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Séminaire Dubreil. Algèbre et théorie des nombres, tome 23, nº 2 (1969-1970), exp. nº DG 5, p. DG1-DG19
<http://www.numdam.org/item?id=SD_1969-1970__23_2_A5_0>

Séminaire DUBREIL-PISOT
(Algèbre et Théorie des nombres)
23e année, 1969/70
Demi-groupes, n° 5, 19 p.

COMMUTATIVE ARCHIMEDEAN CANCELLATIVE SEMIGROUPS WITHOUT IDEMPOTENT

by Takayuki TAMURA

Part 1 : Survey of $\mathfrak{N}$-semigroups.

0. Introduction.

A commutative semigroup S is called archimedean, if, for every $a , b \in S$, there are positive integers m , n and elements $c , d \in S$ such that $a^m = bc$ and $b^n = ad$. Every commutative semigroup is the disjoint union of commutative archimedean semigroups. By an $\mathfrak{N}$-semigroup, we mean a commutative cancellative archimedean semigroup without idempotent. All commutative archimedean semigroups are classified into (cf. [19], [27], [29]) :

(0.1) Nil-semigroups (i. e. some power of every element is 0),

(0.2) Abelian groups,

(0.3) Ideal extensions of a non-trivial abelian group G by a non-trivial semigroup of type (0.1),

(0.4) $\mathfrak{N}$-semigroups,

(0.5) Archimedean, non-cancellative semigroups without idempotent.

$\mathfrak{N}$-semigroups were studied for the first time by the author ([28]) in 1957, but the terminology is due to PETRICH [23].

1. Basic representation.

Let P be the set of all positive integers, and P^0 be the set of all non-negative integers throughout this paper.

(1.0) THEOREM ([6], [28], [31]). - Let G be an abelian group. Let I be a function $G \times G \to P^0$ which satisfies the following conditions :

(1.1) $I(\alpha , \beta) = I(\beta , \alpha)$, for all $\alpha , \beta \in G$;

(1.2) $I(\alpha , \beta) + I(\alpha\beta , \gamma) = I(\alpha , \beta\gamma) + I(\beta , \gamma)$, for all $\alpha , \beta , \gamma \in G$;

(1.3) $I(\varepsilon , \alpha) = 1$, for all $\alpha \in G$, ε being the identity element of G ;

(1.4) For every $\alpha \in G$, there is a positive integer m such that $I(\alpha^m , \alpha) > 0$.

We define an operation on the set $S = P^0 \times G = \{(m , \alpha) : m \in P^0 , \alpha \in G\}$ by $(m , \alpha)(n , \beta) = (m + n + I(\alpha , \beta) , \alpha\beta)$. Then, S is an $\mathfrak{N}$-semigroup. S is denoted by $S = (G ; I)$. Every $\mathfrak{N}$-semigroup can be obtained in this manner. The function $I(\alpha , \beta)$ is called an ϑ-function on G .

Let S be an $\mathfrak{N}$-semigroup, and let $a \in S$. Define a relation τ_a by :

(1.5) $x\tau_a y$ if, and only if, $a^m x = a^n y$ for some $m , n \in P$.

Then, τ_a is a congruence on S , and S/τ_a is a group (see [28], [30]). Each τ_a-class S_λ contains exactly one prime element p_λ with respect to a , i. e. an element p_λ which cannot be divisible by a . Let $G_a = S/\tau_a$. A function $I : G \times G \rightarrow P^0$ is defined by

$$p_\alpha \, p_\beta = a^{I(\alpha,\beta)} \, p_{\alpha\beta} \ .$$

Then, S is isomorphic to $(G_a ; I)$. G_a is called the structure group of S with respect to a ; I is the ϑ-function (of S) with respect to a . The element a is called the standard element of the representation $(G_a ; I)$ for S . Since S is cancellative, S can be embedded into a group.

Let Q be the quotient group of S , and $a \in S$. The structure group G_a is nothing but Q/A , where $A = [a]$ is the cyclic subgroup of Q generated by a . Let

$$Q = \bigcup_{\xi \in G_a} A_\xi$$

be the decomposition modulo A . Then it follows that each coset A_ξ intersects S . Let $\{x_\xi : \xi \in G_a\}$ be a complete representative system of Q modulo A . There is an integer $\delta(\xi)$ such that, for each ξ ,

$$x_\xi \, a^m \in S , \quad \text{if} \quad \delta(\xi) \leq m ; \qquad \text{but} \quad x_\xi \, a^m \notin S , \quad \text{if} \quad \delta(\xi) > m \ .$$

Let $p_\xi = a^{\delta(\xi)} x_\xi$. The element p_ξ is a prime element with respect to the element a for each $\xi \in G_a$ (see [31]).

Remark. - If G is periodic, then (1.4) can be removed, since it is automatically satisfied.

2. ϑ-functions and structure groups.

Given arbitrary abelian group G , there is at least one ϑ-function $I : G \times G \rightarrow P^0$ satisfying (1.1) through (1.4), for example

$$I(\alpha , \beta) = 1 , \qquad \text{for all} \quad \alpha , \beta \in G \ .$$

(2.1) If $I_0(\alpha, \beta) = 1$ for all $\alpha, \beta \in G$, then $(G ; I_0)$ is isomorphic to the direct product of the positive integer additive semigroup P and the group G.

(2.2) How can we determine all ϑ-functions for a given G ? The explicit form of ϑ-functions is obtained in case where G is cyclic group ; and we can describe the ϑ-functions I on the direct product of A and B, in terms of the ϑ-functions I_A on A, and I_B on B (cf. [2]). However, a general case is not known.

(2.3) The structure group and the ϑ-function depend on the choice of a standard element, hence the study of the relation between two ϑ-functions is requested. Suppose two pairs $(G ; I)$, $(G' ; I')$ satisfying (1.1) through (1.4) are given. Under what condition on G, G', I, I', do they determine isomorphic $\mathfrak{R}$-semigroups ? In other words, does there exist an $\mathfrak{R}$-semigroup S such that $G = G_a$ and $G' = G_b$ for some $a, b \in S$? SASAKI studied this problem, and gave a necessary and sufficient condition ([24], [25]), but it seems to be complicated.

3. Generalized representation.

This section is due to DICKENSON [8], SASAKI [26]. Let G be an abelian group, and P^0 the set of all non-negative integers. Let a function $I' : G \times G \to P^0$ which satisfies (1.1), (1.2) and (1.4). Such a function I' is called a generalized ϑ-function, or simply, ϑ'-function. The two conditions (1.1) and (1.2) for ϑ'-function imply $I'(\varepsilon, \alpha) = I'(\varepsilon, \varepsilon)$ for all $\alpha \in G$.

(3.1) Let $I' : G \times G \to P^0$ be an ϑ'-function. We define an operation in the set $S = P^0 \times G$ as in theorem (1.0).

(3.1.1)
$$(m, \alpha)(n, \beta) = (m + n + I'(\alpha, \beta), \alpha\beta) .$$

Then, S is an $\mathfrak{N}$-semigroup. Every $\mathfrak{N}$-semigroup is obtained in this manner. (The second assertion is obvious.)

(3.2) Let I' be a function $G \times G \to P^0$, and let $S = G \times P^0$. If S is an $\mathfrak{N}$-semigroup with respect to the operation (3.1.1), then the function I' satisfies (1.1), (1.2), (1.4).

We call $S = (G ; I')$ a generalized representation of S, while the representation by theorem (1.0) will be called Tamura representation of S.

(3.3) Let $S = (G ; I')$ be a generalized representation of an $\mathfrak{N}$-semigroup S. Let

$$k = I'(\varepsilon, \varepsilon) ,$$

and let

$$H = \{(m, \alpha) \in S ; \ 0 \leqslant m \leqslant k - 1\} .$$

Let (m, α), $(n, \beta) \in H$. There are non-negative integers u, v, $0 \leqslant v \leqslant k - 1$, such that $m + n + I'(\alpha , \beta) = uk + v$. Define a binary operation in H by

$$(m , \alpha)(n , \beta) = (v , \alpha\beta) \quad .$$

Define I on $H \times H$ by

$$I((m , \alpha) , (n , \beta)) = u \quad .$$

Let

$$C_k = \{(m , \varepsilon) ; \quad 0 \leq m \leq k - 1\} \quad .$$

H is an extension of C_k , cyclic group, by the group G . Then, $(H ; I)$ is a Tamura representation with standard element $(0 , \varepsilon)$.

Generalized representations occur when working with ideal of an $\mathfrak{N}$-semigroup and treating right zero composition of $\mathfrak{N}$-semigroups (cf. DICKINSON [8]).

4. Ideals of $\mathfrak{N}$-semigroups.

Ideals of an $\mathfrak{N}$-semigroup are also $\mathfrak{N}$-semigroups. This section is due to [8].

(4.1) Let $S = (G ; I')$ be an $\mathfrak{N}$-semigroup with generalized function. Let J be an ideal of S . Define $\varphi : G \rightarrow P^0$ by

$$\varphi(\xi) = \min\{n ; \quad (n , \xi) \in J\} \quad .$$

Then

$$\text{(4.1.1)} \qquad \varphi(\xi) + I'(\xi , \eta) \geq \varphi(\xi\eta) , \qquad \text{for all } \xi , \eta \in G \quad .$$

A non-negative valued function on G is called an ideal function relative to $(G ; I')$, if it satisfies (4.1.1). Let $\mathcal{K}$ be the set of all ideals of S , and Φ be the set of all ideal functions relative to $(G ; I')$. Then, the map $f : \mathcal{K} \rightarrow \Phi$ defined by $f(J) = \varphi$, $J \in \mathcal{K}$, is surjective. In particular, if $(G ; I')$ is a Tamura representation, then f is bijective.

(4.2) Let S be an $\mathfrak{N}$-semigroup with Tamura representation $(G ; I)$. Let J be an ideal of S . Then J , an $\mathfrak{N}$-semigroup, has generalized representation $(G ; I_J')$, where I_J' is defined by

$$I_J'(\xi , \eta) = I(\xi , \eta) + \varphi(\xi) + \varphi(\eta) - \varphi(\xi\eta) \quad ,$$

φ being the ideal function associated with J relative to $(G ; I)$. I_J' is an ϑ-function if, and only if, $(0 , \varepsilon) \in J$. Let T be the product set $P^0 \times G$. We define

$$(m , \xi)(n , \eta) = (m + n + I_J'(\xi , \eta) , \xi\eta) \quad .$$

Then, J is isomorphic onto T by the map $(n , \xi) \rightarrow (n - \varphi(\xi) , \xi)$.

5. Power joinedness and structure groups.

(5.1) (CHRISLOCK [3], [4] ; HIGGINS [14], [15]). An $\mathfrak{N}$-semigroup S is finitely generated, if, and only if, G_a is finite for all $a \in S$, equivalently for some $a \in S$.

S is power joined, if, for a, $b \in S$, there are positive integers m and n such that $a^m = b^n$.

(5.2) ([3], [4]). An $\mathfrak{N}$-semigroup S is power joined, if, and only if, G_a is periodic for all $a \in S$, equivalently for some $a \in S$.

Also (5.1) and (5.2) are partially due to PETRICH [23].

(5.3) ([14], [15]). S is a finitely generated $\mathfrak{N}$-semigroup, if, and only if, S is isomorphic to a subdirect product of a positive integer additive semigroup and an abelian group.

(5.4) ([32]). S is a power joined $\mathfrak{N}$-semigroup, if, and only if, S is isomorphic to a subdirect product of a positive rational additive semigroup and an abelian group.

Remark. - (5.1) and (5.2) hold in the case without cancellation. In such case, τ_a, defined in paragraph 1, is still a group congruence, and $G_a = S/\tau_a$ is considered as the structure group with respect to a (see LEVIN [20], [21], [22]).

6. Representation in power joined case.

(6.1) ([32]). Let S be a power joined $\mathfrak{N}$-semigroup, $S = (G ; I)$, where G is a periodic abelian group. We define a function $\bar{\varphi} : G \to R_+$ the set of all positive rational numbers ;

(6.1.1) $$\bar{\varphi}(\alpha) = \left(\sum_{i=1}^{s} I(\alpha , \alpha^i)\right) / s ,$$

where s is the order of the element α of G. Then $\bar{\varphi}$ satisfies the following properties :

(6.1.2) $\bar{\varphi}(\varepsilon) = 1$, ε being the identity element of G ;
(6.1.3) $\bar{\varphi}(\alpha) + \bar{\varphi}(\beta) - \bar{\varphi}(\alpha\beta)$ is a non-negative integer ;
(6.1.4) $I(\alpha , \beta) = \bar{\varphi}(\alpha) + \bar{\varphi}(\beta) - \bar{\varphi}(\alpha\beta)$.

Conversely, let $\bar{\varphi}$ be a function, $G \to R_+$, which satisfies (6.1.2) and (6.1.3). If we define I by (6.1.4), then I satisfies (1.1) through (1.4), and (6.1.1).

Thus a power joined $\mathfrak{N}$-semigroup is determined by G, and $\bar{\varphi} : G \to R_+$ with

(6.1.2), (6.1.3). Notice that $\bar{\varphi}$ induces a homomorphism of G onto the additive group of the rational numbers modulo integers.

(6.2) ([14], [15]). If an $\mathfrak{N}$-semigroup is finitely generated, then G is finite, and

$$|G|.\bar{\varphi}(\alpha) = \sum_{\xi \in G} I(\alpha , \xi) \quad .$$

Notice that $|G|.\bar{\varphi}(\alpha)$ is the number of prime elements with respect to $(0 , \alpha) \in S = (G ; I)$.

7. Positive rational semigroups.

This section is due to [32].

(7.1) If S is a commutative power joined semigroup without idempotent, then S is homomorphic into the semigroup R_+ of all positive rational numbers with addition. The homomorphism is unique in the following sense. Let φ and φ_0 be homomorphisms of S into R_+ ; then $\varphi(x) = r.\varphi_0(x)$, where r is a positive rational number, and $r.\varphi_0(x)$ is the usual multiplication of r and $\varphi_0(x)$. Hence, $\varphi(S) \simeq \varphi_0(S)$. In particular, if S and S' are positive rational semigroups with addition, and if φ is a homomorphism of S onto S' , then φ is an isomorphism, and $\varphi(x) = r.x$.

A semigroup S is called power cancellative, if S satisfies

$$a^n = b^n \implies a = b , \qquad n = 2 , 3 , \ldots \quad .$$

(7.2) Let S be a non-trivial commutative semigroup. Then S is power joined and power cancellative, if, and only if, S is isomorphic to a positive rational semigroup. S is a finitely generated, power joined, power cancellative semigroup, if, and only if, it is isomorphic to a positive integer semigroup.

(7.3) A semigroup S is isomorphic to a positive rational semigroup, if, and only if, S is isomorphic to a direct limit of finitely generated commutative, power joined, power cancellative semigroups.

8. $\mathfrak{N}$-semigroups and quotient groups ([31]).

Recall the theory of abelian group extensions (see, for example, [9]). By a factor system f of an abelian group A into an abelian group $B(+)$, we mean a function $A \times A \to B$ such that

$$f(\alpha , \beta) = f(\beta , \alpha) , \qquad \text{for all } \alpha , \beta \in A ,$$

$$f(\alpha , \varepsilon) = 0 , \qquad \text{for all } \alpha \in A , \quad \varepsilon \text{ being the identity element} ,$$

$$f(\alpha , \beta) + f(\alpha\beta , \gamma) = f(\alpha , \beta\gamma) + f(\beta , \gamma) , \qquad \text{for all } \alpha , \beta , \gamma \in A .$$

A factor system f determines the abelian group extension $((B , A ; f))$ of B by A with respect to f, that is,

$$((B , A ; f)) = \{((x , \alpha)) ; \ x \in B , \ \alpha \in A\} ,$$

and the operation is defined by

$$((x , \alpha))((y , \beta)) = ((x + y + f(\alpha , \beta) , \alpha\beta)) .$$

A factor system $g(\alpha , \beta)$ is said to be equivalent to $f(\alpha , \beta)$, if there is a map $\varphi : B \to A$ such that $g(\alpha , \beta) = f(\alpha , \beta) + \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta)$.

We are interested in the case where B is the group Z of all integers with addition. The results are seen in [29].

(8.1) Let S be an $\mathfrak{N}$-semigroup, $S = (G ; I)$. The quotient group Q of S is the abelian extension $((Z , G ; f))$ with respect to f defined by

$$f(\alpha , \beta) = I(\alpha , \beta) - 1 .$$

S can be embedded into Q by the map

$$(x , \alpha) \to ((x + 1 , \alpha)) .$$

The identity element of Q is $((0 , \varepsilon))$, and the inverse element of $((x , \alpha))$ is

$$((x , \alpha))^{-1} = ((- x - f(\alpha , \alpha^{-1}) , \alpha^{-1})) .$$

(8.2) Let Q be an abelian group which is not torsion, S be an $\mathfrak{N}$-subsemigroup of Q, and A be the infinite cyclic subgroup of Q generated by a of S. The following properties are equivalent :

(8.2.1) Q is the quotient group of S ;

(8.2.2) $Q = A.S$;

(8.2.3) S intersects each congruence class of Q modulo A .

(8.3) Let Q be a non-torsion abelian group, and let a be an element of Q which is of infinite order. There exists a maximal $\mathfrak{N}$-subsemigroup S containing a, such that Q is the quotient group of S .

(8.4) Let K be an abelian group, and A be the group of all integers under addition. If $G = ((A , K ; f))$, then there exists a factor system $g(\alpha , \beta)$ such that :

(8.4.1) $g(\alpha , \beta) \geqslant 0$;
(8.4.2) $g(\alpha , \beta)$ is equivalent to $f(\alpha , \beta)$.

This is obtained as the application of (8.3) to the abelian group theory.

(8.5) We can describe $\mathfrak{N}$-subsemigroups of a group Q , when Q is given. Let Q be a non-torsion abelian group, hence $Q = ((A , K ; f))$. Let δ be a map, $K \to A$, satisfying :

(8.5.1) $\delta(\varepsilon) = 1$, ε being the identity element of K ;
(8.5.2) $f(\alpha , \beta) + \delta(\alpha) + \delta(\beta) - \delta(\alpha\beta) \geqslant 0$, for all $\alpha , \beta \in K$;
(8.5.3) For every $\alpha \in K$, there is a positive integer m such that

$$f(\alpha , \alpha^m) + \delta(\alpha) + \delta(\alpha^m) - \delta(\alpha^{m+1}) > 0 \quad .$$

If S is defined by $S = \{((x , \alpha)) ; \; x \geqslant \delta(\alpha) , \; \alpha \in K\}$ as a subsemigroup of Q , then S is an $\mathfrak{N}$-semigroup. Every $\mathfrak{N}$-subsemigroup containing $((1 , \varepsilon))$, whose quotient group is Q , can be obtained in this manner. Notice that δ is obtained in paragraph 1.

(8.6) An $\mathfrak{N}$-semigroup S is power cancellative, if, and only if, Q is torsion free.

9. Translations of $\mathfrak{N}$-semigroups.

This section is due to HALL [13] and DICKINSON [8].

(9.1) Let $S = (G ; I)$ with Tamura representation. Let (m , α) be an element of S such that

$$m + I(\alpha , \xi) - 1 \geqslant 0 , \qquad \text{for all } \xi \in G \quad .$$

Define $f : S \to S$ by

$$f((x , \eta)) = (m + x + I(\alpha , \eta) - 1 , \alpha\eta) \quad .$$

Then, f is a translation of S . Note that $f((0 , \varepsilon)) = (m , \alpha)$. All translations of S can be obtained in this manner.

(9.2) Thus f is associated with (m , α) . Let P^0 be the set of all non-negative integers. Let

$$\tau = \{((m , \alpha)) ; \; m \in P^0 , \; \alpha \in G \text{ such that } m + I(\alpha , \xi) - 1 \geqslant 0 \text{ for all } \xi \in G\} ,$$

and define operation by

(9.2.1) $$((m , \alpha))((n , \beta)) = ((m + n + I(\alpha , \beta) - 1 , \alpha\beta)) \quad .$$

Then, the translation semigroup $T(S)$ of S is isomorphic to J by the map $\varkappa : f \to ((m , \alpha))$ such that $f((0 , \varepsilon)) = ((m , \alpha))$.

Let $T_{in}(S)$ be the inner translation semigroup of S. Then

$$\varkappa(T_{in}(S)) = \{((m+1, \alpha)) ; \ (m, \alpha) \in S\} .$$

For convenience, τ will be denoted by $T(S)$ identifying f with $\varkappa(f)$. Hence

$$T_{in}(S) = \{((n, \alpha)) ; \ n > 0 , \ \alpha \in G\} .$$

(9.3) If A is a commutative cancellative semigroup, then every translation of A is injective, and the translation semigroup $T(A)$ is commutative, cancellative. Furthermore, $T_{in}(A)$ is isomorphic with A in the natural way.

(9.4) [13]. Let $S = (G ; I)$. Then, $T_{in}(S)$ is contained in a single archimedean component of $T(S)$. Let $T_0(S)$ denote the archimedean component of $T(S)$ containing $T_{in}(S)$. Then

$$T_0(S) = T_{in}(S) \cup \{((0, \alpha)) ; \ I(\alpha, \xi) > 0 \ \text{for all} \ \xi \in G , \ \text{and} \ I(\alpha^m, \alpha) > 1 \ \text{for some} \ m \in P\} .$$

(9.5) ([8], [13]). Let $T_g(S)$ denote the unique maximal subgroup of $T(S)$. Then

$$\begin{aligned} T_g(S) &= \{((0, \alpha)) ; \ I(\alpha, \xi) = 1 \ \text{for all} \ \xi \in G\} \\ &= \{((0, \alpha)) ; \ I(\alpha, \xi) > 0 , \ I(\alpha^{-1}, \xi) > 0 \ \text{for all} \ \xi \in G , \ \text{and} \ I(\alpha, \alpha^{-1}) = 1\} . \end{aligned}$$

$T_g(S)$ is isomorphic with a subgroup of G. We have one of invariant properties of the structure groups : Every structure group of S contains a subgroup which is isomorphic to $T_g(S)$.

(9.6) [13]. $$T_g(S) = \{((0, \alpha)) ; \ I(\alpha, \xi) > 0 \ \text{for all} \ \xi \in G , \ I(\alpha^m, \alpha) = 1 \ \text{for all} \ m \in P\} ,$$

if G is periodic.

(9.7) Observation of translations from quotient group [8].

Let S be a commutative cancellative semigroup, and Q be the quotient group of S. We assume $S \subset Q$ disregarding the inclusion map $j : S \to Q$. If $f \in T(S)$, then f can be extended a translation f^* of Q as follows : for $x \in Q$, $x = ab^{-1}$,

$$f^*(x) = f(a) \ f(b)^{-1} .$$

(9.8) Let $i : S \to T(S)$ be defined by $i(a) = f_a$, where $f_a(x) = ax$. Let $h : T(S) \to T(Q)$ be given by $h(f) = f^*$. Let $k : T(Q) \to Q$ be given by $k(f^*) = f^*(e)$, where e is the identity element of Q. Let $j : S \to Q$. Then,

$j(a) = k(h(i(a)))$ for $a \in S$. Each map is injective. k is bijective.

(9.9) $j(S) \subset kh(T(S)) \subset Q$, and $kh(T(S))$ is the idealizer of $j(S)$ in Q, that is,

$$kh(T(S)) = \{q \in Q \ ; \ qx \in j(S) \text{ for all } x \in j(S)\} .$$

Therefore $((m , \alpha))$, $((x , \beta))$ in (9.2) are regarded as elements of Q. Also, $T_{in}(S) = h^{-1} k^{-1}(j(S))$.

Part 2 : Cancellative congruences on $\mathfrak{N}$-semigroups.

10. Congruences.

Since commutativity and archimedeaness are preserved by homomorphisms, every homomorphic image of an $\mathfrak{N}$-semigroup has one of types (0.1) through (0.5). At the present time, we are most interested in the study of the homomorphisms of type (0.2) and (0.4), but we will not deal with the remaining cases. As seen in HALL [13], the homomorphisms of type (0.4) play an important part in the construction of commutative cancellative or separative semigroups. On the other hand, we know periodic $\mathfrak{N}$-semigroups are homomorphic to semigroups of type (0.2) and (0.4) in their subdirect decompositions (§ 5). What can we say about a similar conjecture for a general case ? These motive the investigation of homomorphisms, congruences of the above type. Most of this part will be published in [33].

Let S be an $\mathfrak{N}$-semigroup, and ρ be a congruence on S. If S/ρ is a group, ρ is called a group-congruence on S ; if S/ρ is an $\mathfrak{N}$-semigroup, ρ is called an $\mathfrak{N}$-congruence on S ; if S/ρ is cancellative, ρ is called a cancellative congruence. If ρ is not the universal relation, $S \times S$, then ρ is a cancellative congruence, if, and only if, it is either a group-congruence or an $\mathfrak{N}$-congruence.

11. Group-congruences.

Let S be an $\mathfrak{N}$-semigroup. All structure groups are the group-homomorphic images of S, and hence every τ_a defined by (1.5) is a group-congruence on S.

(11.1) If τ is a group-congruence on S, then $\tau_a \subseteq \tau$ for some $a \in S$.

As well known, DUBREIL [9] established the theory of group-congruences on arbitrary semigroups, but we can directly obtain it in our special case.

(11.2) Définition. - Let S be a commutative semigroup. A subsemigroup C of S is called cofinal in S, if, for every $x \in S$, there is an element $y \in S$ such that $xy \in C$. A subsemigroup A of S is called unitary in S, if $x \in S$, $a \in A$, and $ax \in A$ implies $x \in A$.

(11.3) Let S be a commutative semigroup, and A a cofinal subsemigroup of S. Define a relation τ_a on S by $s\tau_A y$, if, and only if, $ax = by$ for some $a, b \in A$. Then, τ_A is a group-congruence on S, and $A \subseteq \mathrm{Ker}\ \tau_A$, where $\mathrm{Ker}\ \tau_A$ denotes the kernel of the homomorphism $S \to S/\tau_A$. The kernel U of the homomorphism $S \to S/\tau_A$ is equal to A, if, and only if, A is unitary.

Thus the map $A \to \tau_A$ from the join semilattice of all unitary subsemigroups of S onto the join semilattice of all group-congruences on S is a join semilattice isomorphism.

The argument becomes simpler in the case of archimedeaness.

(11.4) Every subsemigroup of a commutative archimedean semigroup is cofinal. If every subsemigroup of a commutative semigroup S is cofinal, then S is archimedean.

Now let us return to $\mathfrak{N}$-semigroups.

(11.5) Let A be a non-void subsemigroup of an $\mathfrak{N}$-semigroup S. Let Q be the quotient group of S. Then the following are equivalent :

(11.5.1) A is unitary in S ;
(11.5.2) $A = S \cap K$, for some subgroup K of Q ;
(11.5.3) τ_A is a group-congruence on S ;
(11.5.4) τ_A is a congruence on S such that S/τ_A has identity element.

K, described in (11.5.2), is called a group-kernel of S. Clearly, the cyclic subsemigroup $[a]$ generated by an element a is unitary in S.

12. $\mathfrak{N}$-congruences, cancellative congruences, and quotient group.

$\mathfrak{N}$-congruences on an $\mathfrak{N}$-semigroup S can be treated from the point of the quotient group of S, analogously to (11.5.2).

(12.1) Let S and S' be $\mathfrak{N}$-semigroups, and suppose $h : S \to S'$ be a homomorphism of S onto S'. Let Q and Q' be the quotient groups of S and S' respectively. Then, the homomorphism h can be extended to a homomorphism $\bar{h} : Q \to Q'$ of Q onto Q', that is, if $z \in S$, $\bar{h}(z) = h(z)$. Accordingly, an $\mathfrak{N}$-congruence ρ on S induces a congruence $\bar{\rho}$ on Q.

A subgroup K of Q is called an $\mathfrak{N}$-kernel of S, if the congruence $\bar{\rho}$ on Q determined by K induces an $\mathfrak{N}$-congruence ρ on S, that is, $\rho = \bar{\rho}|S$.

(12.2) A subgroup K of Q is an $\mathfrak{N}$-kernel of S, if, and only if, $S \cap K = \emptyset$. There is one to one correspondence between all $\mathfrak{N}$-congruences ρ on S and all $\mathfrak{N}$-kernels K of S. For example, the torsion subgroup of Q is an $\mathfrak{N}$-kernel of S.

(12.3) Let $\mathfrak{L}$ be the set of all cancellative congruences on S, $\mathfrak{L}_g$ the set of all group-congruences on S, and $\mathfrak{L}_n$ the set of all $\mathfrak{N}$-congruences on S. Then, $\mathfrak{L} = \mathfrak{L}_g \cup \mathfrak{L}_n$ is disjoint union. $\mathfrak{L}$ is the lattice which is isomorphic to the lattice of congruences on Q, equivalently the lattice of subgroups of Q. $\mathfrak{L}_g$ is a join subsemilattice of $\mathfrak{L}$; $\mathfrak{L}_n$ is a meet subsemilattice of $\mathfrak{L}$. Furthermore, $\mathfrak{L}_g$ is a join-ideal of $\mathfrak{L}$, but $\mathfrak{L}_n$ need not be a meet-ideal of $\mathfrak{L}$.

(12.4) If K is an $\mathfrak{N}$-kernel of an $\mathfrak{N}$-semigroup S, and if C is an infinite cyclic subgroup of Q generated by an element $a \in S$, then

$$C \cap K = \{e\}, \qquad e \text{ being the identity element of } Q.$$

We have a generalization of (5.4) and (5.5) as follows :

(12.5) PROPOSITION. - <u>If an</u> $\mathfrak{N}$-<u>semigroup</u> S <u>is properly homomorphic onto an</u> $\mathfrak{N}$-<u>semigroup</u> S', <u>then</u> S <u>is isomorphic onto a subdirect product of</u> S' <u>and every structure group of</u> S.

(12.6) By using Zorn's lemma, we can prove that there exists a maximal subgroup K_0 of Q which is disjoint from S. The $\mathfrak{N}$-congruence ρ_0 on S, associated with $\mathfrak{N}$-kernel K_0, is a maximal $\mathfrak{N}$-congruence.

(12.7) If S is a power joined $\mathfrak{N}$-semigroup, then the torsion subgroup of Q is a unique maximal $\mathfrak{N}$-kernel of S.

The next step is to study how to determine $\mathfrak{N}$-congruences more explicitly, and to characterize the structure of $\mathfrak{N}$-semigroups which occur as the homomorphic images of an $\mathfrak{N}$-semigroup induced by a maximal $\mathfrak{N}$-congruence.

13. $\mathfrak{N}$-kernels.

Following the notations in paragraph 8, the quotient group Q of $S = (G\,;\,I)$ is $((Z, G\,;\,f))$, where $f = I(\alpha, \beta) - 1$. Let K be an $\mathfrak{N}$-kernel of S. It is obvious $((0, \varepsilon)) \in K$.

(13.1) If $((x, \alpha)) \in K$, then $x \leqslant 0$. In particular, if $((x, \varepsilon)) \in K$, then $x = 0$.

(13.2) Let $H = \{\alpha \in G\,;\ ((x, \alpha)) \in K \text{ for some } x \leqslant 0\}$. Then, H is a subgroup of G.

(13.3) If $((x, \alpha))$ and $((y, \alpha))$ are in K, then $x = y$. Thus, x of $((x, \alpha))$ is uniquely determined by $\alpha \in H$ in $((x, \alpha)) \in K$.

(13.4) Let $\varphi(\alpha) = -x$. Then, $\varphi(\alpha) \geqslant 0$ for all $\alpha \in H$, and $\varphi(\varepsilon) = 0$, and the $\mathfrak{N}$-kernel K is obtained by

(13.4.1) $$K = \{((-\varphi(\alpha), \alpha)) ; \ \alpha \in H\} ,$$

where φ satisfies

(13.4.2) $$\varphi(\alpha) \geqslant 0 , \qquad \varphi(\varepsilon) = 0 ,$$

$$\varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta) = I(\alpha, \beta) - 1 , \qquad \text{for all } \alpha, \beta \in H .$$

Conversely, assume a subgroup H of G and a map, $\varphi : H \to P^0$, satisfy (13.4.2), and define K by (13.4.1). Then, K is an $\mathfrak{N}$-kernel of S. The couple (H, φ) satisfying (13.4.2) is called an $\mathfrak{N}$-couple of $S = (G ; I)$.

(13.5) Given $S = (G ; I)$, and an $\mathfrak{N}$-couple (H, φ) of S, we define a relation ρ on S by

$$(m, \xi) \ \rho \ (n, \eta) \quad \text{if, and only if,} \quad \xi \equiv \eta \pmod H ,$$

$$m - n = I(\eta, \eta^{-1}) - I(\xi, \eta^{-1}) - \varphi(\xi\eta^{-1}) .$$

Then, ρ is an $\mathfrak{N}$-congruence on S. All $\mathfrak{N}$-congruences on S are obtained in this manner.

14. Analyzing of $\mathfrak{N}$-congruences.

We will attack $\mathfrak{N}$-congruences by the direct method without using the quotient group.

(14.1) Let ρ be an $\mathfrak{N}$-congruence on an $\mathfrak{N}$-semigroup $S = (G ; I)$. Given ρ, a congruence σ on G is defined by :

(14.1.1) $\alpha\sigma\beta$ if, and only if, $(m, \alpha) \ \rho \ (n, \beta)$ for some $m, n \in P^0$. Let H be the subgroup of G induced by σ. H will be called the kernel of σ.

(14.2) $(m, \alpha) \ \rho \ (n, \beta)$ implies $(m + i, \alpha) \ \rho \ (n + i, \beta)$ for $i \geqslant -\min\{m, n\}$.

(14.3) If $(m, \alpha) \ \rho \ (n, \beta)$, then $m = n$.

(14.4) If $(0, e) \ \rho \ (n, \alpha)$, e being the identity element of G, then $n = 0$.

(14.5) For each $\alpha \in H$, there is a non-negative integer ℓ such that :

(14.5.1) $(i, \alpha) \ \rho \ (\ell + i, \varepsilon)$, $i = 0, 1, 2, \ldots,$ and

(14.5.2) If $(m, \alpha) \ \rho \ (n, \varepsilon)$, then $n - m = \ell$.

(14.6) If $(m_1, \alpha) \ \rho \ (n_1, \beta)$ and $(m_2, \alpha) \ \rho \ (n_2, \beta)$, then $n_1 - m_1 = n_2 - m_2$.

(14.7) Given ρ , relations on S , and G are defined by

$$(m, \alpha) \lessdot (n, \beta) \quad \text{if, and only if,} \quad (m, \alpha)\ \rho\ (n, \beta) \ \text{ and } \ m \lessdot n ,$$

$$\alpha \lessdot \beta \quad \text{if, and only if,} \quad (m, \alpha) \lessdot (n, \beta) \ \text{ for some } \ m, n \in P^0 .$$

Recall Z , in (14.8) below, denotes the set of integers.

(14.8) $\xi \lessdot \varepsilon$ for all $\xi \in H$. Hence $d(\xi , \varepsilon) \geqslant 0$ for all $\xi \in H$.

(14.9) THEOREM. - <u>If</u> ρ <u>is an</u> $\mathfrak{N}$<u>-congruence on</u> $S = (G ; I)$, <u>then a congruence</u> σ <u>on</u> G <u>and a function</u> $d : \sigma \to Z$ <u>are determined such that the following conditions are satisfied</u> :

(14.9.1) $d(\alpha , \beta) = - d(\beta , \alpha)$, <u>if</u> $\alpha\sigma\beta$;
(14.9.2) $d(\alpha , \beta) = d(\alpha , \gamma) + d(\gamma , \beta)$, <u>if</u> $\alpha\sigma\gamma$, <u>and</u> $\gamma\sigma\beta$;
(14.9.3) $d(\alpha\delta , \beta\delta) = d(\alpha , \beta) - I(\alpha , \delta) + I(\beta , \delta)$, <u>if</u> $\alpha\sigma\beta$ <u>and</u> $\delta \in G$.

Conversely, assume that a congruence σ on G and a function $d : \sigma \to Z$ satisfy the above three conditions. Define a relation ρ on S by :

(14.9.4) $(m , \alpha)\ \rho\ (n , \beta)$ if, and only if, $\alpha\sigma\beta$ and $n - m = d(\alpha , \beta)$.

Then, ρ is an $\mathfrak{N}$-congruence on S . Every $\mathfrak{N}$-congruence is obtained in this manner.

(14.10) Suppose S has a non-trivial $\mathfrak{N}$-congruence ρ (i. e. $\rho \neq \iota$). Let $S = (G ; I)$ and $\tau = S|\rho$. Let τ be the group-congruence on S defined by

$$(x , \xi)\ \tau\ (y , \eta) \quad \text{if, and only if,} \quad \xi = \eta .$$

Then, we can easily prove $\rho \cap \tau = \iota$ by using (14.9.1). Thus we have a same result as (12.5).

15. d-functions.

Thus $\mathfrak{N}$-congruences are determined by the d-functions. Now, the next question is how to characterize the d-functions in other terms.

(15.1) If $\alpha_1\, \sigma\beta_1$, $\alpha_2\, \sigma\beta_2$, and if $\alpha_1\, \gamma_1 = \alpha_2\, \gamma_2$ and $\beta_1\, \gamma_1 = \beta_2\, \gamma_2$ for some $\gamma_1 , \gamma_2 \in G$, then

$$d(\alpha_1 , \beta_1) - I(\alpha_1 , \gamma_1) + I(\beta_1 , \gamma_1) = d(\alpha_2 , \beta_2) - I(\alpha_2 , \gamma_2) + I(\beta_2 , \gamma_2) .$$

(15.2) A d-function with (14.9.1), (14.9.2), and (14.9.3), is determined by the restriction d_1 of d to $H \times H$, where H is the kernel of σ . This means that, if d and d' satisfy (14.9.1), (14.9.2), (14.9.3), and if $d|H \times H = d'|H \times H$, then $d = d'$.

(15.3) THEOREM. - Let $S = (G\,;\,I)$, and assume there is a subgroup H of G with a function $\varphi : H \to P^0$ satisfying :

(15.3.1) $\varphi(\varepsilon) = 0$, ε being the identity element ;

(15.3.2) $\varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta) = I(\alpha\,,\,\beta) - 1$, for all $\alpha\,,\,\beta \in H$.

If we define d_1 by $d_1(\alpha\,,\,\beta) = \varphi(\alpha) - \varphi(\beta)$, then d_1 is a function $H \times H \to Z$ which satisfies (14.9.1), (14.9.2), and (14.9.3), in the restricted sense. (In other words, d_1 satisfies the axioms (14.9.1) through (14.9.3), except replacing " $\alpha\sigma\beta$ ", " $\alpha\sigma\gamma$ ", " $\gamma\sigma\beta$ ", " $\delta \in G$ ", by " $(\alpha\,,\,\beta) \in H \times H$ ", " $(\alpha\,,\,\gamma) \in H \times H$ ", " $(\gamma\,,\,\beta) \in H \times H$ ", " $\delta \in H$ " respectively.) All d_1-functions are determined by φ in this manner.

(15.4) A d-function and ρ can be described in terms of I and φ :

For $\xi\,,\,\eta \in G$ such that $\xi \equiv \eta \pmod H$,

$$d(\xi\,,\,\eta) = \varphi(\xi\eta^{-1}) + I(\xi\,,\,\eta^{-1}) - I(\eta^{-1}\,,\,\eta) = \varphi(\xi\eta^{-1}) - I(\xi\eta^{-1}\,,\,\eta) + 1 \ .$$

We arrive at the same conclusion as (13.5).

The following is a restatement of (15.3).

(15.5) THEOREM. - Let $S = (G\,;\,I)$. Assume a subgroup H of G and a factor system f of H into Z satisfy :

(15.5.1) $f(\alpha\,,\,\beta) \geqslant -1$;

(15.5.2) There is $\varphi : H \to P^0$ such that

$$f(\alpha\,,\,\beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha\beta)\,, \qquad \text{for all } \alpha\,,\,\beta \in H \ .$$

Then, we can define d_1 by $d_1(\alpha\,,\,\beta) = \varphi(\alpha) - \varphi(\beta)$.

16. Tamura representation of $\mathfrak{N}$-homomorphic images.

Let $S = (G\,;\,I)$, and ρ be an $\mathfrak{N}$-congruence on S, and $\bar{S} = S|\rho$.

(16.1) Let X be a ρ-class of S. The following are equivalent :

(16.1.1) X is a prime element of $\bar{S}$;

(16.1.2) If $(x\,,\,\pi) \in X$, then $x = 0$;

(16.1.3) $X = \{(0\,,\,\pi) \in S\,;\ d(\pi\,,\,\xi) = 0 \text{ for all } \xi \geqslant \pi\}$,

where d is the d-function associated with ρ.

(16.1.3') $X = \{(0\,,\,\pi) \in S\,;\ d(\pi\,,\,\xi) \leqslant 0 \text{ for all } \xi \text{ such that } \xi\sigma\pi\}$.

Such $\pi \in G$ is called a ρ-maximal element of G. α is ρ-maximal in G, if, and only if, $(m\,,\,\alpha)$ is maximal with respect to $\leqslant$ for each $m \in P^0$.

(16.2) Let $(\overline{m, \alpha})$ denote the element of $\overline{S}$, i. e. the ρ-class containing (m, α) of S. A congruence $\overline{\sigma}$ of $\overline{S}$ is associated with σ of G in the following way :

$$(\overline{k, \xi})\ \overline{\sigma}\ (\overline{\ell, \eta}) \quad \text{if, and only if,} \quad \xi\sigma\eta .$$

Let $\alpha \in G$, and let $(\overline{0, \pi})$ be a prime element of the $\overline{\sigma}$-class containing $(\overline{0, \alpha})$. π is a ρ-maximal element of G associated with α. Define $\tilde{\varphi} : G \to P^0$ by :

(16.2.1) $\tilde{\varphi}(\alpha) = d(\alpha, \pi)$.

Then, $\tilde{\varphi}$ is an extension of φ to G, and $\tilde{\varphi}$ is determined by φ as follows :

(16.2.2) $\tilde{\varphi}(\alpha) = \varphi(\alpha\pi^{-1}) + I(\alpha, \pi^{-1}) - I(\pi^{-1}, \pi)$,

where π is ρ-maximal associated with α. (16.2.2) can be adopted as the definition of $\tilde{\varphi}$.

Also, we have :

(16.2.3) $\tilde{\varphi}(\alpha) = \max\limits_{\alpha\sigma\xi} d(\alpha, \xi) \leqslant I(\alpha, \alpha^{-1})$.

(16.3) THEOREM. - <u>Let</u> $S = (G ; I)$, <u>and</u> ρ <u>be an</u> $\mathfrak{N}$-<u>congruence on</u> S. <u>Let</u> σ <u>on</u> G, <u>and</u> $d : \sigma \to Z$ <u>be induced by</u> ρ <u>(as theorem</u> (14.9)) ; <u>let</u> $\tilde{\varphi}$ <u>be defined by</u> (16.2.1). <u>Let</u> $\overline{G} = G/\sigma$, <u>and let</u> $\overline{\pi}$ <u>be an element of</u> $\overline{G}$. <u>Now</u>

$$\overline{I} : \overline{G} \times \overline{G} \to P^0$$

<u>is defined by</u>

$$\overline{I}(\overline{\pi}_1, \overline{\pi}_2) = I(\pi_1, \pi_2) + \tilde{\varphi}(\pi_1 \pi_2) ,$$

<u>where</u> $\tilde{\varphi}(\pi_1 \pi_2) = d(\pi_1 \pi_2, \alpha)$, α <u>being</u> ρ-<u>maximal associated with</u> $\pi_1 \pi_2$.

<u>Then,</u> $S|\rho$ <u>is isomorphic to</u> $(\overline{G} ; \overline{I})$.

17. $\mathfrak{N}$-congruences on power joined $\mathfrak{N}$-semigroups.

(17.1) Let $S = (G ; I)$ be power joined. A subgroup H of G has a function $\varphi : H \to P^0$ satisfying (15.3.1) and (15.3.2), if, and only if, H is a subgroup of G such that $\overline{\varphi}(\xi)$ is a positive integer for all $\xi \in H$, where $\overline{\varphi}$ is defined in paragraph 6. In this case, φ is uniquely determined by H, that is,

$$\varphi(\xi) = \overline{\varphi}(\xi) - 1 .$$

A subgroup H of G, on which $\overline{\varphi}$ is positive integer valued, is called $\mathfrak{N}$-subgroup of G with respect to $\overline{\varphi}$.

(17.2) Let H_0 denote the set of all elements α of G such that $\overline{\varphi}(\alpha)$ is a positive integer, equivalently,

$$H_0 = \{\alpha \in G \; ; \; \sum_{i=1}^{s} I(\alpha , \alpha^i) \text{ is a multiple of } s , \; s = |\alpha|\} .$$

Then, H_0 is a subgroup of G, and every subgroup of H_0 is an $\mathfrak{N}$-subgroup of G.

(17.3) We can characterize $\mathfrak{N}$-congruences on a power joined $\mathfrak{N}$-semigroup in the two ways : in terms of subgroups of the torsion subgroup of Q, in the sense of (12.7), and in terms of subgroups of H_0, as the above.

(17.4) The set of all $\mathfrak{N}$-congruences on a power joined $\mathfrak{N}$-semigroup S is a lattice, which is isomorphic to the lattice of subgroups of H_0, and to the lattice of subgroups of the torsion subgroup of Q.

18. Irreducible $\mathfrak{N}$-semigroups.

(18.1) An $\mathfrak{N}$-semigroup S is called irreducible, if there is no $\mathfrak{N}$-congruence on S except the identity relation ι. Every $\mathfrak{N}$-semigroup is homomorphic to an irreducible $\mathfrak{N}$-semigroup. It is induced by a maximal $\mathfrak{N}$-kernel (§ 12).

(18.2) Let S be an $\mathfrak{N}$-semigroup. S is irreducible, if, and only if, for every $a , b \in S$, there is $m \in P$ and an element $x \in S$ such that either $a^m = b^m x$ or $b^m = a^m x$.

(18.3) THEOREM. - <u>An irreducible $\mathfrak{N}$-semigroup is isomorphic to a semigroup of positive real numbers with addition.</u>

(18.4) THEOREM. - <u>Every $\mathfrak{N}$-semigroup is isomorphic to a subdirect product of an additive positive real semigroup and an abelian group.</u>

<u>Remark</u>. - To prove (18.3) and (18.4), we use [1], [5], [12], [17]. Irreducible $\mathfrak{N}$-semigroups are closely related to $\mathfrak{N}$-semigroups satisfying the divisibility chain condition. A commutative semigroup S is said to satisfy the divisibility chain condition, if, for any distinct elements $a , b \in S$, either $a = bx$ or $b = ax$ for some $x \in S$. The information related to such semigroups are obtained in CLIFFORD [5], FUCHS [12], and ETTERBEEK [10].

ADDENDUM. - Recently, R. P. DICKINSON has determined all congruences ρ of type (0.5) on an $\mathfrak{N}$-semigroup $S = (G ; I)$ in terms of a certain function $\varphi : \sigma \to P^0$, when an $\mathfrak{N}$-congruence ρ_1 is associated with d, σ such that $S|\rho_1$ is the greatest cancellative homomorphic image of $S|\rho$.

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(Texte reçu le 15 septembre 1970)

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