Séminaire Dubreil.
Algèbre et théorie
des nombres

Vlastimil Dlab

Filtered vector spaces (abstract)

Séminaire Dubreil. Algèbre et théorie des nombres, tome 28, nº 1 (1974-1975), exp. nº 5, p. 1-6
<http://www.numdam.org/item?id=SD_1974-1975__28_1_A2_0>

Séminaire P. DUBREIL
(Algèbre)
28e année, 1974/75, n° 5, 6 p.

9 et 16 décembre 1974

FILTERED VECTOR SPACES

(Abstract)

by Vlastimil DLAB

Let K be a central subfield of a division ring F , and let $[F : K] < \infty$. By a K-<u>structure</u> over F , we understand a finite (partially) ordered set $S=\{i,j,\dots\}$ together with an order preserving map from S to the lattice of all subdivision rings of F containing K : $i \longmapsto F_i$. A <u>representation</u> (W , W_i) of S , i. e. an S-filtered vector space, or briefly an S-space, is a vector F-space W_F together with F_i-subspaces W_i , $i \in S$, such that $i \leqslant j$ implies $W_i \subseteq W_j$. The category of all S-spaces (with obvious morphisms) is additive ; S is said to be of finite type if there is only a finite number of finite dimensional indecomposable S-spaces.

Filtered vector spaces appear very naturally in the study of representations of algebras (cf. [14], [7], [8] and [3]). However there are also other areas of mathematics where they play an essential rôle. In particular, GEL'FAND and PONOMAREV offered in [9] a conceptual proof of Kronecker theorem on normal forms of pairs of matrices and related results through a complete classification of the S-spaces, where S consists of four unrelated points.

The concepts of an K-structure S and an S-space are closely related to the concepts of a K-<u>species</u> and a representation of a K-species (cf. [7], [8] and [3]). These embody a certain amount of information extracted from the corresponding K-algebras which proves to be in some instances (e. g. for zero square radical or hereditary K-algebras) sufficient to determine the behaviour of the respective module categories (see [8], [3]).

GABRIEL's result on characterization of all quivers of finite representation type [7] and his proof based on the classification of "classical" K-structures (i. e. those for which $F_i = F$ for all $i \in S$) of finite type given KLEJNER, NAZAROVA and ROJTER [10], [11], is generalized in [3] by the following theorem.

THEOREM 1. - <u>A K-structure is of finite type if, and only if, its "weighted" width is</u> < 4 <u>and if it does not contain any K-substructure of the following form</u> :

(i) $I_2 \cup I_2 \cup I_2$;

(ii) $I_1 \cup I_3 \cup I_3$;

(iii) $I_1 \cup I_2 \cup I_5$;

(iv) $N \cup I_4$;

(v) $I_2 \cup I_2(G)$ <u>with</u> $[F : G] = 2$;

(vi) $I_1 \cup I_3(G)$ with $[F : G] = 2$ and

(vii) $I_2(G)$ with $[F : G] = 3$.

Here, I_n or $I_n(G)$ denotes the K-structure $\{1 < 2 < \ldots < n\}$ with $F_i = F$ for all $1 \leqslant i \leqslant n$ or $F_i = G$ for all $1 \leqslant i \leqslant n$, respectively, N denotes the classical K-structure $\{i < j > k < \ell\}$ and $\cup$ the disjoint union of ordered sets.

The proof of sufficiency given in [3] is "elementary" : It is based on the decomposition theorems for the K-structures $I_m(G)$, $I_n \cup I_1(G)$ and $I_1 \cup I_2(G)$ with $[F : G] = 2$ and for $I_1(6)$ with $[F : G] = 3$. The proof of necessity extends in [3] to a proof of the 2nd Brauer-Thrall conjecture :

THEOREM 1'. - If a K-structure S is not of finite type, then it is of strongly unbounded type in the sense that denoting by n_d the number of indecomposable S-spaces of dimension d , $n_d \neq 0$ for arbitrarily large d , $n_\infty \neq 0$ and, if K is infinite, $n_d = \infty$ for arbitrarily large d .

The proof in [3] uses some arguments from algebraic geometry and is to a large extent constructive : it depends on the following simple categorical :

CONSTRUCTION. - Let $\mathcal{C}$ be a Grothendieck category and I , I' two its indecomposable objects.

(a) If $0 = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_{i-1} \subseteq W_i \subseteq \ldots \subseteq W = \bigcup_{i=1}^{m} W_i$ for $m = 1,2,\ldots,\infty$ is a sequence of subobjects $W_i \in \mathcal{C}$ of W such that $W_i/W_{i-1} \approx I$ and every morphism $I \dashrightarrow W/W_{i-1}$ factors through W_i/W_{i-1} , then W is indecomposable.

(b) If $0 = W'_0 \subseteq W'_1 \subseteq \ldots \subseteq W'_{i-1} \subseteq W'_i \subseteq \ldots \subseteq W'_n = W'$ is a sequence of subobjects $W'_i \in \mathcal{C}$ of W' such that $W'_i/W'_{i-1} \approx I'$, and if $I \not\approx I'$, then $W \not\approx W'$.

The following theorem generalizing Gabriel's classification of quivers [7] can be deduced from the above classification of the filtered vector spaces (see [3]).

THEOREM 2. - A K-species is of finite type if, and only if, its diagram is a finite disjoint union of Dynkin diagrams. A K-species which is not of finite type is of strongly unbounded type.

A case-by-case inspection reveals immediately that the numbers of indecomposable representations of a K-species corresponding to a particular Dynkin diagram equals to its number of positive roots. An explanation of this phenomenon is offered by an elegant method of Coxeter functors deviced by BERNŠTEJN, GEL'FAND and PONOMAREV [1] : These are endofunctors in the (abelian) category $\mathfrak{L}(Q)$ of all representations of a K-species Q which allow, in the case when the diagram of Q is Dynkin, to construct all indecomposable representations from the simple representations in the same way as Coxeter transformations allow to construct all positive roots from the simple ones (for this general result see [5], or [4]). The method of Coxeter functors was also successfully applied in [5] to all extended Dynkin diagrams. In combination with the recent results of RINGEL in [16], one can substantially improve theorem 2 ; we shall formulate the result, without loss of generality, for "irreducible" K-species (cf. [13] for representations of quivers).

THEOREM 3.

(i) A K-species Q is of finite type if and only if its diagram Γ is a Dynkin diagram. The dimension function induces a bijection between the indecomposable representations of Q and the positive roots of Γ .

(ii) A K-species Q is of tame (classifiable) type if, and only if, its diagram Γ is an extended Dynkin diagram. The dimension function induces a bijection between the discrete indecomposable representations of Q and the positive roots of Γ . The subcategory of all homogeneous representations is a product of uniserial subcategories with a single simple object ; their dimension types are integral multiples of a fixed vector.

Thus, all other K-species Q are of wild (non-classifiable) type in the sense that there exists a full axact embedding of the category of all finite-dimensional modules over the free associative algebra generated by two variables over some commutative field K' into the category $\mathcal{L}(Q)$ of all representations of Q . The term "tame" in the formulation of the last theorem is, of course, used in the sense "not finite and not wild".

The last result demonstrates clearly the trichotomy (in contrast to Brauer-Thrall dichotomy) which dominates the theory of representation of K-species, and thus also of K-algebras.

Now, the results on K-species of finite type obtained in this elegant way can easily be translated back to K-structures, i. e. we can deduce from here the classification of K-structures of finite type (theorem 1). An obvious question, still open in general, is to classify K-structures of tame type. The first step in this direction, was recently obtained by NAZAROVA [15] for "classical" K-structures (cf. also [2]).

THEOREM 4. - A "classical" K-structure is of tame type if, and only if, its width is < 5 and if it does not contain any K-substructure of the following form :

(i) $I_1 \cup I_1 \cup I_1 \cup I_2$;

(ii) $I_2 \cup I_2 \cup I_3$;

(iii) $I_1 \cup I_3 \cup I_4$;

(iv) $I_1 \cup I_2 \cup I_6$ and

(v) $N \cup I_5$.

NAZAROVA's proof of theorem 4 is based on the generalized notion of "differentiation" of [11] and on the results obtained in [12].

The applications of the theory of filtered vector spaces to some problems of linear algebra given by GEL'FAND and PONOMAREV in [9] can be extended to "non-classical" situation. A typical problem in this direction is the classification of all real linear transformations between two exemples vector spaces. Here, we have a complex vector space $X \times Y$ filtered by two complex subspaces $X \times 0$ and $0 \times Y$ together with a real subspace : the graph of the transformation. This problem can be translated to the problem of classification of all indecomposable representations of the $\underset{\sim}{R}$-species $(\underset{\sim}{C}\ ,\ \underset{\sim}{C}\ ,\ {}_{\underset{\sim}{C}}\underset{\sim}{C}_{\underset{\sim}{R}} \otimes {}_{\underset{\sim}{R}}\underset{\sim}{C}_{\underset{\sim}{C}})$ and, using methods developed in [16] one can establish (see [6]) :

THEOREM 5. – <u>An indecomposable real linear transformation between two complex vector spaces can be brought, by a suitable choice of bases, to one of the following four types</u> :

$$(i)\ \begin{pmatrix} E_\infty & 0 & 0 & & & \\ E_0 & E_\infty & 0 & & 0 & \\ 0 & E_0 & E_\infty & & & \\ & & \ddots & \ddots & & \\ & 0 & & E_0 & E_\infty \\ & & & 0 & E_0 \end{pmatrix}, \qquad (ii)\ \begin{pmatrix} E_\infty & E_0 & 0 & & & \\ 0 & E_\infty & E_0 & & 0 & \\ 0 & 0 & E_\infty & \ddots & & \\ & & & \ddots & E_0 & 0 \\ & 0 & & & E_\infty & E_0 \end{pmatrix},$$

$$(iii)\ \begin{pmatrix} E_a & E_\infty & 0 & & & \\ 0 & E_a & E_\infty & & 0 & \\ 0 & 0 & E_a & & & \\ & & & \ddots & & \\ & 0 & & & E_a & E_\infty \\ & & & & 0 & E_a \end{pmatrix} \text{ \underline{with} } |a| \leqslant 1$$

<u>and</u>

$$(iv)\ \begin{pmatrix} E_1 & E_{ab} & 0 & 0 & & & & \\ E_{-1} & E_1 & 0 & E_\infty & & & 0 & \\ 0 & 0 & E_1 & E_{ab} & & & & \\ 0 & 0 & E_{-1} & E_1 & & & & \\ & & & & \ddots & & & \\ & & & & E_1 & E_{ab} & 0 & 0 \\ & 0 & & & E_{-1} & E_1 & 0 & E_\infty \\ & & & & 0 & 0 & E_1 & E_{ab} \\ & & & & 0 & 0 & E_{-1} & E_1 \end{pmatrix} \text{ \underline{with either} } b > 0 \text{ \underline{or} } b = 0 \text{ \underline{and} } a < 0 .$$

Here, E_a, E_∞ and E_{ab} are real 2×2 matrices $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, respectively. Every real linear transformation between two complex vector spaces is a product (unique up to the order of factors) of the linear transformations of the type (i)-(iv).

One should perhaps add that the proof of theorem 5 provides also an immediate argument which yields Jordan normal form and Kronecker normal form of pairs of matrices.

REFERENCES

[1] BERNŠTEJN (I. N.), GEL'FAND (I. M.) and PONOMAREV (V. A.). - Coxeter functors and Gabriel's theorem [in Russian], Uspekhi Mat. Nauk, t. 28, 1973, n° 2, p. 19-33.

[2] BRENNER (Sheila). - Decomposition properties of some small diagrams of modules, "Symposia Matematica", Vol. 13, p. 127-141. - London, New York, Academic Press, 1974 (Publicazione dell'Istituto nazionale di alta Matematica).

[3] DLAB (V.) and RINGEL (C. M.). - On algebras of finite representation type, J. of Algebra, t. 33, 1975, p. 306-394.

[4] DLAB (V.) and RINGEL (C. M.). - Représentation des graphes valués, C. R. Acad. Sc. Paris, t. 278, 1974, Série A, p. 537-540.

[5] DLAB (V.) and RINGEL (C. M.). - Representations of graphs and algebras, Carleton Lecture Notes, n° 8, 1974.

[6] DLAB (V.) and RINGEL (C. M.). - Normal forms of real matrices with respect to complex similarity (to appear).

[7] GABRIEL (P.). - Unzerlegbare Darstellungen, I., Manuscripta Math., t. 6, 1972, p. 71-103.

[8] GABRIEL (P.). - Indecomposable representations, II., Symposia Mathematica, Vol. 11, p. 81-104. - London, New York, Academic Press, 1973 (Publicazione dell'Istituto nazionale di Alta Matematica).

[9] GEL'FAND (I. M.) and PONOMAREV (V. A.). - Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space, "Hilbert space operators and operator algebras" [1970. Tihany], p. 163-237. - Amsterdam, London, North-Holland publishing Company, 1972 (Colloquia Mathematica Societatis Janos Bolyai, 5).

[10] KLEJNER (M. M.). - Partially ordered sets of finite type [in Russian], Zap. Naučn. Semin. Leningrad. Otdel. Mat. Inst. Steklova, t. 28, 1972, p. 32-41.

[11] NAZAROVA (L. A.) and ROJTER (A. V.). - Representations of partially ordered sets [in Russian], Zap. Naučn. Semin. Leningrad. Otdel. Mat. Inst. Steklova, t. 28, 1972, p. 5-31.

[12] NAZAROVA (L. A.) and ROJTER (A. V.). - On a problem of I. M. Gel'fand [in Russian], Funkcional. Anal., t. 7, 1973, n° 4, p. 54-69.

[13] NAZAROVA (L. A.). - Representation of quivers of infinite type [in Russian], Izv. Akad. Nauk SSSR, Ser. mat., t. 37, 1973, p. 752-791.

[14] NAZAROVA (L. A.) and ROJTER (A. V.). - Categorical matrix problems and the Brauer-Thrall conjecture. - Kiev, Akad. Nauk USSR, 1973 (Preprint).

[15] NAZAROVA (L. A.). - Partially ordered sets with an infinite number of indecomposable representations, "Proceedings of the International Conference on representations of algebras" [1974. Ottawa], p. 20-01 - 20-11.

[16] RINGEL (C. M.). - Representations of K-species and bimodules (to appear).

(Texte reçu le 17 mars 1975)

Vlastimil DLAB
Dept of Mathematics
Carleton University
OTTAWA, Ontario (Canada)