Séminaire Dubreil.
Algèbre et théorie
des nombres

Robert M. Fossum

# Minimal injective resolutions

*Séminaire Dubreil. Algèbre et théorie des nombres*, tome 28, nº 1 (1974-1975), exp. nº 16, p. 1-5
<http://www.numdam.org/item?id=SD_1974-1975__28_1_A10_0>

Séminaire P. DUBREIL
(Algèbre)
28e année, 1974/75, n° 16, 5 p. 

7 avril 1975

# MINIMAL INJECTIVE RESOLUTIONS

by Robert M. FOSSUM

## 0. Introduction.

The many results about commutative noetherian rings which have a non -zero module of finite type with finite injective dimension seem to indicate that the minimal injective resolution of a module of finite type should contain a great amount of information about the module. See for example PESKINE and SZIRO's paper [5]. In this report, I will outline a proof of a result due principally to FOXBY and GRIFFITH (and proved independently by P. ROBERTS using differents methods) which states :

*If $A$ is a noetherian local ring with maximal ideal $\mathfrak{m}$ and if $M$ is an $A$-module of finite type, then $\mathrm{Ext}^j_A(A/\mathfrak{m}, M) \neq 0$ for all $j$ in the range*

$$\operatorname{depth} M \leq j \leq \operatorname{id}_A M .$$

This can be interpreted as a rigidity result. It also gives information about the minimal injective resolution of $M$. For if

$$0 \to M \to I^0 \to I' \to \ldots$$

is a minimal injective resolution of $M$, then the result states that the injective envelope of the residue class field is a direct summand of $I^j$ for those integers $j$ in the range $\operatorname{depth} M \leq j \leq \operatorname{id} M$. An interesting aspect of the proof is that is uses HOCHSTER's result establishing the existence of a maximal Cohen-Macaulay module (not necessarily of finite type) for a local ring of characteristic $p$ (see HOCHSTER [4]), while the result itself is independent of characteristic.

Complete details can be found in a paper by FOSSUM, FOXBY, GRIFFITH and REITEN [2].

## 1. Preliminary results.

Let $A$ be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue class field $k = A/\mathfrak{m}$. Let $M$ be an $A$-module of finite type. It is standard that

$$\operatorname{depth}_A M = \inf\{i \ ; \ \mathrm{Ext}^i_A(k, M) \neq 0\}$$

and

$$\operatorname{id}_A M = \sup\{i \ ; \ \mathrm{Ext}^i_A(k, M) \neq 0\} .$$

So the question is : what happens to $\mathrm{Ext}^j_A(k, M)$ for $j$ in the interval between $\operatorname{depth}_A M$ and $\operatorname{id}_A M$ ?

BASS reported two results [1].

PROPOSITION 1. – *If* $\mathrm{id}_A\, M < \infty$, *then* $\mathrm{id}_A\, M = \mathrm{depth}_A\, A$.

PROPOSITION 2. – *If* $\mathrm{id}_A\, M = \infty$, *then* $\mathrm{Ext}_A^j(k, M) \neq 0$ *for all* $j$ *with* $j \geqslant \dim A$.

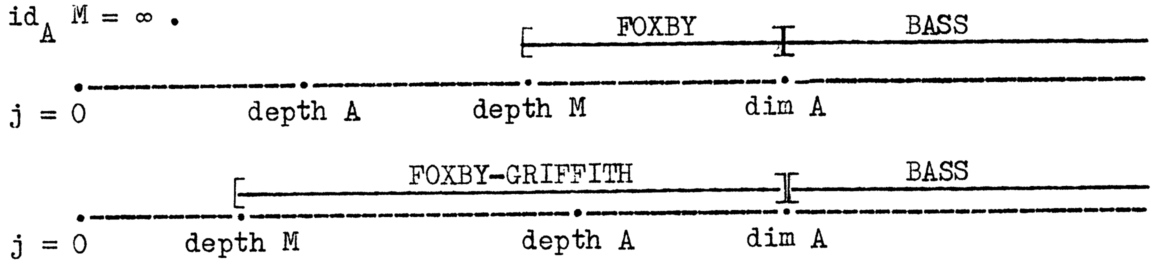
Much later FOXBY [3] extended the range in which $\mathrm{Ext}_A^j(k, M) \neq 0$ for very special modules.

PROPOSITION 3. – *If* depth $A \leqslant$ depth $M$, *then* $\mathrm{Ext}_A^j(k, M) \neq 0$ *for those* $j$ *with*

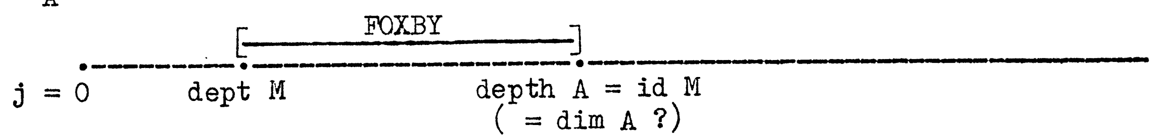
$$\mathrm{depth}\, M \leqslant j \leqslant \mathrm{id}\, M .$$

A diagram explains these results. The intervals with solid lines indicate the range of $j$ where $\mathrm{Ext}_A^j(k, M) \neq 0$.

1° $\mathrm{id}_A\, M = \infty$.




2° $\mathrm{id}_A\, M < \infty$




2. Main theorem.

The main theorem, which is stated in the local case in the introduction, follows :

THEOREM 1. – *Let* $A$ *be a noetherian ring and* $M$ *an* $A$*-module of finite type. Let*

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

*be a minimal injective resolution of* $M$. *If* $j$ *is an integer, if* $\mathfrak{p} \in \mathrm{spec}\, A$, *and if* $\mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \leqslant j \leqslant \mathrm{id}_{A_\mathfrak{p}} M_\mathfrak{p}$, *then the injective envelope of* $A/\mathfrak{p}$ *is a direct summand of* $I^j$.

It is clear that we may assume that $A$ is local and even complete, if necessary. Furthermore we can assume that depth $M <$ depth $A$ by FOXBY's result.

*Reduction.* – Let $f_1, \ldots, f_r$ be a set of elements in $A$ that forms a regular $M$-sequence and a regular $A$-sequence. Let $\mathfrak{f}$ denote the ideal generated by these elements. Since

$$\mathrm{Ext}_A^j(k, M) \cong \mathrm{Ext}_{A/\mathfrak{f}}^{j-r}(k, M/\mathfrak{f} M) ,$$

it may be assumed that depth $M = 0$ .

## 3. The main lemma.

The principal lemma that connects the maximal Cohen-Macaulay modules with the problem under consideration follows. Let $E(k)$ denote the injective envelope of $k$ as an $A$-module. The functor, that to $M$ associates $\mathrm{Hom}_A(M , E(k))$ is denoted by $M^{\vee}$ .

LEMMA 1. - *Suppose* $N$ *is an* $A$-*module (not necessarily of finite type), suppose* $x_1 , \ldots , x_n$ , *is a regular* $N$-*sequence such that the annihilator*

$$\mathrm{Ann}(N/(x_1 , \ldots , x_n)N)$$

*is proper and* $\mathfrak{m}$-*primary. If* $M$ *is an* $A$-*module of finite type with* depth $M = 0$ , *then*

$$\mathrm{Ext}^i_A(N/(x_1 , \ldots , x_j)N , M) \neq 0$$

*for all* $i$ *and* $j$ *with* $0 \leqslant i \leqslant j$ .

*Proof*. - The proof goes by induction on $j$ . Suppose $j = 0$ . Since

$$\mathrm{Hom}_A(N , k) \simeq \mathrm{Hom}_A(N , \mathrm{Hom}_A(k , E(k)) \cong \mathrm{Hom}_A(N \otimes_A k , E(k)) ,$$

it is sufficient to show that $N \otimes_A k \neq 0$ . But it is assumed that

$$\mathrm{Ann}(N/(x_1 , \ldots , x_n)N)$$

is $\mathfrak{m}$-primary and therefore $\mathfrak{m}(N/(x_1 , \ldots , x_n)N) \neq N/(x_1 , \ldots , x_n)N$ . Hence $N/\mathfrak{m}N \neq 0$ . The assumption depth $M = 0$ is equivalent to the assumption that $k$ is isomorphic to a submodule of $M$ . Therefore $\mathrm{Hom}_A(N , k) \neq 0$ implies $\mathrm{Hom}_A(N,M) \neq 0$ .

The induction step uses the isomorphisms

$$\mathrm{Ext}^i_A(N/(x_1 , \ldots , x_j)N , M) \simeq (\mathrm{Tor}^i_A(N/(x_1 , \ldots , x_j) , M^{\vee})^{\vee}$$

and the exact sequences

$$0 \longrightarrow N/(x_1 ,\ldots, x_{j-1})N \xrightarrow{.x_j} N/(x_1 ,\ldots, x_{j-1})N \longrightarrow N/(x_1 ,\ldots, x_j)N \longrightarrow 0$$

to show that $\mathrm{Tor}^A_i(N/(x_1 , \ldots , x_j)N , M^{\vee}) \neq 0$ and therefore

$$\mathrm{Ext}^i_A(N/(x_1 , \ldots , x_j)N , M) \neq 0 .$$

LEMMA 2. - *If* $\mathrm{Ext}^j_A(k , M) = 0$ , *then* $\mathrm{Ext}^j_A(T , M) = 0$ *for all* $A$-*modules* $T$ *with* $\mathrm{Supp}\, T \leqslant \{\mathfrak{m}\}$ .

*Proof*. - By induction on length, it is clear that $\mathrm{Ext}^j_A(T , M) = 0$ for all $A$-modules $T$ of finite length. Otherwise write $T = \varinjlim T_\alpha$ where each $T_\alpha$ has finite length. Then

$$\mathrm{Ext}^j_A(T , M) = \varprojlim \mathrm{Ext}^j_A(T_\alpha , M) .$$

## 4. Proof of the theorem.

We assume, which we may, that $A$ is a complete local ring and that depth $M = 0$. Suppose $j$ is an integer in the range $0 < j < \dim A$. Let $d = \dim A$.

Suppose $p$ is the characteristic of the residue class field. Let $R = A/pA$. We now quote a result due to HOCHSTER [4].

THEOREM 2. - *If* $R$ *is an equi-characteristic local ring of dimension* $t$, *then there is an* $R$-*module* $T$ (not necessarily of finite type) *such that if* $x_1, \dots, x_t$ *is a system of parameters of* $R$, *then* $(x_1, \dots, x_t)T \neq T$ *and these elements form a regular* $T$-*sequence. Such a* $T$ *is called a maximal Cohen-Macaulay module.*

Apply this theorem to the ring $R$ above. If $\dim R = d - 1$, pick elements $x_1, \dots, x_{d-1}$ in $A$ that form a system of parameters in $R$ and if $\dim R = \dim A$, pick $x_1, \dots, x_d$ in $A$ forming a system of parameters in $R$. Let $T$ be a maximal Cohen-Macaulay module for $R$. Set $N = T/x_d T$ (where $x_d = 0$ in case $\dim R = -1 + \dim A$). Then $x_1, \dots, x_{d-1}$ is a regular $N$-sequence and $\mathrm{Ann}\, N/(x_1, \dots, x_{d-1})N$ is $\mathfrak{m}$-primary. Apply lemma 1 to get

$$\mathrm{Ext}_A^r(N/(x_1, \dots, x_{d-1})N, M) \neq 0 \quad \text{for } 0 \leqslant r \leqslant d-1 .$$

Apply lemma 2 to get $\mathrm{Ext}_A^r(k, M) \neq 0$ for $0 \leqslant r \leqslant d - 1$. This proves the theorem.

COROLLARY 1. - *If* $j$ *is an integer with* $j > \text{depth } M$, *then* $\mathrm{Ext}_A^j(k, M) = 0$ *if, and only if,* $\mathrm{id}_A M < j$.

COROLLARY 2. - *If* $0 \to M \to I^0 \to I^1 \to \dots$ *is an injective resolution of* $M$, *then* $E(k)$ *is a direct summand of* $I^j$ *if, and only if,* $\text{depth } M \leqslant j \leqslant \text{id } M$.

*Remark* 1. - It does not follow, nor as examples show is it even true, that the local cohomology modules $H^j_{\mathfrak{m}}(M) \neq 0$.

*Remark* 2. - If $M$ is a nonzero module of finite type and finite injective dimension, then $\text{id } M = \text{depth } A$. If $\text{depth } A \leqslant \text{depth } M$, then $A$ is Cohen-Macaulay. If $\text{depth } M < \text{depth } A$, then it is clear from the last paragraph of the proof that $\dim A - \text{depth } A \leqslant 1$. In particular, if $\dim A = \dim(A/pA)$, then the proof shows that $A$ is Cohen-Macaulay. But this also is easily obtained from [5]

### REFERENCES

[1] BASS (H.). - On the ubiquity of Gorenstein rings, Math. Z., t. 82, 1963, p. 8-28.

[2] FOSSUM (R.), FOXBY (H.-B.), GRIFFITH (P.) and REITEN (I.). - Minimal injective resolutions with applications to dualizing modules and Gorenstein modules, University of Illinois, 1974.

[3] FOXBY (H.-B.). - On the $\mu^i$ in a minimal injective resolution, Math. Scand., t. 29, 1971, p. 175-186.

[4] HOCHSTER (M.). - Deep local rings, Aarhus Universitets Matematiske Institut, Preprint Series n° 8, 1973/74.

[5] PESKINE (C.) and SZPIRO (L.). - Dimension projective finie et cohomologie locale. - Paris, Presses universitaires de France, 1973 (Institut des Hautes Etudes Scientifiques. Publications mathématiques, 42, p. 47-119).

(Texte reçu le 8 avril 1975)

Robert M. FOSSUM
Dept of Mathematics
University of Illinois
URBANA, Illinois 61801
(Etats-Unis)