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ALGEBRAS OF IMPLICATION AND SEMI-LATTICES

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In this paper, we discuss algebraic systems with a single binary operation modeled on the boolean operation $a \rightarrow b = a' \cup b$, where a' is the boolean complement of a. Such algebras satisfy three basic equalities motivitated from laws of implication in the absolute propositional calculus, and are called implication algebras. We first develop some of the elementary arithmetic of these systems, and use it to determine the free algebra with two generators. Implication algebras are then related to partially ordered sets and semi-lattices. We next define a semi-boolean algebra, and establish a one-to-one correspondence between implication algebras and semi-boolean algebras. Boolean algebra is then obtained as a special case by postulating the existence of a distinguished element satisfying a fourth postulate. We therefore obtain a new characterization of boolean algebra in terms of a single binary operation plus a constant ; therefore, more analogous to the theory of groups than that of rings. The elements of algebraic structure theory are then outlined, and a ply operation is introduced for filters. Generalizations of the characterization of implication algebras in terms of semi-boolean algebra are then given, which, on the one hand, give an algebraic structure to the algebra of filters and, on the other hand, give a new characterization of brouwerian algebra. We conclude with examples and applications.

1. Basic theory.

We define an implication algebra as a system, $\langle I , . \rangle$, defined on a set I closed under a single binary operation, ab, satisfying :

(I.1) $(ab)a = a$ (contraction),

(I.2) $(ab)b = (ba)a$ (quasi-commutative),

(I.3) $a(bc) = b(ac)$ (exchange).

On the basis of these three axioms, we prove the following lemmas and theorems.

LEMMA 1. - $a(ab) = ab$.

LEMMA 2. - $aa = (ab)(ab)$.

THEOREM 1. - $\exists\, 1 \in I$ <u>such that</u> :

(i) $aa = 1$,

(ii) $1a = a$,

(iii) $a1 = 1$, $\forall\, a \in I$.

THEOREM 2. - <u>In any implication algebra, the following identities hold</u> :

(i) $a(ba) = 1$;

(ii) $((ab)b)b = ab$;

(iii) $((ab)b)a = ba$;

(iv) $a((ab)b) = b((ab)b) = 1$;

(v) $(ab)(ba) = ba$.

As a corollary, we can determine the multiplication table for the free algebra with two generators. It will contain the six elements,

$$I = \{a \, , \, b \, , \, 1 \, , \, ab \, , \, ba \, , \, (ab)b\} \, .$$

	1	a	b	ab	ba	(ab)b
1	1	a	b	ab	ba	(ab)b
a	1	1	ab	ab	1	1
b	1	ba	1	1	ba	1
ab	1	a	(ab)b	1	ba	(ab)b
ba	1	(ab)b	b	ab	1	(ab)b
(ab)b	1	ba	ab	ab	ba	1

<u>Table</u> 1

We next associate a partially ordered set with every implication algebra as in the following theorem.

THEOREM 3. - <u>Every implication algebra,</u> $\langle I \, , \, . \rangle$, <u>determines a partially ordered set,</u> $\langle I \, , \leqslant \rangle$, <u>with greatest element</u> 1 <u>under</u> $a \leqslant b \iff ab = 1$, <u>which is left isotone and right antitone with respect to implication, i. e.</u> :

(i) $a \leqslant a$;

(ii) $a \leqslant b$ <u>and</u> $b \leqslant a \implies a = b$;

(iii) $a \leqslant b$ *and* $b \leqslant c \implies a \leqslant c$;

(iv) $a \leqslant 1 , \quad \forall a$;

(v) $a \leqslant b \implies ca \leqslant cb$;

(vi) $a \leqslant b \implies bc \leqslant ac$.

On the basis of this theorem, we can construct a diagram for an implication algebra as shown, for example, for table 1 in figure 1. The figure shows that not every implication algebra is a lattice, but the next two theorems indicate the relationship of implication algebras to lattices.

THEOREM 4. - $\langle I , . \rangle$ *is a union semi-lattice under* $a \cup b = (ab)b$, *i. e.* :

(i) $a , b \leqslant (ab)b$;

(ii) $a , b \leqslant c \implies (ab)b \leqslant c$.

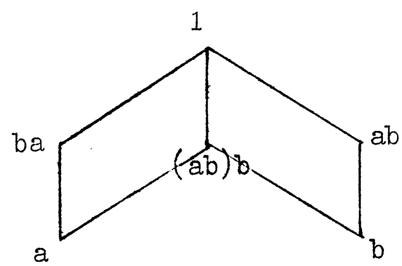

Figure 1

THEOREM 5. - *Every pair of elements* a , b *which have a lower bound, have a greatest lower bound given by* $a \cap b = (a(bc))c$, *where* c *is any element satisfying* $ca = cb = 1$.

We now define a principal filter generated by an element a by

$$[a]\uparrow = \{x \ ; \ a \leqslant x\} \ .$$

It then follows that $[a]\uparrow$ can be alternately characterized as the set of left multiples of a , i. e.,

$$[a]\uparrow = \{x \ ; \ x = ya , \ y \in I\} \ ,$$

as given by the following theorem.

THEOREM 6. - $a \leqslant b \iff b = xa$.

Theorem 5 now states that every principal filter in I is a lattice. The main result of this section is that this lattice is a boolean algebra, i. e., a complemented, distributive lattice. Thus :

THEOREM 7. - Every principal filter of an implication algebra is a boolean algebra, in which $x'_a = xa$ is the complement of x within the filter $[a]\uparrow$, i. e. :

(i) $x \cup x'_a = (x(xa))(xa) = 1$;

(ii) $x \cap x'_a = (x((xa)a))a = a$.

We define a semi-boolean algebra as a union semi-lattice in which every principal filter is a boolean algebra. Theorem 7, then states that every implication algebra is associated with a semi-boolean algebra. Conversely, every semi-boolean algebra determines an implication algebra, and these two correspondences are inverses, as given in the following theorem.

THEOREM 8. - If $\langle I , \cup \rangle$ is a semi-boolean algebra, then $\langle I , \cup \rangle$ determines an implication algebra, $\langle I , . \rangle$, under the definition $ab = (a \cup b)'_b$, where x'_b designates the boolean complement of x within the principal filter $[b]\uparrow$.

Thus the theory of implication algebra is identical with the theory of semi-boolean algebra. Boolean algebra may then be obtained as a special case of a semi-boolean algebra with a least element. In terms of implication algebra, we therefore have :

THEOREM 9. - A boolean algebra is a system $\langle I , . , 0 \rangle$ satisfying (I.1)-(I.3) and :

(I.4) $\exists\, 0 \in I$ such that $0a = 1$, $\forall\, a \in I$.

2. Algebraic structure theory.

The usual algebraic concepts, such as, sub-algebra, homomorphism, and direct product, can now be defined for implication algebras in terms of the single operation, ab . The facts that order, the greatest element, union and meet (when it exists), are preserved, follow as corollaries. The concept of the kernel is defined for an homomorphism Φ as the set $\Phi^{-1}(1)$, where 1 is the unit of the image algebra. A congruence relation is defined in the usual way as an equivalence relation with the left and right substitution properties, with respect to implication. A filter is defined, either :

(a) as a set F such that :

(i) $a \in F$ and $a \leqslant x \implies x \in F$,

(ii) $a , b \in F \implies a \cap b \in F$ (provided it exists),

or :

(b) as a set F such that :

(i) $1 \in F$,

(ii) a , $ab \in F \implies b \in F$ (modus ponens).

The kernel of a homomorphism is then a filter. Conversely, given a filter, F , we define a congruence relation mod F by

$$a \equiv b \iff ab \text{ , } ba \in F \quad ,$$

and show that this is a congruence relation using (I.1)-(I.3). The congruence relation then determines a quotient algebra in the usual way and a natural homomorphism such that correspondence between filters and homomorphisms is one-to-one and inverse. Thus the lattice of homomorphisms, $\mathcal{H}(I)$, the lattice of congruence relations, $\Theta(I)$, and the lattice of filters, $\mathfrak{F}(I)$, are all isomorphic. The lattice of filters then contains a subset, the principal filters, which is an implication algebra dually isomorphic to I itself. All of this is standard, but we are now led to characterize the principal filter $[ab]\uparrow$ in terms of the two filters, $[a]\uparrow$ and $[b]\uparrow$. This is turn leads to a generalization to an implication product, $H \circ K$, of two arbitrary filters, H and K , given by

$$H \circ K = \{k \in K \ ; \ hk = k \ , \ \forall\, h \in H \ \text{(or alternatively, } h \cup k = 1 \ , \ \forall\, h \in H)\} \quad .$$

This product satisfies (I.1) and (I.3), but fails to satisfy (I.2) which is replaced by the inequality

$$(H \circ K) \circ K \supseteq H \cap K \quad .$$

Hence, we seek an abstract characterization of systems with a ply operation similar to $H \circ K$ for filters.

3. Generalizations.

The characterization theorem for implication algebras in terms of semi-boolean algebras, theorem 8, can be generalized by relaxing the conditions of complementation within the principal filters, using either lower or upper pseudo-complements. Thus, in an arbitrary distributive lattice with 0 and 1 , a complement of an element a satisfies the two equations :

(i) $a \cup a' = 1$;

(ii) $a \cap a' = 0$.

An upper, respectively lower, pseudo-complement of a is defined as a best solution of (i), respectively (ii), in the sense that an upper pseudo-complement, a^* , satisfies

$$a \cup a^* = 1 \ \text{ and } \ a \cup x = 1 \implies a^* \leqslant x \quad ,$$

while a lower pseudo-complement, a_* , satisfies

$$a \cap a_* = 0 \quad \text{and} \quad a \cap x = 0 \implies x \leqslant a_* \,.$$

If we now consider a union semi-lattice, $\langle I , \cup \rangle$, in which every principal filter is upper, respectively, lower, pseudo-complemented, then we can define ply operations by

$$a \circ b = (a \cup b)^*_b \qquad \text{and} \qquad a \overset{*}{} b = (a \cup b)_{*b} \;,$$

similar to ordinary implication in the semi-boolean case. The first of these defines an abstract algebra which characterizes the algebra of filters of a boolean algebra (implication algebra), in which o-ply is the filter product defined above, while the second gives a new characterization of a brouwerian algebra, i. e., $a \overset{*}{} b$ is intuitionist implication.

4. Examples and applications.

A boolean algebra $\langle I , \cup , \cap , ' , 0 , 1 \rangle$, determines an implication algebra, $\langle I , . \rangle$, under either boolean implication,

$$ab = a \rightarrow b = a' \cup b \;,$$

or boolean subtraction,

$$ab = b - a = a' \cap b \;.$$

This leads to two classes of implication algebras which are dual to each other, but not self dual. Taking sub-algebras, homomorphic images and direct products therefore produce further examples. For example, every upper section (subset containing with any a every x such that $a \leqslant x$) of a boolean algebra is a sub-implication algebra under boolean implication, while every lower section is a "subtraction" algebra under boolean subtraction. For example, if X is any set, then the set of :

(i) non-empty,
(ii) proper,
(iii) finite,
(iv) infinite,
etc.

subsets of X is an implication algebra, but not generally a boolean algebra. The most important examples come from topology where the set $\mathfrak{N}$ of neighborhoods (sets with non-empty interiors) of a topological space X form an implication algebra under set implication, $AB = A' \cup B$, since if A and B have non-empty interiors, then so does $A' \cup B$. If x is a fixed point of X , then the neighborhoods of x

form a filter, $\mathfrak{N}_x$, the neighborhood filter of x , which is therefore the kernel of an implication homomorphism, and therefore defines a quotient algebra, $\mathfrak{N}/\mathfrak{N}_x$. Such quotient algebras characterize the local behavior of the space at x . Thus, applying an algebraic structure to the neighborhood system of a topological space, enables us to formulate many topological questions including the definition of a topological space, in algebraic terms.

Further examples are found in axiomatic set theory, in which the class of all sets defines an implication algebra which is not boolean. Similarly, boolean rings determine implication algebras under $ab = a \times b + b$, where $+$ and $\times$ are the ring operations, and enable us to obtain a simple formulation of the relationship between boolean algebras and generalized boolean algebras and boolean with or without identities. Finally, implication algebras serve as Lindenbaum-Tarski algebras for various forms of the absolute propositional calculus without negation. Systems of the type involving the operation $a \circ b$, defined above, serve to shed new light on the relations in various algebras of logic.

Résumé en français

Dans cette conférence, nous considérerons des algèbres munies d'une seule opération binaire suggérée par l'opération booléenne

$$a \rightarrow b = a' \vee b \quad ,$$

où a' est le complément booléen de a . De telles algèbres vérifient trois égalités fondamentales dues aux lois d'implication du calcul absolu des propositions, et sont appelées algèbres d'implication. Nous développons d'abord les propriétés arithmétiques élémentaires de ces systèmes, et les utilisons pour déterminer l'algèbre libre à deux générateurs. Les algèbres d'implication sont alors rattachées aux ensembles particulièrement ordonnés et aux demi-treillis. Nous définissons ensuite une algèbre semi-booléenne, et établissons une bijection entre algèbres d'implication et algèbres semi-booléennes. Les algèbres de Boole sont alors obtenues comme cas particulier, en postulant l'existence d'un élément distingué vérifiant un quatrième axiome (existence d'un plus petit élément). Nous obtenons ainsi une nouvelle caractérisation des algèbres de Boole au moyen d'une seule opération binaire et d'une opération constante, théorie plus proche de celle des groupes que de celle des anneaux. Nous développons enfin les éléments de la théorie algébrique de cette structure, et l'appliquons à la définition d'une opération d'implication dans l'ensemble des filtres d'une algèbre d'implication. On indique alors des généralisations

de la caractérisation des algèbres d'implication au moyen d'algèbres semi-booléennes qui donnent, d'une part une structure algébrique à l'algèbre des filtres, et d'autre part une nouvelle caractérisation de l'algèbre brouwerienne.

Nous terminons par quelques exemples et applications.