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NOTES ON AN EXTENSION OF KRULL'S PRINCIPAL IDEAL THEOREM

by David EISENBUD (*)

In this note, we will propose a generalization of Krull's principal ideal theorem which we can prove to be correct, for example, for rings containing a field. We will then sketch an application to the theory of determinantal ideals ; roughly speaking, our theorem implies a more precise version of the theorems of MACAULAY and EAGON on the heights of determinantal ideals. We also mention a speculative connection between our conjecture and the intersection conjectures of SERRE and PESKINE-SZPIRO. Details will appear elsewhere.

1. The generalized principal ideal theorem.

Throughout this paper, all rings will be assumed commutative and noetherian.

Krull's principal ideal theorem [5] states that an element a in the maximal ideal of a local ring R generates an ideal of height at most one (the apparently sharper statement that the minimal primes all have height at most one follows trivially by localization). Regarding a as a homomorphism $R \to R$, and noting that the rank of R, as an R-module, is 1, one might be lead to conjecture that something "similar" can be said about homomorphisms from an arbitrary module into R. To be more precise, one needs first the right notion of the rank of a module. Since we wish to work with homomorphisms to the ring, it is not unreasonable to require that an R-module M should have rank 0 if, and only if, $M^* = \mathrm{Hom}(M,R) = 0$. As with all notions of rank, a module M should have rank $\leq k$ if, and only if, its $(k+1)^{th}$-exterior power has rank 0. These conditions uniquely specify a notion of the rank of a module, which can be more simply put as follows :

Definition. - Let $\mathfrak{U}$ be the set of nonzerodivisors of R. The rank of a finitely generated R-module M is the minimal number of generators of $M_{\mathfrak{U}}$ as an R-module.

Of course, if R is a domain, this is the usual notion of rank. We can now state our conjecture :

Generalized principal ideal conjecture. - Let R be a local ring with maximal ideal J, and let M be a finitely generated R-module of rank n. Let

$$M^* = \mathrm{Hom}(M , R) ,$$

and let φ be an element of JM^*. Then the height of the ideal $\varphi(M)$ is at most n.

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It is easy to see that, if $M = R^n$, this conjecture becomes the version of the Krull principal ideal theorem which states that the height of a proper n-generator ideal is at most n .

We can prove a weakned form of the conjecture, in which "height" is replaced by "depth", and we can prove the conjecture itself in many cases.

Recall that if $I \subseteq R$ is an ideal, and N is a finitely generated R-module, then depth (I , N) is the length of a maximal N-sequence in I .

THEOREM 1. - *Let* R *be a noetherian local ring with maximal ideal* J . *Let* M *be a finitely generated* R-*module, and let* $\varphi \in JM^*$.

(a) *If* N *is a finitely generated* R-*module, then* $\mathrm{depth}(\varphi(M), N) \leq \mathrm{rank}\ M$.

(b) *If* R *has (possibly not finitely generated) Cohen-Macaulay modules in the sense of HOCHSTER* [4], *then*

$$\mathrm{height}\ \varphi(M) \leq \mathrm{rank}\ M .$$

The extra hypothesis of (b) is known to be fulfilled if R contains a field, and is conjectured to be true in general [4].

It is perhaps amusing to note that our conjecture can be reformulated as giving a condition, in terms of the punctered spectrum, for an element of a module to be part of a minimal system of generators :

Conjecture (second version). - Let (R , J) be a local ring of dimension d , and let M be a module of rank < d . Suppose that a is an element of M such that, for every prime ideal $P \neq J$, a generates a free summand of M_P . Then a is part of a minimal system of generators for M .

2. Determinantal ideals.

One of the earliest generalizations of the principal ideal theorem was the theorem of MACAULAY [6] that (for polynomial rings) the height of the ideal of $p \times p$ minors of a $p \times q$ matrix, if the ideal is proper, is at most $q - p + 1$. This was generalized by EAGON in 1960, who showed (for a general noetherian ring) that the height of the ideal of $k \times k$ minors of a $p \times q$ matrix, if the ideal is proper, is at most $(p - k + 1)(q - k + 1)$ (There is a very elegant proof of this in [3]). On the basis of the conjecture made in the last section, we can extend this result to say something about what happens to the ideal of $k \times k$ minors when an extra column is added to the matrix.

Before stating our result, we remark on a result that can be proved by the technique of [3].

PROPOSITION. - *Let* φ *be a* $p \times q$ *matrix over a ring* R , *and suppose that the* $(\ell + 1) \times (\ell + 1)$ *minors of* φ *are all* 0 . *Then the ideal generated by the* $\ell \times \ell$

minors of φ has height at most

$$p + q - 2\ell + 1 .$$

Now suppose that R is local and that we adjoin a new column, with entries in the maximal ideal, to a $p \times q$ matrix φ, obtaining a $p \times (q + 1)$ matrix φ'. Suppose that the $k \times k$ minors of φ are all 0. What can the height h of the ideal of $k \times k$ minors of φ' be ? Of course, it is contained in the ideal of $(k - 1) \times (k - 1)$ minors of φ, and also in the ideal generated by the p entries of the new column, so one obtains a bound from the proposition :

$$h \leqslant \min(p , p + q - 2k + 3) .$$

The next theorem shows that one can do better (at least much of the time !) :

THEOREM. - Suppose that R is a local ring satisfying the generalized principal ideal conjecture. (For example, suppose that R contains a field.) Let φ be a $p \times q$ matrix over R whose $k \times k$ minors are all 0, and let φ' be a matrix obtained from φ by adjoining a column whose entries are in the maximal ideal. Then the height of the ideal of $k \times k$ minors of φ' is at most $p - k + 1$.

As a consequence of the theorem and the proposition, we can prove a result which generalizes a "rigidity" theorem of BUCHSBAUM and RIM [1], which, in turn, generalized the result that if n elements $f_1 , \dots , f_n$ of a local ring generate an ideal of height n, then any k of them generate an ideal of height k :

COROLLARY. - Suppose that R is a local ring satisfying the generalized principal ideal conjecture, and that φ is a $p \times q$ matrix over R, with coefficients in the maximal ideal, such that the ideal of $k \times k$ minors of φ has height $(p - k + 1)(q - k + 1)$, the largest possible value. Then for every integer $\ell \geqslant k$, and every $s \times t$ submatrix $\overline{\varphi}$ of φ, the height of the ideal of $\ell \times \ell$ minors of $\overline{\varphi}$ is

$$(s - \ell + 1)(t - \ell + 1) ,$$

again the largest possible value.

In particular, no $\ell \times \ell$ minor of φ is 0.

To see that the hypothesis about coefficients being in the maximal ideal are necessary for this, consider the following matrix over $F[x , y]$, when F is a field :

$$\begin{pmatrix} 0 & 0 & 1 \\ x & y & 0 \end{pmatrix} .$$

Here the height of the ideal of 2×2 minors is $2(= (3 - 2 + 1)$, but the first 2×2 minor is 0.

3. A remark on intersection theory.

The remark is easy and speculative : Suppose that M and N are modules of ranks m and n over a local ring (R, J) (containing a field, say). Suppose that $\varphi \in JM^*$ and $\psi \in JN^*$, and write

$$X = \varphi(M)$$
$$Y = \psi(N) .$$

Then the ideal $X + Y$ can be written as $(\varphi , \psi)(M \oplus N)$, so $X + Y$ has height at most $m + n$. This gives some hold on the "intersection theory" of ideals of the form $\varphi(M)$. For example, if one could prove that for every prime ideal P of a regular local ring R , J there exists a module M with rank $M = \text{ht } P$ and an element $\varphi \in JM^*$ with $\varphi(M) = P$, then one could deduce Serre's intersection theorem ([7], ch. V, theorem 3).

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