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SEMIMETRICS, SEMIÉCARTS IN ORDERED SEMIGROUPS

by Viakalathur S. KRISHNAN (*)

Completing a metric space is a classic construction. Starting with the space of rationals Q (or Q^n) with its metric taking values in the ordered semigroup of positive rationals, its completion R (or R^n) has a metric in the completion, in a suitable sense, of Q^+, which is the ordered semigroup of positive reals. That this result admits natural generalizations is the main contention of this paper.

First, we abstract the properties of the semigroups Q^+ or R^+ in the definition of the "Abelian perfectly ordered semigroup".

Definition. - An Abelian perfectly ordered semigroup (or Apo-semigroup, for short) is a triple $(S , + , \leqslant)$ consisting of a set S, a binary operation $+$ defined on S under which $(S , +)$ is a commutative semigroup with zero element 0, and a relation of partial order $\leqslant$ defined on S such that the following conditions are satisfied :

(a) For arbitrary x , y , z from S, $x \leqslant y$ if, and only if, $(x + z) \leqslant (y + z)$

(b) The set $H = (x \in S ; 0 < x)$ is down-directed and weakly divisible : that is, if x , y are in H, there is a z in H which is $\leqslant x , \leqslant y$; and if x is in H, there is a x' in H such that $x' + x' \leqslant x$.

It is not hard to show that the last condition (b) is true for H if, and only if, it is true of some coinitial subset of $(H , \leqslant)$.

We next define a semimetric or semiécart for a set X into such a Apo-semigroup.

Definition. - Given a set X, and an Apo-semigroup $(S , + , \leqslant)$, a mapping d of $X \times X$ in S is called a semimetric for X in $(S , + , \leqslant)$ (is called a semiécart for X in $(S , + , \leqslant)$), if it satisfies the following condition : for any x , y , z from X, $d(x , z) \leqslant d(x , y) + d(y , z)$ (if it satisfies the following condition : given h in H there is h' in H such that for arbitrary x , y , z from X, $d(x , y) \leqslant h'$ and $d(y , z) \leqslant h'$ imply $d(x , z) \leqslant h$).

In view of our assumption (b) for H, it follows that a semimetric d for X in $(S , + , H)$ is ipso facto a semiécart for X in $(S , + , \leqslant)$.

(*) Conférence prononcée à Nice, en septembre 1970, à la session du Séminaire consacrée aux Demi-groupes.

Since the condition (a) for the ordered semigroup implies that it is cancellative (being also abelian), the semigroup $(S , +)$ can be isomorphically imbedded as a subsemigroup of a group $(G , +)$ of differences ; and we can also now extend the partial order $\leqslant$ from the subsemigroup (of elements of the form $x - 0$) to the whole group, by setting : $(x - x') \leqslant (y - y')$ if, and only if, $(x + y') \leqslant (x' + y)$ in $(S , \leqslant)$. Then, it is seen that $(G , + , \leqslant)$ is also an Apo-(semi-)group. We call it the "group-completion" of the Apo-semigroup.

Given the Apo-semigroup $(S , + , \leqslant)$, the set S has a "intrinsic" semimetric in the Apo-group $(G , + , \leqslant)$ which is the group completion of $(S , + , \leqslant)$; namely d , given by $d(x , x') = x' - x$.

Note also that when d is a semimetric (or semiécart) for X in $(S , + , \leqslant)$, there is a conjugate semimetric d' given by $d'(x , y) = d(y , x)$ for any x , y from X .

We pass on to define the "semiuniform spaces" and their "completions".

Definition. - A family $\mathcal{U} = (U_j ; \; j \in j)$ of binary relations on a set X (indexed by a set J) is called a semiuniformity (or semiuniform structure) for X if the following conditions are true :

(U1) For each x of X and each j of J , $(x , x) \in U_j$, that is all the relations U_j are reflexive.

(U2) Given $j \in J$, there is a $j' \in J$ such that the relational product $U_{j'} \circ U_{j'}$ is contained in U_j .

We may call the family $\mathcal{U}$ a transitive family of relations when (U2) holds, and

(U3) For $j , j' \in J$, there is a $j'' \in J$, such that $U_{j''}$ is contained in both U_j and $U_{j'}$.

The semiuniformity is called a quasiuniformity, if it satisfies also the following "symmetry" condition :

(U4) Given $j \in J$, there is $j'' \in J$, such that the reverse relation $U_{j''} \circ U_{j''}$ is contained in U_j .

And finally, the quasiuniformity is a uniformity (in the sense of A. WEIL), if the intersection of the U_j is the identity relation on X .

A semiuniformity $\mathcal{U}$ for X determines a "conjugate" semiuniformity

$$\mathcal{U}^{-1} = (U_j^{-1} ; \; j \in J)$$

obtained by taking the reverse relations for all the U_j . $\mathcal{U}$ (and its conjugate) also determine a "symmetric" associate semiuniformity (or quasiuniformity)

$$S(\mathcal{U}) = (U_j \cap U_j^{-1} \; ; \; j \in J) \; .$$

A semiuniformity $\mathcal{U}$ for X determines a topology $T(\mathcal{U})$ for X when we take as a base of neighbourhoods at a point x of X the sets $(U_j(x) \; ; \; j \in J)$ where, as usual, $U_j(x)$ consists of the points y of X for which $(x , y) \in U_j$. The topology $T(S(\mathcal{U}))$ determined by the symmetric associate $S(\mathcal{U})$ of $\mathcal{U}$, we shall call the "star topology" determined by $\mathcal{U}$, and denote it by $T^*(\mathcal{U})$.

If now $(D , \leqslant)$ is any down-directed (indexing) set, a function s of D in X is called a $(D , \leqslant)$-sequence in X. Such a sequence is said to converge to a point x of X under a topology T for X if, for each neighbourhood $N(x)$ of x in T, we can find a d in D such that $s(e)$ belongs to $N(x)$ for each e (of D) which is $\leqslant d$. And such a $(D , \leqslant)$-sequence s in X is called a Cauchy sequence of the semiuniform space $(X , \mathcal{U})$ if, for each U_j in $\mathcal{U}$, we can find a d in D such that $(s(e) , s(e')) \in U_j$ whenever e , e' (of D) are $\leqslant d$. Clearly, a Cauchy sequence of $(X , \mathcal{U})$ is also a Cauchy sequence of $(X , S(\mathcal{U}))$, and vice-versa. It can be shown that any $(D , \leqslant)$-sequence of X, which converges to some point of X under $T^*(\mathcal{U})$, is a Cauchy sequence of $(X , \mathcal{U})$. The semi-uniform space $(X , \mathcal{U})$ is called a <u>complete</u> semi-uniform space if every Cauchy sequence of $(X , \mathcal{U})$ converges to some point of X under $T^*(\mathcal{U})$. It follows that $(X , \mathcal{U})$ is complete if, and only if, $(X , S(\mathcal{U}))$ is complete.

We state then the main theorem regarding completing a semiuniform space (which I have proved elsewhere).

THEOREM 1. – <u>Given a semiuniform space</u> $(X , \mathcal{U})$ <u>there is an associated complete semiuniform space</u> $(X^* , \mathcal{U}^*)$, <u>which we call the canonical completion of</u> $(X , \mathcal{U})$, <u>such that : there is a bi-uniform bijection between</u> $(X , \mathcal{U})$ <u>and a semiuniform subspace of</u> $(X^* , \mathcal{U}^*)$; <u>and every point of</u> X^* <u>is a limit of a Cauchy sequence of</u> $(X^* , \mathcal{U}^*)$ <u>consisting of points of this subspace only, the convergence being under the star topology of</u> X^* <u>determined by</u> $\mathcal{U}^*$.

When we consider a set X with a semiécart (or semimetric) d in Apo-semigroup $(S , + , \leqslant)$, we get an associated semiuniformity $\mathcal{U} = (U_h \; ; \; h \in H)$ for X, when we set $((x , y) \in U_h) \Longleftrightarrow (d(x , y) \leqslant h)$. This semiuniformity is symmetric if the semimetric is symmetric. In particular, for a Apo-semigroup the intrinsic semimetric for S in the "group completion" $(G , + , \leqslant)$ gives rise to an intrinsic semiuniformity for $(S , + , \leqslant)$. Then we have the following main results.

THEOREM 2. – <u>The completion of an Apo-semigroup is also an Apo-semigroup ; the completion of an Apo-group is an Apo-group. Upto an order- and semigroup-isomorphism, the semiuniform completion of the group completion of an Apo-semigroup</u>

is the same as the group completion of the semiuniform completion of the Apo-semigroup.

If a set X has a semiécart d in a Apo-semigroup (S , + , $\leqslant$) , then its canonical completion (as a semiuniform space) has its semiuniformity derivable from a semiécart in the canonical completion of the Apo-semigroup. This can also be treated as a semiécart in the group completion of this last complete semigroup.

If X has a semimetric in a totally ordered Apo-semigroup or group, its canonical completion, as a semiuniform space, has its semiuniformity derivable from a semimetric in the canonical completion of the Apo-semigroup or group, which would also be totally ordered.

Details of proofs would be appearing in a paper shortly in the Proceedings of the Czechoslovak Akademy of Sciences [under a report of a Topology Conference, held at Kanpur (India)] .

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