Séminaire Dubreil.
Algèbre et théorie
des nombres

Štefan Schwarz

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Séminaire Dubreil. Algèbre et théorie des nombres, tome 23, nº 2 (1969-1970), exp. nº DG 2, p. DG1-DG5
<http://www.numdam.org/item?id=SD_1969-1970__23_2_A2_0>

Séminaire DUBREIL-PISOT
(Algèbre et Théorie des nombres)
23e année, 1969/70
Demi-groupes, n° 2, 5 p.

ON THE STRUCTURE OF DUAL SEMIGROUPS

by Štefan SCHWARZ

Let $S \neq 0$ be a semigroup with zero 0. For a non-empty set A, the left annihilator of A is defined by $\ell(A) = \{x \mid xA = 0\}$. The right annihilator $r(A)$ is defined analogously.

A semigroup is called <u>dual</u> if, for every left ideal L of S, we have

$$\ell[r(L)] = L \quad ,$$

and, for every right ideal R of S, we have

$$r[\ell(R)] = R \quad .$$

The lattices of left and right ideals of such a semigroup are antiisomorphic. Some properties follow from this symetry ; e. g. :

(a) $\ell(S) = r(S) = 0$;

(b) If A, B are two left ideals, we have not only

$$r(A \cup B) = r(A) \cap r(B)$$

(which holds in any semigroup), but also

$$r(A \cap B) = r(A) \cup r(B) \quad ;$$

(c) If L_0 is a 0-minimal left ideal of S, $r(L_0)$ is a maximal right ideal of S.

Ten years ago, I developed a structure theory for dual semigroups under the assumption of some minimal conditions (see [3]). In the meantime, NUMAKURA [2] proved that, in the very special case of dual semigroups without nilpotent ideals, some of these minimal conditions can be omited.

It is the first purpose of this talk to underline that almost all results which I have proved earlier are valid without any minimal condition. Of course, the proofs differ rather essentially from those given in [3]. As a matter of fact, it turns out that the condition for a semigroup to be dual implies itself some rather strong minimal conditions.

The second purpose is to give some decomposition theorems which enable to reduce the study of dual semigroups to some special types of dual semigroups.

The proofs of the results stated below will appear in [5].

The basic statement is :

THEOREM 1. - Any left ideal of a dual semigroup contains a 0-minimal left ideal.

An analogous statement holds for right and two-sided ideals. This implies that, in a dual semigroup, every left ideal is contained in a maximal left ideal, and any two-sided ideal is contained in a maximal two-sided ideal.

Before going further, we mention a lemma on (non-necessarily dual) semigroups having maximal ideals (see [1], [4]).

LEMMA. - Let $\{M_\alpha \mid \alpha \in \Lambda\}$ be the set of all maximal two-sided ideals of a semigroup S. Denote $P_\alpha = S - M_\alpha$ and $M^* = \bigcap_\alpha M_\alpha$. Then $P_\alpha \cap P_\beta = \emptyset$ for $\alpha \neq \beta$, and S can be written as a union of disjoint sets, $S = [\bigcup_\alpha P_\alpha] \cup M^*$.

Hereby :

(a) Every P_α $(\alpha \in \Lambda)$ is a Green's J-class ;

(b) $P_\alpha P_\beta \subset M^*$;

(c) M^* is non-empty ;

(d) The difference semigroup S/M^* is a 0-direct union

$$S/M^* = \bigcup_\alpha \overline{P}_\alpha , \qquad \overline{P}_\alpha = P_\alpha \cup \overline{0} ,$$

where each $\overline{P}_\alpha$ is either a 0-simple semigroup or a null semigroup of order two ($\overline{0}$ has an obvious meaning).

Suppose in the following that S is dual. We have :

THEOREM 2.

(a) M^* does not contain non-zero idempotents.

(b) To any $a \in S$, there are uniquely determined idempotents e, f such that $a = ae = fa$.

(c) In particular, if $a \in P_\alpha$, then $e, f \in P_\alpha$.

Note that the second statement proves the existence of non-zero idempotents in any dual semigroup S. Denote by E the set of all non-zero idempotents $\in S$.

The principal left ideal Se, generated by an $e \in E$, has the following properties.

THEOREM 3.

(a) Se contains a unique non-zero idempotent (namely e itself).

(b) Se contains a unique 0-minimal left ideal of S .

(c) $Se_1 \cap Se_2 = 0$ for $e_1 \neq e_2$, $e_1 , e_2 \in E$.

These results imply the following decomposition theorem.

THEOREM 4. - Any dual semigroup can be written as a union of mutually quasidisjoint principal left ideals generated by idempotents :

$$S = \bigcup_{e \in E} Se \ .$$

Analogously, we have a decomposition into principal right ideals :

$$S = \bigcup_{e \in E} eS \ .$$

The following theorem gives further informations concerning the idempotents.

THEOREM 5. - For two different idempotents $e_1 \neq e_2$, we have $e_1 . e_2 = 0$. Moreover, $P_\alpha . P_\beta = 0$ for $P_\alpha \neq P_\beta$.

This implies the following corollary.

COROLLARY. - Any commutative dual semigroup is a 0-direct union of dual semigroups, each of which contains a unique idempotent.

It is easy to see that, in this case, each of the dual semigroups Se , $e \in E$, contains a unique maximal ideal M_e^* , and the complement $Se - M_e^*$ is a group with e as unit element. The study of commutative dual semigroups can be reduced to the study of dual semigroups containing a unique non-zero idempotent.

Theorems 2-5 imply also the following theorem.

THEOREM 6. - For any dual semigroup, the difference semigroup S/M^* is a 0-direct union of completely 0-simple dual semigroups.

Note that this result has been obtained without any minimal condition.

The following corollary is of independent interest.

COROLLARY. - Any 0-simple dual semigroup is completely 0-simple.

There is also an intimate connection between the 0-minimal one-sided ideal of S and the 0-minimal one-sided ideals of $\overline{P}_\alpha$. We shall not enter into these details, and return rather to the semigroup S as a whole, whereby the emphasis will be on the non-commutative case.

The decomposition into left ideals gives rise to the natural question concerning a decomposition of S into two-sided ideals. It would be nice to find decompositions into two-sided quasidisjoint ideals in which each of the summands is dual.

The first step to obtain such a decomposition is the following. If some of the left ideals Se (in theorem 4) is itself 0-simple, the corresponding two-sided ideal SeS is dual (and completely 0-simple) and has a zero intersection with M^*. Denote by H the set of all idempotents for which SeS is completely 0-simple. We then have a decomposition into a 0-direct union,

$$S = [\bigcup_{e\in H} SeS] \cup S_1 \ .$$

Here the first term (if it is non-empty) is a 0-direct union of completely 0-simple dual semigroups, and S_1 (if it is not empty) is itself a dual semigroup which has the property that any two-sided ideal of S_1 has a non-zero intersection with M^*.

This implies (among others) the following corollary.

COROLLARY. – A dual semigroup is a 0-direct union of (completely) 0-simple dual semigroups if, and only if, $M^* = 0$.

Can S_1 be further decomposed ? It can be proved that, if SeS , $e \in E \cap S_1$, contains a unique 0-minimal two-sided ideal, then SeS is a 0-direct component of S_1 , and both summands SeS and its complement in S_1 (with 0 adjoined) are dual.

Denote by F the set of all idempotents $\in E - H$ for which SeS , $e \in F$, contains a unique 0-minimal two-sided ideal. Then S_1 can be written as a 0-direct union in the form

$$S_1 = [\bigcup_{e\in F} SeS] \cup S_2 \ .$$

Each term in the first summand (if it is not empty) contains a unique 0-minimal two-sided ideal, while S_2 (if it is not empty) is a dual semigroup having the property that each principal two-sided ideal, generated by an idempotent, contains at least two 0-minimal two-sided ideals.

COROLLARY. – A dual semigroup is a 0-direct union of dual semigroups each of which contains a unique 0-minimal two-sided ideal if, and only if, for any two idempotents e , $f \in S$, we have either $SeS = SfS$ or $SeS \cap SfS = 0$.

Summarily : the decompositions just mentioned enable, in some cases, to reduce the study of dual semigroups to some "simpler" types of dual semigroups.

Here are some comments. The structure of a completely simple dual semigroup can be fully described by means of matrices over a group with zero (see [3]). The same is true for a 0-direct union of such semigroups. Though the results mentioned above denote a considerable step forward, there remains open the problem of their representation, or otherwise expressed, to find methods how to construct all dual semigroups.

As a first step, it would be desirable to find some kind of representations of dual semigroups S containing a unique 0-minimal two-sided ideal (here, of course, the interest is on the case that S is not 0-simple). Even, this (rather special) problem is far from to be easy. Roughly speaking, this is due to the fact that we have to deal with ideals contained in M^*, and the obvious difficulties with nilpotent ideals arise.

But nevertheless I would like to end with the following remark. Since the structure of any dual semigroup has (at least formally) some resemblence with the structure of completely 0-simple dual semigroups, the problem just mentioned seems not to be quite hopeless.

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(Texte reçu le 22 septembre 1970)

Štefan SCHWARZ
Prof. Techn. Univ.
Gottwaldovo nám. 2
BRATISLAVA (Tchécoslovaquie)