Séminaire Dubreil.
Algèbre et théorie
des nombres

Rolando E. Peinado

Finite semigroups admitting ring structure

Séminaire Dubreil. Algèbre et théorie des nombres, tome 23, nº 2 (1969-1970), exp. nº DG 13, p. DG1-DG3
<http://www.numdam.org/item?id=SD_1969-1970__23_2_A12_0>

Séminaire DUBREIL-PISOT
(Algèbre et Théorie des nombres)
23e année, 1969/70
Demi-groupes, n° 13, 3 p.

FINITE SEMIGROUPS ADMITTING RING STRUCTURE

by Rolando E. PEINADO

A semigroup S is said to admit a ring structure if S is isomorphic to the multiplicative structure of some (associative) ring R (in case S has no zero elements, we consider S^0), where R belongs to a class of rings $\mathcal{R}$, we call S an $\mathcal{R}$-semigroup. It is clear that the class of $\mathcal{R}$-semigroup is not empty since the multiplicative structure of any ring is such a semigroup. By considering left (right) zero semigroup with more than one element, it is clear that not all semigroups are $\mathcal{R}$-semigroups. The problem of characterizing the class of all $\mathcal{R}$-semigroups, where $\mathcal{R}$ is the class of all rings, was first formally stated by P. DUBREIL [1] and KOGALOVSKIJ in [2], shows that it is impossible to give an axiomatic characterization of such class.

Here we proposed to give a brief account of results obtained by several people on the specific case of finite $\mathcal{R}$-semigroup. For a more complete survey of the general case (includes unique addition R-semigroups) and a more complete bibliography, see R. E. PEINADO [3], to appear in Semigroup Forum.

If S is a finite R-semigroup without zero (S^0 admits a ring structure), and suppose R is a ring whose multiplicative structure is isomorphic to S^0 . Then R has no zero divisors, and hence it is cancellative and being finite, it is a finite integral domain, and hence R a field, moreover it is a Galois field of order p^n. This implies that S has order $p^n - 1$ and is a cyclic group. The converse is obvious.

THEOREM 1. - <u>A finite semigroup</u> S <u>without zero is an</u> R-<u>semigroup if, and only if,</u> S <u>is a cyclic group of order</u> $p^n - 1$ <u>for some prime</u> p .

This theorem enables us to restrict our attention to finite semigroups with zero element and zero divisors. In considering finite $\mathcal{R}$-semigroup, the following results are important.

THEOREM 2. - <u>Let</u> R <u>be a finite ring of order</u> n , <u>where</u>

$$n = \prod p_i^{t_i} , \quad i = 1 , \ldots , k , \quad t_i > 0 .$$

<u>Then</u> R <u>is isomorphic to the direct sum of</u> p_i-<u>rings</u> R_i , $i = 1 , \ldots , k$ <u>of order</u> $p_i^{t_i}$

THEOREM 3. - Let S be a finite semigroup of order n ,

$$n = \prod p_i^{t_i} , \qquad i = 1 , \dots , k .$$

S is an $\mathcal{R}$-semigroup if, and only if, S is the direct sum of R-semigroups S_i , of order $p_i^{t_i}$, $i = 1 , \dots , k$.

THEOREM 4. - Let S be a finite cyclic semigroup ; then S is an $\mathcal{R}$-semigroup if, and only if, S is of order 1 or 2 .

If $n = p$, a prime number, then there exists two non-isomorphic rings of order p . The ring with trivial multiplication, all products igual to zero and the Galois field of order p .

THEOREM 5. - A finite semigroup S with zero of order p , a primer number, is an $\mathcal{R}$-semigroup if, and only if, S is a cyclic group with zero or a zero semigroup

COROLLARY. - A finite semigroup S with zero, of square-free order is an $\mathcal{R}$-semigroup if, and only if, S is a direct product of zero semigroups and/or cyclic groups with zero.

THEOREM 6. - A finitely generated commutative semigroup S is an $\mathcal{R}$-semigroup if, and only if, S is finite.

THEOREM 7. - A finite $\mathcal{R}$-semigroup of cube-free order is commutative.

For the integers modulo n , Z_n , we can obtain the following interesting result.

THEOREM 8. - Let $\bar{Z}_{p^t} = Z_{p^t} - \{0\}$. Then the multiplicative semigroup $\bar{Z}_{p^t}$ contains a subgroup $G(p^t)$ isomorphic to $\bar{Z}_p$, given by $G(p^t) = \langle X_{p^t} \rangle$, where, for the prime $p \neq 2$ and $t > 1$,

$$X_{p^t} \equiv X_{p^{t-1}} \quad (\text{modulo } p^{t-1})$$

and

$$(X_p)^{\frac{p-1}{2}} \equiv p^t - 1 \quad (\text{modulo } p^t) ,$$

for $p = 2$, $G(2^t) = \langle 1 \rangle$,
for $t = 1$, $G(p) = \bar{Z}_p$.

REFERENCES

[1] DUBREIL (P.). - Quelques problèmes d'algèbre liés à la théorie des demi-groupes, Colloque d'algèbre supérieure [1956. Bruxelles], p. 29-44. - Louvain, Ceuterick, 1957 (Centre belge de Recherches mathématiques).

[2] KOGALOVSKIJ (S. R.). - On the multiplicative semigroups of rings [in Russian], Doklady Akad. Nauk. S. S. S. R., t. 140, 1961, p. 1005-1007.

[3] PEINADO (R. E.). - On semigroups admitting ring structure, Semigroup Forum (to appear).

Rolando PEINADO
University of Puerto Rico
MAYAGUEZ (Porto-Rico)

(Texte reçu le 23 septembre 1970)