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ON SIEVE METHOD AND GOLDBACH PROBLEM

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The historical origin of Sieve method is the well-known "sieve of Eratosthènes". ERATOSTHÈNES noted that the prime numbers between $n^{1/2}$ and n can be isolated by removing from the sequence $2, 3, \ldots, n$ every number which is multiple of a prime not exceeding $n^{1/2}$. Let $\pi(x)$ denote the number of primes $\leqslant x$ and $\prod_{p\leqslant\sqrt{n}} p = \Pi$, where p denote a prime number. Then

$$1 + \pi(n) - \pi(n^{1/2}) = \sum_{a\leqslant n} \sum_{d|(a,\Pi)} \mu(d) = \sum_{d|\Pi} \mu(d)[\frac{n}{d}] .$$

If we use $\frac{n}{d} + \theta$ instead of $[\frac{n}{d}]$, then it will cause an error term

$$O(2^{\pi(\sqrt{n})})$$

in above formula, so the sieve of Eratosthènes with its large error term is almost useless.

It was not a great achievement when V. BRUN, in 1919, devised his new sieve method and applied it successfully to several difficult and important problems in number theory. In 1947, A. SELBERG gave another sieve method which is much simpler and leads to more precise results than the more complicated Brun's method. Indeed these methods represent an indispensable tool in number theory.

The essence of the methods of BRUN and SELBERG is to use some inequalities instead of

$$\sum_{d/n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 , \\ 0 & \text{if } n > 1 , \end{cases}$$

in order to decrease the error term. For example, for any given set of real numbers λ_d 's with $\lambda_1 = 1$, then

$$\sum_{d|n} \mu(d) \leqslant (\sum_{d|n} \lambda_d)^2 .$$

Choose suitable λ_d 's . Then we obtain the Selberg's upper bound method.

Two famous conjectures are connected with the sieve method :

(a) Goldbach conjecture : Every even integer $n(\geqslant 4)$ is a sum of two primes

(b) Prime twins conjecture : There exists infinitely many prime twins $(p , p+2)$.

These two problems may be treated similarly by sieve method, so we only state the Goldbach problem.

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Let $A = \{a_\nu\}$ be a set of integers. Let P denote a set of r primes $p_1 < \dots < p_r$. Further let $S(A, P)$ denote the number of elements in A which is unsifted by the sequence P. Take $A = \{\nu(n - \nu), v = 1, 2, \dots, n\}$ and P = the set of all primes $\leqslant n^{1/(\ell+1)}$, where ℓ is a natural number. Suppose that we can obtain a positive lower estimation for $S(A, P)$ when n is large. Then it follows that

(a') every large even integer n is the sum of 2 numbers each being a product of at most ℓ prime factors.

The proposition (a') is denoted by (ℓ, ℓ). Similarly, we may define (ℓ, m) for $\ell \neq m$.

BRUN was the first who proved $(9, 9)$. Brun's method and his result were improved by several mathematicians, for examples,

$(7, 7)$ (H. RADEMACHER, 1924),

$(6, 6)$ (T. ESTERMANN, 1932),

$(5, 7)$, $(4, 9)$, $(3, 15)$, $(2, 366)$ (G. RICCI, 1937),

$(5, 5)$, $(4, 4)$ (A. A. BUCHSTAB, 1938-1940),

(a, b) $(a + b \leqslant 6)$ (P. KUHN, 1953-1954).

WANG proved, in 1956-1957, that

$$(3, 4), (3, 3), (a, b) \quad (a + b \leqslant 5), (2, 3)$$

in which $(3, 3)$ was proved by A. I. VINOGRADOV independently, in 1956, and $(2, 3)$ was announced by SELBERG but no proof of $(2, 3)$ had been appeared.

If we take $A = \{n - p, p \leqslant n\}$, where p denotes a prime number and P the set of all primes $\leqslant n^{1/(\ell+1)}$, then a positive lower estimation of $S(A, P)$ leads the proposition $(1, \ell)$.

In 1932, ESTERMANN first proved $(1, 6)$ under the assumption of GRH (Grand Riemann hypothesis). Without any improved hypothesis, A. RENYI provod 1, c) in 1948, where c is a constant. In Renyi's proof, a mean value theorem for $\pi(x ; k, \ell)$ is proved by means of the so called large sieve of LINNICK and RENYI that may be used instead of Quasi-RH, namely

$$(1) \qquad \sum_{k \leqslant x^\delta} \max_{(\ell,k)=1} \left|\pi(x ; k, \ell) - \frac{\mathrm{li}\, x}{\varphi(k)}\right| = O\left(\frac{x}{\log^3 x}\right),$$

where $\pi(x ; k, \ell) = \sum_{p \leqslant x, p \equiv \ell (\mathrm{mod}\ k)} 1$, $\mathrm{li}\, x = \int_2^x \frac{dt}{\log t}$, $\varphi(k)$ the Euler function and δ a certain positive constant. Notice that $\pi(x ; k, \ell)$ should be replaced by a sum with a weight in his original paper. If (1) holds with $\delta = \frac{1}{2} - \varepsilon$ for any pre-assigned positive number ε, it may be used instead of GRH in the proof of Estermann's $(1, 6)$.

WANG improved the 6 to 3 , that is, (1 , 3) under the assumption of GRH or (1) for $\delta = \frac{1}{2} - \varepsilon$.

In 1961, M. B. BARBAN proved that (1) holds for $\delta = \frac{1}{6}$. PAN Cheng Dong proved independently that (1) is true for $\delta = \frac{1}{3}$ and derived (1 , 5) . BARBAN and PAN also proved (1) with $\delta = \frac{3}{8}$ and (1 , 4) . Finally, E. BOMBIERI and VINOGRADOV established independently the formula (1) with $\delta = \frac{1}{2} - \varepsilon$ so they proved (1 , 3) . More precisely, Bombieri's formula may be stated as follows

$$\sum_{k \leq x^{1/2}/(\log x)^B} \max_{(\ell,k)=1} \left|\pi(x ; k , \ell) - \frac{\mathrm{li}\, x}{\varphi(k)}\right| = O\left(\frac{x}{\log^A x}\right) ,$$

where A is a given positive constant and $B = B(A)$. Bombieri's result is slightly stronger than (1) with $\delta = \frac{1}{2} - \varepsilon$. Bombieri's formula has many important applications in number theory.

In 1966, CHEN gave the previous method an important improvement. So he proved (1 , 2) .

THEOREM 1 (CHEN). - Every large even integer is a sum of a prime and a product of at most 2 primes.

There are several simplified proofs of Chen's theorem of which one was given by DING Xia-Xi, PAN and WANG [1]. It may be sketched as follows.

Let x denote an even integer and $S(x , q , x^{1/\alpha})$ denote the number of primes such that

$$2 \leq p \leq x , \quad x - p \not\equiv 0 (\mathrm{mod}\ p') , \quad x - p \equiv a (\mathrm{mod}\ q) , \quad 2 \leq p' \leq x^{1/\alpha} , \quad p' \nmid q .$$

Let

$$(2) \quad M = S(x , 2 , x^{1/10}) - \frac{1}{2} \sum_{x^{1/10} < p \leq x^{1/3}} S(x , 2p , x^{1/10}) - \frac{1}{2} \sum_{(p_{1,2}), p_3 \leq x/p_1 p_2, x-p=p_1 p_2 p_3} 1 ,$$

where p , p_1 , p_2 , p_3 are primes, and $(p_{1,2})$ denotes the condition

$$x^{1/10} < p \leq x^{1/3} \leq p_2 \leq \left(\frac{x}{p_1}\right)^{1/2} .$$

Then a positive lower estimation for M when x is large implies (1 , 2) , since there exists a prime p such that $x - p$ has at most 1 prime factor in the interval $[x^{1/10} , x^{1/3}]$ and 1 prime factor $> x^{1/3}$ or $x - p$ has only prime factors $> x^{1/3}$. The first and second terms in the right hand side of (2) may be evaluated by the sieve method of Selberg and the Bombieri's formula. The third term may be estimated directly by the following mean value theorem which is similar to that of Bombieri.

Let $2 \leq y \leq x$. Let

$$\pi(y , a , q ; \ell) = \sum_{p \leq y/a, ap \equiv \ell (\mathrm{mod}\ q)} 1 .$$

Then we have the following theorem.

THEOREM 2. - <u>For any given positive constant</u> A <u>and positive number</u> ε , <u>the estimation</u>

$$\sum_{q\leq x^{\frac{1}{2}}/(\log x)^B} \max_{g\leq x} \max_{(\ell,q)=1} \left|\sum_{A_1<a\leq A_2} f(a)\left(\pi(y,a,q,\ell) - \frac{\mathrm{li}\ y/a}{\varphi(q)}\right)\right| = O\left(\frac{x}{\log^A x}\right)$$

holds for $(\log y)^{2B} < A_1 \leq A_2 < y^{1-\varepsilon}$, where $|f(a)| \leq 1$, $B = A + 7$ and the constant implied by the symbole "O" depends only on ε and A .

REFERENCE

[1] PAN Cheng-Dong, DING Xia-Xi and WANG Yuan. - On the representation of every large even integer as a sum of a prime and an almost prime, Scientia Sinica, t. 5, 1975, p. 599-610.