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THE STRUCTURE OF ω-REGULAR SEMIGROUPS

by Janet AULT and Mario PETRICH

1. - Finding the complete structure of regular semigroups of a certain class has succeeded only when sufficiently strong conditions on idempotents and (or) ideals have been imposed. On the one hand, there is the theorem of REES [7], giving the structure of completely 0-simple semigroups, and its successive generalizations to primitive regular semigroups [2], and $\mathfrak{J}$- and $\mathfrak{J}_1$-regular semigroups [4]. On the other hand, with very different restrictions, REILLY [8] has determined the structure of bisimple ω-semigroups and, independently of each other, KOCHIN [1] of inverse simple ω-semigroups, and MUNN [5] of inverse ω-semigroups.

An ω-chain with zero is a poset $\{e_i \mid i \geqslant 0\} \cup 0$, with $e_i > e_j$ if $i < j$, and $0 < e_i$ for all i, j. We call a regular semigroup S ω-regular, if S has a zero, and the poset of its idempotents is an orthogonal sum [2] of ω-chains with zero. We announce here the complete determination of the structure of such semigroups, including various special cases thereof, and briefly mention their isomorphisms.

2. - An ω-regular semigroup can be uniquely written as an orthogonal sum of ω-regular prime (i. e., with 0 a prime ideal) semigroups. This reduces the problems of structure and isomorphism to ω-regular prime semigroups. We distinguish three cases :

(i) 0-simple,

(ii) Prime with a proper 0-minimal ideal,

(iii) Prime without a 0-minimal ideal.

Case (i) is the most difficult (and interesting), and includes a variety of special cases some of which reduce to those constructed by REILLY [8], KOCHIN [1], and MUNN [5], [6].

3. - Let A be a nonempty set, d be a positive integer, V be a semigroup which is a chain of d groups $G_0 > G_1 > \dots > G_{d-1}$, and σ be a homomorphism of V into G_0. Let $w : A \to \{0 , 1 , \dots , d-1\}$ be any function, denoted by $w : \alpha \to w_\alpha$. For $\alpha \in A$, $0 \leqslant i , j < d$, define $\langle \alpha , i \rangle$ by

$$\langle \alpha , i \rangle \equiv w_\alpha + i \pmod d , \qquad 0 \leqslant \langle \alpha , i \rangle < d ,$$

and define $[i , \alpha , j]$ to satisfy

$$[i , \alpha , j] d = (i - j) - (\langle \alpha , i \rangle - \langle \alpha , j \rangle) .$$

Construction 1. – On the set

$$S = \{(\alpha , m , g , n , \beta) \mid \alpha , \beta \in A , \; m , n \geqslant 0 , \; g \in V\} \cup 0 ,$$

define a multiplication by, for $g_i \in G_i$, $g_j \in G_j$, $v = n - s - [i , \beta , j]$,

$$(\alpha , m , g_i , n , \beta)(\beta , s , g_j , t , \gamma) = \begin{cases} (\alpha , m - [i , \alpha , j] - v , (g_i \sigma^{-v}) g_j , t , \gamma) , & \text{if } v < 0 , \\ & \text{or } v = 0 , \; i \leqslant j , \\ (\alpha , m , g_i (g_j \sigma^{v}) , t + [i , \gamma , j] + v , \gamma) , & \text{if } v > 0 , \\ & \text{or } v = 0 , \; i > j , \end{cases}$$

and all other products are equal to 0 . The set S , with this multiplication, will be denoted by $\Theta(A , w ; V , \sigma)$.

Construction 2. – On the set

$$S' = \{(\alpha , m , g , n , \beta) \mid \alpha , \beta \in A , \; m - w_\alpha \equiv n - w_\beta \equiv i \pmod d) , \; g \in G_i\} \cup 0 ,$$

define a multiplication by, for $g_i \in G_i$, $g_j \in G_j$, $v = n' - s' - [i , \beta , j]$, where $n = n'd + n''$, $s = s'd + s''$, $0 \leqslant n''$, $s'' < d$,

$$(\alpha , m , g_i , n , \beta)(\beta , s , g_j , t , \gamma) = \begin{cases} (\alpha , m + s - n , (g_i \sigma^{-v}) g_j , t , \gamma) , & \text{if } n \leqslant s , \\ (\alpha , m , g_i (g_j \sigma^{v}) , t + n - s , \gamma) , & \text{if } n > s , \end{cases}$$

and all other products are equal to 0 . The set S' , with this multiplication, will be denoted by $\Theta[A , w ; V , \sigma]$.

The following is our fundamental result.

THEOREM 1. – For a groupoid S , the following statements are equivalent :

(i) S is a 0-simple ω-regular semigroup ;
(ii) S is isomorphic to $\Theta(A , w ; V , \sigma)$;
(iii) S is isomorphic to $\Theta[A , w ; V , \sigma]$.

The proof of "(i) $\Longrightarrow$ (ii)" consists of "introducing coordinates" into various $\mathcal{L}$- and $\mathcal{R}$-classes, and of constructing the homomorphism σ ; it is quite long, and is broken into a sequence of lemmas. For "(ii) $\Longrightarrow$ (iii)", one finds a suitable isomorphism, while "(iii) $\Longrightarrow$ (i)" consists of a verification of the defining properties of a 0-simple ω-regular semigroup.

Define the *top* of S in the theorem by $\mathfrak{J}(S) = \{a \in S \mid e \,\mathcal{L}\, a\,,\ a \,\mathcal{R}\, f$ for some maximal idempotents $e\,,\ f\} \cup 0$. Then $\mathfrak{J}(S)$ is a primitive inverse semigroup. It follows from the proof that we can always suppose that $w_\alpha = 0$ for some $\alpha \in A$. Call S *balanced*, if any two maximal idempotents of S are $\mathcal{D}$-equivalent.

THEOREM 2. – *The following conditions on a* 0-*simple* ω-*regular semigroup* S *are equivalent* :

(i) S *is balanced* ;

(ii) S *admits a representation as in theorem* 1, *with* $w_\alpha = 0$ *for all* $\alpha \in A$;

(iii) $\mathfrak{J}(S)$ *is a Brandt semigroup* ;

(iv) S *is isomorphic to a Rees matrix semigroup* $\mathfrak{M}^0(K\,;\,A\,,\,A\,;\,\Delta)$ *over a simple inverse* ω-*semigroup* K , Δ *is the identity matrix.*

The structure of the semigroup K in theorem 2 was determined by KOCHIN [1] and MUNN [5], the Rees matrix semigroups over bisimple inverse semigroups were studied in [3] (for the 0-simple case in the theorem, cf. [3], cor. 5.7, and [6], th. 4.2). Various other special cases include : 0-bisimple, combinatorial, balanced, and combinations thereof.

4. – For the remaining cases, we will need the following.

Construction 3. – Let Y be a tree semilattice satisfying one of the two conditions :

(1) Y has a zero ζ , and all elements of Y are of finite height ;

(2) Y has no zero, and is of locally finite length.

To every non-zero element α of Y , associate a Brandt semigroup S_α , suppose that the family $\{S_\alpha\}$ is pairwise disjoint, and that a homomorphism $\varphi_\alpha : S_\alpha \to S_{\bar\alpha}$ is given, where $\bar\alpha$ is the unique element of Y covered by α , with the properties :

(i) $S_\alpha \varphi_\alpha \cap S_\beta \varphi_\beta = 0$, if $\bar\alpha = \bar\beta$;

(ii) For every infinite ascending chain $\alpha_1 < \alpha_2 < \dots$ in Y , and every $a \in S_{\alpha_1}$, there exists α_k such that $a \notin S_{\alpha_k} \varphi_{\alpha_k} \varphi_{\alpha_{k-1}} \cdots \varphi_{\alpha_2}$.

Let $\psi_{\alpha,\alpha}$ be the identity mapping on S_α , and for $\alpha > \beta$, let

$$\psi_{\alpha,\beta} = \varphi_\alpha \varphi_{\alpha_1} \cdots \varphi_{\alpha_n} , \qquad \text{where } \alpha > \alpha_1 > \dots > \alpha_n > \beta .$$

Let

$$S = \Big(\bigcup_{\alpha \in Y \setminus \zeta} (S_\alpha \setminus 0_\alpha) \Big) \cup 0 \quad ,$$

where ζ is the zero of Y (if Y has one), and 0 is an element not contained in any S_α ; and on S define the multiplication $\star$ by

$$a \star b = (a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta}) \ , \qquad \text{if } \alpha\beta \neq \zeta \quad \text{and} \quad (a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta}) \neq 0_{\alpha\beta} \ \text{ in } S_{\alpha\beta} \ ,$$

and all other products are equal to 0 . The set S , with this multiplication, will be called a <u>Brandt tree</u>, if Y has a zero and a <u>rooted Brandt tree</u> otherwise.

THEOREM 3. - <u>A semigroup S is prime ω-regular and has a proper 0-minimal ideal if, and only if, S is an ideal extension of a 0-simple ω-regular semigroup I by a Brandt tree T determined by a 0-restricted homomorphism of T into the top of I</u> .

Such a homomorphism is completely determined by its restriction to the socle $\mathfrak{S}(T)$ of T , so all such homomorphisms are given by 0-restricted homomorphisms of $\mathfrak{S}(T)$ into $\mathfrak{J}(I)$, both of which are primitive inverse semigroups, and are easy to find explicitly.

THEOREM 4. - <u>A groupoid S is a prime ω-regular semigroup without 0-minimal ideals if, and only if, S is a rooted Brandt tree.</u>

5. - The semigroups $\mathcal{O}(A , w ; V , \sigma)$ and $\mathcal{O}[A , w ; V , \sigma]$ do not seem to admit a neat isomorphism theorem, except in special cases. In the balanced case, using theorem 2, ([3], 4.1) and ([1], theor.4), we derive a satisfactory isomorphism theorem. A direct proof does the same in the case these semigroups are combinatorial. Isomorphisms of the semigroups in construction 3 are similar to those in [4], théorème 3.1, while isomorphisms of the semigroups in theorem 3 can be expressed by isomorphisms of I and T satisfying a commutative diagram.

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