_j} (-1)^k \binom{m_j}{k} x(s + kte_j)\Big\|^r_{L^p(\Omega)} dt\right\}^{1/r}$$

$$\sim \|x\|_{L^p(\Omega)} + \sum_{j=1}^{n}\left\{\int_0^{1} t^{-r\tau_j - 1} \Big\| \frac{\partial^{\nu_j} x(s)}{\partial s_j^{\nu_j}} - \frac{\partial^{\nu_j} x(s+te_j)}{\partial s_j^{\nu_j}}\Big\|^r_{L^p(\Omega)} dt\right\}^{1/r},$$

where $\sigma_j = \nu_j + \tau_j$ with $\nu_j \in \mathbb{Z}$, $0 < \tau_j \leq 1$ and the last $\| \ \|$ should be replaced by

$$\Big\| \frac{\partial^{\nu_j} x(s)}{\partial s_j^{\nu_j}} - 2\frac{\partial^{\nu_j} x(s+te_j)}{\partial s_j^{\nu_j}} + \frac{\partial^{\nu_j} x(s+2te_j)}{\partial s_j^{\nu_j}}\Big\|_{L^p(\Omega)}$$

when $\tau_j = 1$.

Thus our definition of Besov spaces coincides with that of Besov [2] when $\Omega = \mathbb{R}^n$.

It follows from the proposition in § 5 that if $\sigma = (\sigma, \sigma, \cdots, \sigma)$ with $\sigma > 0$ and if $m > \sigma$ is an integer, we have

(6.10) $$B^{\sigma}_{p*,r}(\Omega) = (L^{p*}(\Omega), W^m_{p*}(\Omega))_{\sigma/m,r},$$

where $W^m_{p*}(\Omega)$ is the Sobolev space $\{x \in L^{p*}(\Omega);\ D^{\alpha}x \in L^{p*}(\Omega), |\alpha| \leq m\}$. In particular, $B^{\sigma}_{p*,r}(\Omega)$ does not depend on the choice of affine coordinate system.

Combining the imbedding theorem of § 5 with a theorem of Muramatu [14] on the range of $A_+^{(1)\alpha_1} \cdots A_+^{(n)\alpha_n}$, we obtain the following imbedding theorem.

<u>Theorem.</u> We assume that $1 \leqq p < p' \leqq \infty$, $1 \leqq q, q' \leqq \infty$ or $q, q' = \infty-$, $1 \leqq r \leqq r' \leqq \infty$ or $r \leqq r' \leqq \infty-$, $\sigma = (\sigma_1, \cdots, \sigma_n)$ with $\sigma_j > 0$, $k = (k_1, \cdots, k_n)$ with $k_j \geqq 0$ integer and

$$(6.11) \qquad \sum_{j=1}^{n} \frac{k_j}{\sigma_j} < 1 .$$

Let $x \in B^{\sigma}_{p*,r}(\Omega)$ and set

$$(6.12) \qquad y = \frac{\partial^{|k|}}{\partial s_1^{k_1} \cdots \partial s_n^{k_n}} x ,$$

$$(6.13) \qquad \mu = \left(\frac{1}{p} - \frac{1}{p'}\right) \sum_{j=1}^{n} \frac{1}{\sigma_j} + \sum_{j=1}^{n} \frac{k_j}{\sigma_j} ,$$

$$(6.14) \qquad p'' = \frac{\sum_{j=1}^{n} \frac{1}{\sigma_j}}{\frac{1}{p} \sum_{j=1}^{n} \frac{1}{\sigma_j} + \sum_{j=1}^{n} \frac{k_j}{\sigma_j} - 1} .$$

(i) If $\mu < 1$, then $y \in B^{(1-\mu)\sigma}_{p'*',r'}(\Omega)$.

(ii) If $\mu > 1$ or if $\mu = 1$ and $p' < \infty$, then $y \in L^{(p'',r')}(\Omega)$

(iii) If $\mu = 1$, $p' = \infty$ and $r = 1$, then $y \in BUC(\Omega)$.

(iv) If $\mu = 1$, $p' = \infty$ and $r > 1$, then $y \in L^u(\Omega)$, $p < u < \infty$

and

$$(6.15) \qquad \sup_{p<u_0\leqq u<\infty} u^{-1+1/r} \|y\|_{L^u(\Omega)} \leqq C \|x\|_{B^{\sigma}_{p*,r}(\Omega)} .$$

For details see [12].

In the case where $\sigma = (\sigma, \sigma, \cdots, \sigma)$, $B^{\sigma}_{p*,r}(\Omega)$ is stable under sufficiently smooth coordinate transformations, so that we can prove the interpolation theorem (6.10) and the imbedding theorem for more general domains Ω by piecing together domains satisfying our strong cone condition.

Muramatu [15] gives a more direct treatment for general domains Ω .

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ERRATA

p.1	ℓ.8 from the bottom	$L_p^{m'+\alpha}(\Omega)$	$\Longrightarrow$ $Lip^{m'+\alpha}(\Omega)$
p.5	ℓ.2 from the bottom	L^p	$\Longrightarrow$ L^r
p.6	ℓ.6 from the bottom	$t^{-\sigma}(AT(t))^m x$	$\Longrightarrow$ $T^{-\sigma}(tA)^m T(t)x$
p.7	ℓ.1 from the bottom	$D_1^{Re\,\alpha(A)}$	$\Longrightarrow$ $D_1^{Re\,\alpha}(A)$
p.8	ℓ.5 from the top	$0 < \pi \leq 1$	$\Longrightarrow$ $0 < \tau \leq 1$
p.10	ℓ.6 from the top	semi-group	$\Longrightarrow$ semi-groups
p.12	ℓ.6 from the top	$(X_{\theta_0 *}, X_{\theta_1 *})$	$\Longrightarrow$ $(X_{\theta_0 *}, X_{\theta_1 *'})$
p.12	ℓ.3 from the bottom	place a period at the end of this line.	
p.14	ℓ.5 from the top	$D(A^{(1)\alpha_1^{(k)}} \cdots A^{(n)\alpha_n^{(k)}}))_{\theta,r}$	$\Longrightarrow$ $D(A_+^{(1)\alpha_1^{(k)}} \cdots A_+^{(n)\alpha_n^{(k)}}))_{\theta,r}$
p.14	ℓ.7 from the bottom	§4	$\Longrightarrow$ §3
p.14	ℓ.5 from the bottom	of class	$\Longrightarrow$ <u>of class</u> (underline)
p.15	ℓ.3 from the top	functions	$\Longrightarrow$ function
p.19	ℓ.1 from the top ~ ℓ.6 from the bottom	underline	
p.20	ℓ.11 from the top	Aerm.	$\Longrightarrow$ Amer.
p.21	ℓ.12 from the bottom	Product	$\Longrightarrow$ Products
p.21	ℓ.7 from the bottom	On the imbedding, continuity	$\Longrightarrow$ On imbedding, continuation
p.21	ℓ.6 from the bottom	in several variables	$\Longrightarrow$ of several variables